# Effect of finite-bandwidth squeezing on inhibition of atomic-phase decays

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The inhibition of atomic-phase decays by squeezed light, and consequent line narrowing, is reexamined for the case of finite-bandwidth squeezing, as produced in the output of a degenerate parametric amplifier. The extension of previous work to incorporate the effect of nonwhite, or colored, noise is accomplished via an adjoint equation, which gives the time evolution of a matrix functional of the incoming field. Simple equations for atomic variables are derived in which noise operators are defined so as to commute with each other and with all other quantities. This facilitates the reduction of a purely quantum-mechanical problem to a classical problem, solvable by classical stochastic-approximation methods. In particular, small-noise-approximation methods and simulations are carried out for a wide range of parameters describing the squeezed-field input. The method of simulation is extended to the computation of correlation functions and spectra for a realistically achievable set of parameters, with consideration given to reflections of the input in the emitted light. The fundamental effect is confirmed, and modifications to previous results are given.

## I. INTRODUCTION

Although squeezed light has now been successfully produced in the laboratory,[1] squeezing is only over a finite bandwidth. Thus effects predicted by the use of broadband theories should be carefully analyzed to ensure that they are still present when a realistic source of squeezed light is used. In this paper, we shall investigate the suppression of atomic-phase decays as first treated by Gardiner.[2]

For the broadband (squeezed-light) case considered, the equations of motion for the atomic "spin" operators were found to be

$$\begin{aligned}\langle \dot{S}_x \rangle &= -\gamma(N+M+\tfrac{1}{2})\langle S_x \rangle \equiv -\gamma_x \langle S_x \rangle \ , \\ \langle \dot{S}_y \rangle &= -\gamma(N-M+\tfrac{1}{2})\langle S_y \rangle \equiv -\gamma_y \langle S_y \rangle \ , \\ \langle \dot{S}_y \rangle &= -\gamma(2N+1)\langle S_z \rangle - \gamma \equiv -\gamma_z \langle S_z \rangle - \gamma \ ,\end{aligned} \tag{1.1}$$

where $\gamma$ is the natural linewidth of the atom. In the case of ideal broadband squeezing one can choose $M=\sqrt{N(N+1)}$, so that for sufficiently large $N$ (high squeezing)

$$\gamma_y \approx \frac{1}{8N}, \quad \gamma_x \approx \gamma_z \approx 2N+1 \ . \tag{1.2}$$

That is, while $\gamma_x$ and $\gamma_z$ become large, $\gamma_y$ can be made as small as we please. Hence the projection of the original orientation of the Bloch vector on the direction of the low-noise quadrature is preserved. In the original paper it was recognized that in practice the finite squeezing bandwidth may limit the validity of the approach used, and that alternative methods for computing the spectrum should be investigated. Squeezing experiments have since borne out such a need.

In a second paper Gardiner *et al.*[3] considered this problem in more detail, developing an input-output formalism that yielded a general quantum Langevin equation, from which the master equation and field-system correlations could be computed using methods adapted from classical stochastics. The question of validity was examined more carefully in this work, but, despite improving on previous effects, the theory was still limited to squeezing bandwidths larger than $\gamma_x$ and $\gamma_y$. For large squeezing, when the time scale of fluctuations in one quadrature becomes very slow (as in the output of a degenerate parametric amplifier[4]), it is clear that another approach is required. It is the aim of this paper to develop methods of dealing with this situation and to give predictions for experimentally accessible values of the parameters.

## II. ADJOINT EQUATION AND QUANTUM LANGEVIN EQUATIONS

Following Gardiner *et al.*[3] we can derive quantum Langevin equations for the atomic system operators

$$\begin{aligned}\dot{S}_x &= -\Omega S_y \ , \\ \dot{S}_y &= \Omega S_x + (\hbar\Omega)^{-1/2}[\xi(t), S_z]_+ \ , \\ \dot{S}_z &= -\gamma - (\hbar\Omega)^{-1/2}[\xi(t), S_y]_+ \ ,\end{aligned} \tag{2.1}$$

where $\xi(t)=\sqrt{2\gamma c}\,\dot{A}_{\rm in}(0,t)$ is the incoming electric field evaluated at the atom. We move to a frame rotating at frequency $\Omega$ and introduce the quadrature phase operators of the incoming field $X_{\rm in}(t)$ and $Y_{\rm in}(t)$ as follows:

$$\xi(t) = -2\sqrt{2\gamma c}\,[Y_{\rm in}(t)\cos(\Omega t) + X_{\rm in}(t)\sin(\Omega t)] \ . \tag{2.2}$$

Rapidly rotating terms are then dropped from the equations of motion. Using the methods outlined in the Appendix, an adjoint equation can be developed for a quantity $\mu(t)$, which is a $2\times 2$ matrix functional of the incoming electric field operator $\xi(t)$. Defining

$$\bar{S}_i(t) = {\rm Tr}_{\rm atom}[S_i \mu(t)] \tag{2.3}$$

 

as the atomic average of spin operators, we can derive from the adjoint equation equations of motion for the $\bar{S}_i(t)$, which are

$$\begin{aligned}\dot{\bar{S}}_x &= -c_0\alpha_X(t)\bar{S}_z \ , \\ \dot{\bar{S}}_y &= -c_0\alpha_Y(t)\bar{S}_z \ , \\ \dot{\bar{S}}_z &= -\gamma + c_0[\alpha_X(t)\bar{S}_x + \alpha_Y(t)\bar{S}_y] \ ,\end{aligned} \tag{2.4}$$

where $c_0=(8\gamma c/\hbar\Omega)^{1/2}$ and

$$\alpha_X(t)\mu \equiv \tfrac{1}{2}[X_{\rm in}(t),\mu]_+ , \quad \alpha_Y(t)\mu \equiv \tfrac{1}{2}[Y_{\rm in}(t),\mu]_+ \ . \tag{2.5}$$

With these definitions, $\alpha_X(t)$ and $\alpha_Y(t)$ can be shown to commute with each other and with everything else. This implies that the equations can be treated as classical $c$-number equations, and so we need only specify the statistics of the operators $\alpha_X$ and $\alpha_Y$, which behave simply like classical random quantities.

## III. STATISTICS OF THE SQUEEZED-LIGHT SOURCE

The most successful source of squeezed light at present is the output from a degenerate parametric amplifier (paramp).[4] The correlation functions in this case are[5]

$$\begin{aligned}\langle \alpha_X(t)\alpha_X(t')\rangle &= \frac{\hbar\Omega}{2c}\frac{1}{4}\left[(\lambda^2-\mu^2)\frac{e^{-\mu|t-t'|}}{2\mu} + \delta(t-t')\right] , \\ \langle \alpha_Y(t)\alpha_Y(t')\rangle &= \frac{\hbar\Omega}{2c}\frac{1}{4}\left[-(\lambda^2-\mu^2)\frac{e^{-\lambda|t-t'|}}{2\lambda} + \delta(t-t')\right] .\end{aligned} \tag{3.1}$$

The first term in each expression gives the effect of squeezing, while the second $\delta$-correlated term represents vacuum fluctuations. That this is, in fact, a valid description of the output has been confirmed by the experiments of Kimble and co-workers.[1] The parameters $\lambda$ and $\mu$ are related to experiment by

$$\lambda = \tfrac{1}{2}\gamma_c + \epsilon, \quad \mu = \tfrac{1}{2}\gamma_c - \epsilon \ , \tag{3.2}$$

where $\gamma_c$ is the cavity damping of the paramp and $\epsilon$ is the amplifier driving strength. The practical limitations of the broadband theory are highlighted by the fact that typically $\gamma_c \approx 10^{-8}$ sec$^{-1}$ and $\epsilon \approx 0.3\gamma_c$ (at best), which are of the order of an atomic lifetime $\gamma^{-1}$. The problem becomes more extreme if we consider the limiting case $\epsilon \to \frac{1}{2}\gamma_c$. In this limit one achieves arbitrarily large squeezing in the $Y$ quadrature, and its correlation time $\lambda^{-1}$ remains finite. Meanwhile, the $X$ quadrature becomes extremely large, and varies on a very slow time scale; in fact, its correlation time $\mu^{-1}$ approaches infinity. This kind of behavior will always be present in light from a system which depends on squeezing produced very close to an instability.

It is useful to make a connection between the correlation functions (3.1) and the broadband theory used in previous works. If $\lambda$ and $\mu$ are *both* very large compared to $\gamma$, then we can approximate the correlation functions by $\delta$ functions

$$\begin{aligned}\langle \alpha_X(t)\alpha_X(t')\rangle &\approx \frac{\hbar\Omega}{2c}\frac{\lambda^2}{4\mu^2}\delta(t-t') \quad (\gamma\lambda^2 \ll 8\mu^3) \\ \langle \alpha_Y(t)\alpha_Y(t')\rangle &\approx \frac{\hbar\Omega}{2c}\frac{\mu^2}{4\lambda^2}\delta(t-t') \quad (\gamma\mu^2 \ll 8\lambda^3) \ ,\end{aligned} \tag{3.3}$$

where the precise conditions for validity are given in parentheses, and have been derived previously.[3] This means that white-noise techniques can be applied to get the original broadband equations, with

$$\begin{aligned}2N+1 &= \tfrac{1}{2}(\mu^2/\lambda^2 + \lambda^2/\mu^2) \ , \\ 2M &= \tfrac{1}{2}(\lambda^2/\mu^2 - \mu^2/\lambda^2) \ .\end{aligned} \tag{3.4}$$

These serve as a benchmark for other more accurate methods of solution.

## IV. SOLUTION OF THE EQUATIONS OF MOTION

The other methods of solution we shall consider follow quite naturally from the form of Eqs. (2.4), which we can write as

$$\dot{u} = [A\alpha_X(t) + B\alpha_Y(t)]u + c \ , \tag{4.1}$$

where

$$\begin{aligned}u &= \begin{bmatrix}\bar{S}_x \\ \bar{S}_y \\ \bar{S}_z\end{bmatrix}, \quad A = c_0\begin{bmatrix}0&0&-1\\0&0&0\\1&0&0\end{bmatrix} , \\ B &= c_0\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix}, \quad c = \begin{bmatrix}0\\0\\-\gamma\end{bmatrix} .\end{aligned} \tag{4.2}$$

In all, four approaches have been taken to solving these equations, as follows.

### A. Approximation of the squeezed quadrature by zero

If the squeezing is good, it may be justifiable to simply omit the noise in the squeezed quadrature by setting $\alpha_Y(t)=0$. An exact solution of (4.1) is then possible,

$$\begin{aligned}u(t) = &\exp\left[A\int_0^t dt'\alpha_X(t')\right]u(0) \\ &+ \int_0^t dt' \exp\left[A\int_{t'}^t ds\,\alpha_X(s)\right]c \ .\end{aligned} \tag{4.3}$$

Noting the Gaussian nature of $\alpha_X(t')$, and in particular the result

$$\langle \exp(ix)\rangle = \exp(-\tfrac{1}{2}\langle x^2\rangle) \ , \tag{4.4}$$

which holds for $x$ Gaussian with $\langle x\rangle = 0$, we get

$$\langle S_x(t)\rangle = \exp\left[-\gamma\left[\frac{\lambda^2}{2\mu^2}t - \frac{(\lambda^2-\mu^2)}{2\mu^3}(1-e^{-\mu t})\right]\right]\langle S_x(0)\rangle ,$$
$$\langle S_y(t)\rangle = \langle S_y(0)\rangle , \tag{4.5}$$
$$\langle S_z(t)\rangle = \exp\left[-\gamma\left[\frac{\lambda^2}{2\mu^2}t - \frac{(\lambda^2-\mu^2)}{2\mu^3}(1-e^{-\mu t})\right]\right]\langle S_z(0)\rangle - \gamma\int_0^t d\tau \exp\left[-\gamma\left[\frac{\lambda^2}{2\mu^2}\tau - \frac{(\lambda^2-\mu^2)}{2\mu^3}(1-e^{-\mu\tau})\right]\right] ,$$

where $\langle \underline{\mathbf{S}} \rangle$ represents here the average of $\underline{\mathbf{S}}$ over both atom and bath density operators. Although this is a rather drastic approximation, it produces the essential features that characterize all of the approaches. In particular, the decay of $\langle S_x(t)\rangle$ and $\langle S_z(t)\rangle$ is slower than the corresponding white-noise result, and a Gaussian character is now displayed (in contrast to simple exponential). As one might expect, the range of validity of such an approximation theory is limited, and in fact one finds unphysical behavior [$\langle S_z(t\to\infty)\rangle$ less than $-1$] for sufficiently large values of $\gamma$ relative to $\mu$ and $\lambda$.

### B. White-noise approximation to the squeezed quadrature

Our second approach involves moving to an interaction picture defined by

$$u = V(t)v, \quad V(t) = \exp\left[A\int_{t_0}^t \alpha_X(t')dt'\right] . \tag{4.6}$$

The equation of motion for $v(t)$ is then

$$\dot{v} = G(t)v + f(t) , \tag{4.7}$$

in which

$$G(t) = c_0\begin{bmatrix} 0 & \xi_1(t) & 0 \\ -\xi_1(t) & 0 & -\xi_2(t) \\ 0 & \xi_2(t) & 0 \end{bmatrix} , \tag{4.8}$$

$$f(t) = -\gamma\begin{bmatrix} \sin\left[c_0\int_{t_0}^t \alpha_X(t')dt'\right] \\ 0 \\ \cos\left[c_0\int_{t_0}^t \alpha_X(t')dt'\right] \end{bmatrix} , \tag{4.9}$$

and

$$\xi_1(t) = \alpha_Y(t)\sin\left[c_0\int_{t_0}^t \alpha_X(t')dt'\right] ,$$
$$\xi_2(t) = \alpha_Y(t)\cos\left[c_0\int_{t_0}^t \alpha_X(t')dt'\right] . \tag{4.10}$$

The advantage of such a transformation is that now both noises $\xi_1(t)$ and $\xi_2(t)$ are of short correlation time and independent at least in the second moments. The assumption of independence requires that the random functions

$$\sin\left[c_0\int_{t_0}^t \alpha_X(t')dt'\right]$$

and

$$\cos\left[c_0\int_{t_0}^t \alpha_X(t')dt'\right]$$

vary more slowly than $\alpha_Y(t)$. We can estimate the rms value of $c_0\alpha_X(t)$ to be $(\gamma\lambda^2/2\mu)^{1/2}$. If $\mu$ is small, $\alpha_X(t)$ varies slowly, so the cosine and sine functions are functions of an almost constant very large argument. We require the effective frequency of these functions to be much less than the correlation time of $\alpha_Y(t)$,

$$(\gamma\lambda^2/2\mu)^{1/2} \ll \lambda, \quad \text{which implies } \gamma \ll 2\mu . \tag{4.11}$$

Furthermore, if the $Y$ quadrature white-noise condition is satisfied [this is the weaker of the two conditions given in (3.3), $\gamma\mu^2 \ll 8\lambda^3$], then a "partial-white-noise" approximation to the stationary ($t_0\to-\infty$) correlation functions is possible,

$$\langle \xi_1(t)\xi_1(t')\rangle = \langle \xi_2(t)\xi_2(t')\rangle = \frac{\hbar\Omega}{2c}\frac{\mu^2}{8\lambda^2}\delta(t-t') ,$$
$$\langle \xi_1(t)\xi_2(t')\rangle = 0 . \tag{4.12}$$

Even with this approximation the equations are still not exactly soluble. However, since both $\xi_1(t)$ and $\xi_2(t)$ are small when the squeezing is large, small-noise-approximation methods[6] can be employed to yield

$$\langle S_x(t)\rangle = \exp\left[-\gamma\frac{\mu^2}{4\lambda^2}t - \gamma\left[\frac{\lambda^2}{2\mu^2}t - \frac{(\lambda^2-\mu^2)}{2\mu^3}(1-e^{-\mu t})\right]\right]\langle S_x(0)\rangle ,$$
$$\langle S_y(t)\rangle = \exp\left[-\frac{\mu^2}{2\lambda^2}t\right]\langle S_y(0)\rangle , \tag{4.13}$$
$$\langle S_z(t)\rangle = \exp\left[-\gamma\frac{\mu^2}{4\lambda^2}t - \gamma\left[\frac{\lambda^2}{2\mu^2}t - \frac{(\lambda^2-\mu^2)}{2\mu^3}(1-e^{-\mu t})\right]\right]\langle S_z(0)\rangle$$
$$-\gamma\int_0^t d\tau \exp\left[-\gamma\frac{\mu^2}{4\lambda^2}\tau - \gamma\left[\frac{\lambda^2}{2\mu^2}\tau - \frac{(\lambda^2-\mu^2)}{2\mu^3}(1-e^{-\mu\tau})\right]\right] .$$

As expected, the expression for $\langle S_y(t)\rangle$ is simply the white-noise result one obtains from solving (1.1). The results for $\langle S_x(t)\rangle$ and $\langle S_z(t)\rangle$ exhibit the same features found in Sec. IV A and, within the range of validity determined above, $\langle S_z(t)\rangle$ does not take on unphysical values in the long-time limit.

### C. A random-rotating-wave approximation

We might wish to consider the other extreme, i.e., $\gamma \gg 2\mu$. Then

$$\sin\left[c_0\int_{t_0}^{t}\alpha_X(t')dt'\right]$$

and

$$\cos\left[c_0\int_{t_0}^{t}\alpha_X(t')dt'\right]$$

will vary much more rapidly than $\alpha_Y(t)$ and therefore "wash out" the exponentially correlated squeezed noise. However, $\alpha_Y(t)$ has a genuinely white part resulting from vacuum fluctuations, which will not be affected by the rapid oscillation of the sine and cosine terms. Hence we write

$$\langle \xi_1(t)\xi_1(t')\rangle = \langle \xi_2(t)\xi_2(t')\rangle = \frac{\hbar\Omega}{2c}\frac{1}{8}\delta(t-t') , \tag{4.14}$$

and following the same small-noise procedure used above, we find

$$\begin{aligned}
\langle S_x(t)\rangle &= \exp\left[-\frac{\gamma}{4}t-\gamma\left[\frac{\lambda^2}{2\mu^2}t-\frac{(\lambda^2-\mu^2)}{2\mu^3}(1-e^{-\mu t})\right]\right]\langle S_x(0)\rangle ,\\
\langle S_y(t)\rangle &= \exp\left[-\frac{\gamma}{2}t\right]\langle S_y(0)\rangle ,\\
\langle S_z(t)\rangle &= \exp\left[-\frac{\gamma}{4}t-\gamma\left[\frac{\lambda^2}{2\mu^2}t-\frac{(\lambda^2-\mu^2)}{2\mu^3}(1-e^{-\mu t})\right]\right]\langle S_z(0)\rangle\\
&\quad -\gamma\int_0^t d\tau\exp\left[-\frac{\gamma}{4}\tau-\gamma\left[\frac{\lambda^2}{2\mu^2}\tau-\frac{(\lambda^2-\mu^2)}{2\mu^3}(1-e^{-\mu\tau})\right]\right] .
\end{aligned} \tag{4.15}$$

In addition to the condition $\gamma \gg 2\mu$, the familiar rotating-wave-type approximation requires that $2\gamma\mu \ll \lambda^2$.

We see that the decay rate of $\langle S_y(t)\rangle$ returns to the normal (unsqueezed) vacuum value, while the factor $\exp(-\gamma\mu^2 t/4\lambda^2)$ from Sec. IV B is replaced by the $\mu$-independent factor $\exp(-\gamma t/4)$ in the expressions for $\langle S_x(t)\rangle$ and $\langle S_z(t)\rangle$. The important conclusion to be drawn from these results is that, unlike what might have been thought previously, arbitrarily large squeezing will not necessarily result in an arbitrarily slow decay of $\langle S_y(t)\rangle$. This arises from the fact that in the limit of perfect squeezing $\mu/\gamma \to 0$.

### D. Simulation methods

The simplicity of Eqs. (2.4), and the special nature of $\alpha_X(t)$ and $\alpha_Y(t)$, have led us to consider simulations by normal stochastic methods. Indeed, this approach has proved to be the most practical and versatile method of computation. Problems with the simulation of white noise are removed by averaging Eq. (2.4) over the white-noise components of $\alpha_X(t)$ and $\alpha_Y(t)$, i.e., we define

$$\alpha_X(t) = \alpha_X^{\omega}(t) + \alpha_X^{c}(t) , \tag{4.16}$$

where

$$\langle \alpha_X^{\omega}(t)\alpha_X^{\omega}(t')\rangle = \frac{\hbar\Omega}{2c}\frac{1}{4}\delta(t-t') , \tag{4.17}$$

$$\langle \alpha_X^{c}(t)\alpha_X^{c}(t')\rangle = \frac{\hbar\Omega}{2c}\frac{1}{4}(\lambda^2-\mu^2)\frac{e^{-\mu|t-t'|}}{2\mu} , \tag{4.18}$$

and similarly for $\alpha_Y(t)$. The equations of motion become

$$\begin{aligned}
\dot{\tilde{S}}_x &= -\frac{\gamma}{2}\tilde{S}_x - c_0\alpha_X^{c}(t)\tilde{S}_z ,\\
\dot{\tilde{S}}_y &= -\frac{\gamma}{2}\tilde{S}_y - c_0\alpha_Y^{c}(t)\tilde{S}_z ,\\
\dot{\tilde{S}}_z &= -\gamma - \gamma\tilde{S}_z + c_0\alpha_X^{c}(t)\tilde{S}_x + c_0\alpha_Y^{c}(t)\tilde{S}_y .
\end{aligned} \tag{4.19}$$

The operator $\alpha_Y^{c}(t)$ cannot be represented by a real random variable, but instead by a pure imaginary variable, since its variance is negative. This enables the variables $\tilde{S}_x$, $\tilde{S}_y$, and $\tilde{S}_z$ to develop imaginary parts, but these will all average to zero. In practical simulations, we simply take the real part of the computed averages over $\alpha_X^{c}$ and $\alpha_Y^{c}$.

## V. NUMERICAL RESULTS

The approximation methods and simulations have been evaluated and compared with the white-noise results over a wide range of parameters. Comparison with experiment involves the evaluation of time correlation functions and spectra, which are somewhat more involved than the decays of the spin averages since the non-Markov nature of the processes being studied means that the quantum regression theorem is not valid. Methods have been developed to simulate these and the output

spectra, but we have not correspondingly extended the approximation methods.

### A. Comparison of the approximation methods, simulations, and white-noise results

The regions of validity of the various methods are, in summary, the following: white noise (WN), $\gamma\mu^2 \ll 8\lambda^3$ and $\gamma\lambda^2 \ll 8\mu^3$; partial white noise (PWN), $\gamma \ll 2\mu$ and $\gamma\mu^2 \ll 8\lambda^3$; rotating wave (RWA), $\gamma \gg 2\mu$ and $2\gamma\mu \ll \lambda^2$; where it is always understood that $\mu < \lambda$.

The various regions are plotted in Fig. 1 and the locations of the parameters for other figures are given. It can be seen that there is only a small region where no approximation method is expected to have some validity.

#### *1. White noise and partial white noise comparison*

In Fig. 2 are plotted simulations for $\mu/\gamma = 8$, $\lambda/\gamma = 32$ (corresponding to 94% squeezing), compared with white-noise and partial-white-noise theories. It can be seen that the agreement between all three methods is quite good, as expected, since this lies in the region where the white-noise methods should be valid. It is interesting to notice that (i) the fast decays of $\langle S_x \rangle$ and $\langle S_z \rangle$ are given quite accurately by the PWN method—the pure exponential decay predicted from the WN method is not seen except at rather longer times; (ii) the slow decay of $\langle S_y \rangle$ is the same in PWN and WN methods, and is somewhat slower than actually found. Numerically, the predicted decay constant is $\mu^2/2\lambda^2 = 0.031$, while the simulated decay constant is 0.053.

#### *2. Region where partial white noise is applicable, but not white noise*

In Fig. 3 are plotted simulations for $\mu/\gamma = 3$, $\lambda/\gamma = 7$, corresponding to 82% squeezing. The simulations are

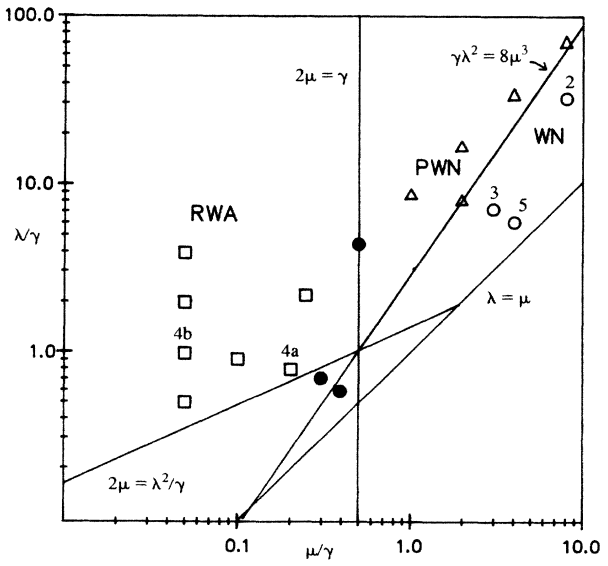


FIG. 1. Range of parameters considered for simulation with regions of validity of the various approximation methods. ○ WN theory, △ PWN, □ RWA, ● no analytical theory.

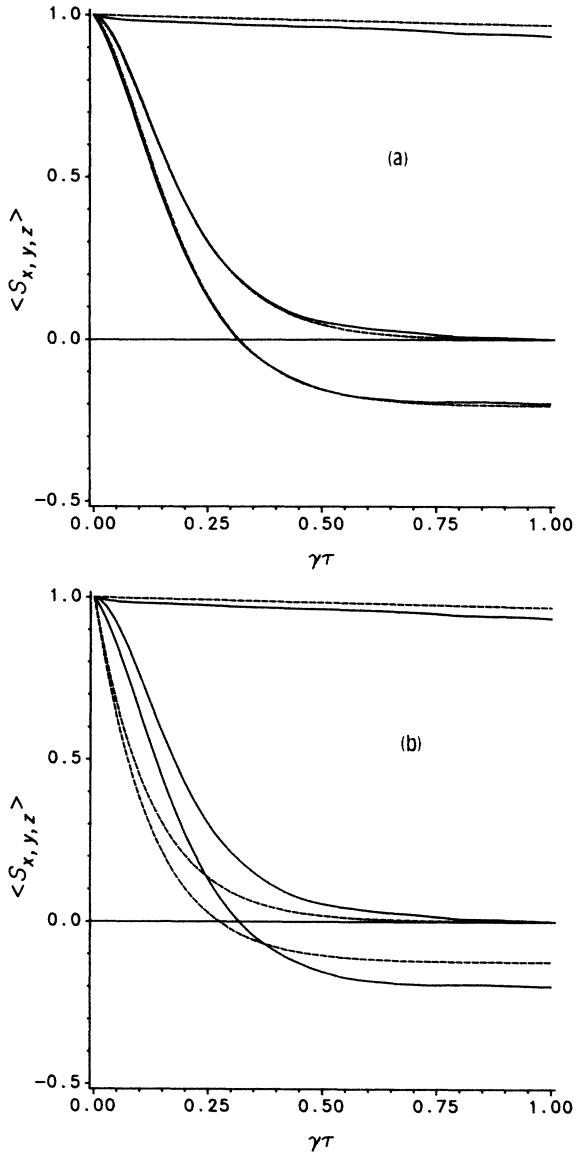


FIG. 2. Decays of the spin averages $\langle S_x \rangle$, $\langle S_y \rangle$, and $\langle S_z \rangle$ for $\mu/\gamma = 8$, $\lambda/\gamma = 32$. Comparison of (a) simulation, 10 000 trials (solid line), and PWN theory (dashed line); (b) simulation and WN theory.

again compared with the PWN and WN methods, and it is clear that this time the WN method is badly in error for the fast decays. Furthermore, the equilibrium value of $\langle S_z \rangle$ is significantly less in the simulations and in the PWN theory than in the WN method. This is expected to happen, since the long correlation time in the unsqueezed quadrature causes a significant reduction in the available power. However, the slow decay is now significantly faster than predicted by the PWN and WN theories (which are identical for this component)—the numerical comparison is between $\mu^2/2\lambda^2 = 0.0918$ and the simulation decay constant 0.15.

#### *3. Random-rotating-wave approximation region*

In Fig. 4 we plot two graphs in the region of validity of this approximation. Agreement is quite good, but by no

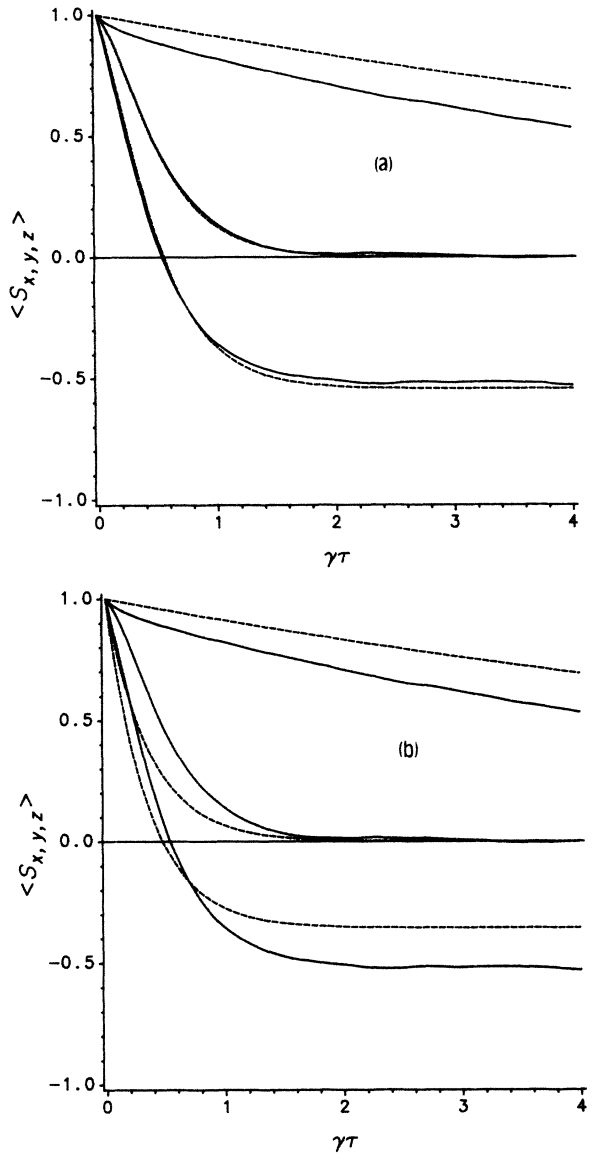


FIG. 3. Decays of the spin averages for $\mu/\gamma=3$, $\lambda/\gamma=7$. Comparison of (a) simulation, 5000 trials (solid line), and PWN theory (dashed line); (b) simulation and WN theory.

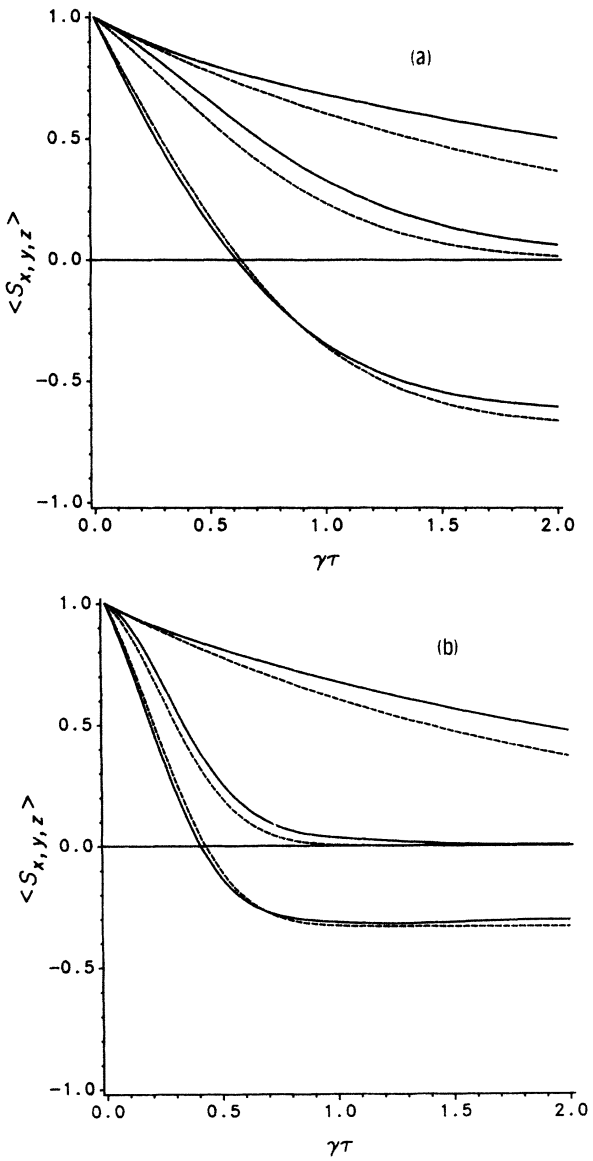


FIG. 4. Comparison of simulation, 5000 trials (solid line), and RWA theory (dashed line) for (a) $\mu/\gamma=0.2$, $\lambda/\gamma=0.8$; (b) $\mu/\gamma=0.05$, $\lambda/\gamma=1.0$.

means perfect. Although the random-rotating-wave-approximation (RWA) method predicts that there should be no inhibition of the $\langle S_y \rangle$ decay, the simulations do show some slight inhibition. On the same scale, the white-noise prediction would give almost a horizontal line, so indeed the essence of the RWA prediction is correct—most of the inhibition is lost in this region, and that which remains is almost certainly unobservable.

#### *4. Situation with modest squeezing*

In Fig. 5 are plotted results for $\mu/\gamma=4$, $\lambda/\gamma=6$. All methods are in good agreement with each other, and even with this relatively low squeezing the difference between the two decay constants is clearly visible. The slow decay has a predicted decay constant of $\mu^2/2\lambda^2=0.222$, while the simulated decay constant is 0.25.

### B. Computation of atomic correlation functions

The simulation of the correlation functions is a little more complicated, and we need to use the result of the Appendix that

$$\langle S_i(t)S_j(t') \rangle = \langle \mathrm{Tr}_s[S_i\mu_j(t,t')] \rangle \ , \tag{5.1}$$

where by $\mathrm{Tr}_s$ we mean the trace over the system and $\mu_j(t,t')$ is a solution of the adjoint equation subject to the initial condition

$$\mu_j(t',t')=S_j\mu(t') \ . \tag{5.2}$$

We now suppose that $\mu(t')$ represents a stationary density operator.

In order to use this formula, we need to represent $\mu(t')$

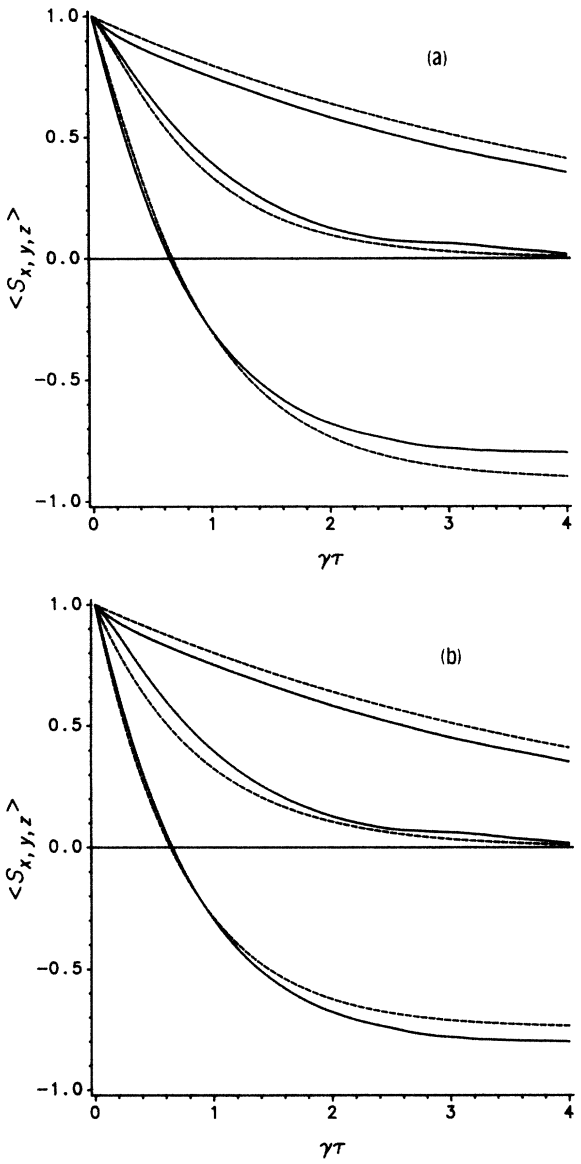


FIG. 5. Decays of the spin averages for $\mu/\gamma=4$, $\lambda/\gamma=6$. Comparison of (a) simulation, 5000 trials (solid line), and PWN theory (dashed line); (b) simulation and WN theory.

in terms of the solutions for $\bar{S}_i(t')$ which can be obtained as individual runs of a simulation. We also need an expression for $\mu_j(t',t')$.

The solutions of the equations of motion (2.4) can be written in the form

$$\bar{S}_i(t)=\sum_j f_{ij}(t,t')\bar{S}_j(t')+g_i(t,t')\bar{I}(t') , \tag{5.3}$$

where $\bar{I}(t)=\bar{I}(t')=\mathrm{Tr}_s[\mu(t)]$, and we do not set $\bar{I}(t)=1$, since $\mu(t)$ may not be normalized to 1 when we use it in the construction of the correlation function, though clearly $\bar{I}(t)$ will be constant. Using

$$\mathrm{Tr}_s(S_iS_j)=2\delta_{ij} , \tag{5.4}$$

we can write

$$\mu(t)=\frac{1}{2}\left[\sum_i S_i\bar{S}_i(t)+\bar{I}(t)\right]$$
$$=\frac{1}{2}\left[\sum_{i,j} f_{ij}(t,t')S_i\bar{S}_j(t')+\left[\sum_i S_ig_i(t,t')+1\right]\bar{I}(t')\right] . \tag{5.5}$$

Thus we have a solution for $\mu(t)$ in terms of the functions $f_{ij}(t,t')$ and $g_i(t,t')$, which are, of course, random functions, since they depend on $\alpha_X(t)$ and $\alpha_Y(t)$.

In the limit that $t\to\infty$ and $\bar{I}(t')=1$, the stationary solution for $\mu(t)$ is

$$\mu_{\mathrm{st}}(t)=\frac{1}{2}\left[\sum_i S_i\bar{g}_i(t)+1\right] , \tag{5.6}$$

since, from Eqs. (2.4), $f_{ij}(t,t')\to 0$ $(t\to\infty)$, and $g_i(t,t')\to\bar{g}_i(t)$, a quantity which no longer depends on the initial conditions, but which is a random function.

We now use the solution for $\mu(t)$ and $\mu_{\mathrm{st}}(t)$ to compute the stationary atomic correlation function $\langle S_i(t)S_j(t')\rangle$. This is done as follows. We first evaluate a quantity $\mu_k(t,t')$ as a solution of the form (5.5) with the initial value chosen as

$$\mu_k(t',t')=S_k\mu_{\mathrm{st}}(t')$$
$$=\frac{1}{2}\left[\sum_i S_kS_i\bar{g}_i(t')+S_k\right] , \tag{5.7}$$

so that

$$\mathrm{Tr}_s[S_l\mu_k(t',t')]=\delta_{kl}+i\sum_i \epsilon_{lki}\bar{g}_i(t') \tag{5.8}$$

and

$$\mathrm{Tr}_s[\mu_k(t',t')]=\bar{g}_k(t') . \tag{5.9}$$

Computing $\mu_k(t,t')$using (5.5), we find that the time correlation function is given by

$$\mathrm{Tr}_s[S_i\mu_k(t,t')]=f_{ik}(t,t')+g_i(t,t')\bar{g}_k(t')$$
$$+i\sum_{l,m}\epsilon_{lkm}f_{il}(t,t')\bar{g}_m(t') . \tag{5.10}$$

This is the correlation function obtained by averaging over the atomic variables. To obtain the measured correlation function one must also average over the bath, thus

$$\langle S_i(t)S_k(t')\rangle=\langle f_{ik}(t,t')\rangle+\langle g_i(t,t')\bar{g}_k(t')\rangle$$
$$+i\sum_{l,m}\epsilon_{lkm}\langle f_{il}(t,t')\bar{g}_m(t')\rangle . \tag{5.11}$$

The final term does not factorize unless the random functions $\alpha_X(t)$ and $\alpha_Y(t)$ represent white noise, i.e., unless the process is a Markov process.

### C. Spectra and correlations of radiated light

As detailed above, computation of the atomic correlations using simulations requires modification of the method used for computing the time development of the

simple means. We first allow Eqs. (4.19) to evolve to a stationary state, thereby obtaining a value for $\bar{g}_i(t')$. A new run is then initiated, with four different sets of initial conditions, so that we may identify the different terms $f_{ij}(t,t')$ and $g_i(t,t')$ that contribute to $\mathrm{Tr}_s[S_i\mu_k(t,t')]$. Typically we carry out 5000 such trials, after which the averaging shown in (5.11) is carried out.

This treatment, of course, gives only the atomic correlations, and hence is sufficient only where the effect of reflections of the input in the output can be neglected. In a practical situation this would correspond to introducing a small port to the system, at the cost, however, of sampling only a very small portion of the emitted light. On the other hand, with the inclusion of reflections one would envisage a setup incorporating a system of isolators to separate the output from the input.

In previous work[3] we have been able to treat reflections, limited though to a large squeezing bandwidth. Simulations again offer a straightforward method for computing the effect of reflections; in particular, correlations such as $\langle S_x(t)\alpha_X^c(t')\rangle$ and $\langle S_y(t)\alpha_Y^c(t')\rangle$. We calculate these quantities by simply averaging products of the two components. An expression for the output-field correlation function in the case where reflections are included has been derived previously[3] and takes the form

$$\langle \dot{A}_{\mathrm{out}}(t)\dot{A}_{\mathrm{out}}(t')\rangle = \langle \dot{A}_{\mathrm{in}}(t)\dot{A}_{\mathrm{in}}(t')\rangle + \frac{\hbar\Omega}{2c}\frac{\gamma}{2}\langle [S_y(t),S_y(t')]_+\rangle + \frac{1}{2}\left[\gamma\frac{\hbar\Omega}{2c}\right]^{1/2}\{\langle [\dot{A}_{\mathrm{in}}(t),S_y(t')]_+\rangle + \langle [\dot{A}_{\mathrm{in}}(t'),S_y(t)]_+\rangle\} . \tag{5.12}$$

By using the expressions (2.2) and (2.5), we are able to identify different frequency components and make connections with the quantities calculated through simulations. Of main interest is the correlation function $\langle E_{\mathrm{out}}^{(-)}(t)E_{\mathrm{out}}^{(+)}(t')\rangle$, from which the fluorescent spectrum is determined. We can show

$$\begin{aligned}\langle E_{\mathrm{out}}^{(-)}(t)E_{\mathrm{out}}^{(+)}(t')\rangle &= \langle \alpha_X^c(t)\alpha_X^c(t')\rangle + \langle \alpha_Y^c(t)\alpha_Y^c(t')\rangle + \frac{\hbar\Omega}{2c}\gamma\langle S^+(\,|t-t'|\,)S^-(0)\rangle \\ &\quad - \frac{\hbar\Omega}{2c}\frac{c_0}{4}[\langle \alpha_X^c(t)S_x(t')\rangle + \langle \alpha_X^c(t')S_x(t)\rangle + \langle \alpha_Y^c(t)S_y(t')\rangle + \langle \alpha_Y^c(t')S_y(t)\rangle \\ &\quad - i\langle \alpha_X^c(t)S_y(t')\rangle + i\langle \alpha_X^c(t')S_y(t)\rangle + i\langle \alpha_Y^c(t)S_x(t')\rangle - i\langle \alpha_Y^c(t')S_x(t)\rangle] .\end{aligned} \tag{5.13}$$

Similar expressions can be determined for $\langle X_{\mathrm{out}}(t)X_{\mathrm{out}}(t')\rangle$ and $\langle Y_{\mathrm{out}}(t)Y_{\mathrm{out}}(t')\rangle$, from which one may calculate homodyne spectra. The advantage of examining these spectra, we find, is that the two distinct time scales characterizing the problem are separated.

Earlier computations of the time development of the atomic means suggest that a significant effect should be observable with fluctuations reduced to approximately 50% of the normal vacuum level, with $\gamma_c \approx 10\gamma$. Hence, for computations of spectra, we have concentrated on the pair of parameters $\mu/\gamma=4$, $\lambda/\gamma=6$, which correspond to a nominal 56% squeezing (at frequency $\Omega$).

Figure 6 displays the atomic correlation functions [the dotted lines give the imaginary parts of $\langle S_x(t)S_y(0)\rangle$ and $\langle S_y(t)S_x(0)\rangle$] for $\mu/\gamma=4$, $\lambda/\gamma=6$. The decays we observe are very similar to the decays of the simple atomic means, with the two different decay rates again clearly distinguished. The similarity with previous results holds for all values of $\mu/\gamma$ and $\lambda/\gamma$ considered. Analytical expressions for the correlations can be computed for the various approximation methods used. We do not plot any of these results, but their agreement with simulations follows the same pattern found in analyses of the simple spin averages.

The atom-field correlations display a similar separation of time scales, but possess an inherently noisier character than the atomic correlations, as shown in Fig. 7. Such reflection terms are clearly very significant if included in the output field.

The field-field correlations $\langle \alpha_X^c(t)\alpha_X^c(t')\rangle$, $\langle \alpha_Y^c(t)\alpha_Y^c(t')\rangle$ have been checked and shown to agree with theoretical expressions, which, for convenience, we use from now on. The further assumption that cross

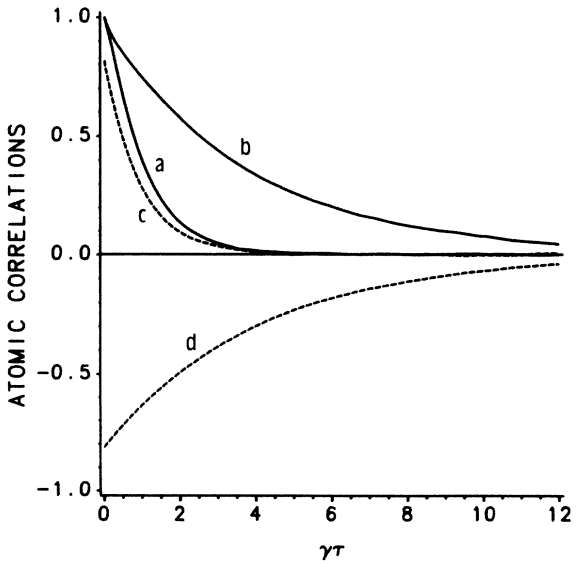


FIG. 6. Atomic correlation functions, simulation, 60 000 trials, $\mu/\gamma=4$, $\lambda/\gamma=6$: *a*, $\langle S_x(\tau)S_x(0)\rangle$; *b*, $\langle S_y(\tau)S_y(0)\rangle$; *c*, $\mathrm{Im}[\langle S_x(\tau)S_y(0)\rangle]$; *d*, $\mathrm{Im}[\langle S_y(\tau)S_x(0)\rangle]$.

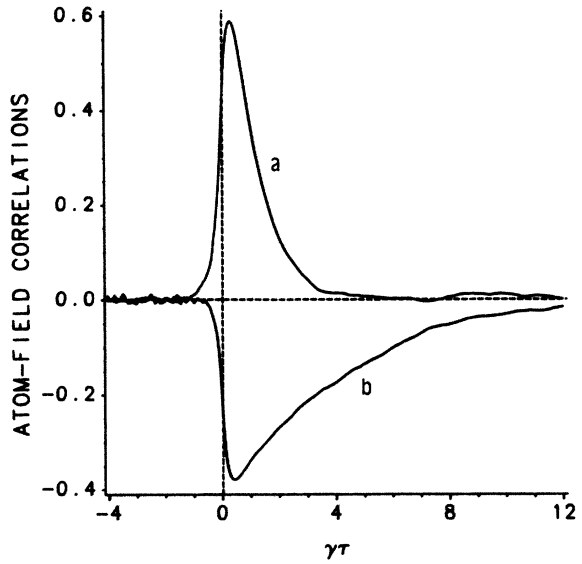


FIG. 7. Atom-field correlations, simulation, 60 000 trials, $\mu/\gamma=4$, $\lambda/\gamma=6$: *a*, $\langle S_x(\tau)\alpha_X^c(0)\rangle$; *b*, $\langle S_y(\tau)\alpha_Y^c(0)\rangle$.

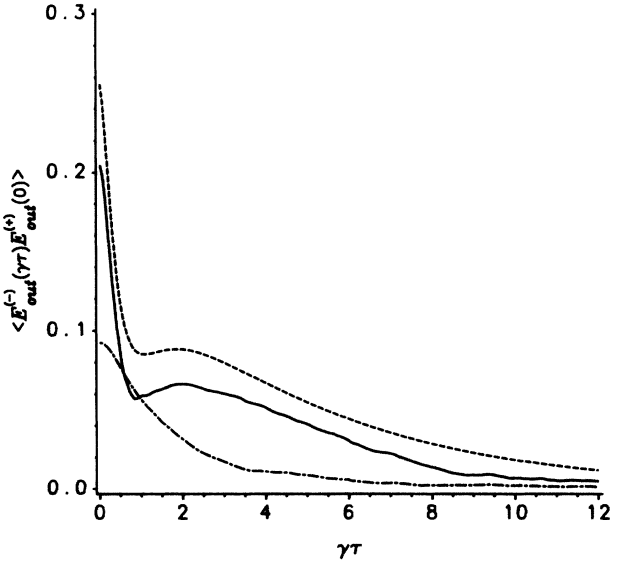


FIG. 8. Correlation functions of emitted light, $\mu/\gamma=4$, $\lambda/\gamma=6$. (i) CWN theory, Eq. (5.14) (dashed line); (ii) simulation, 60 000 trials, reflections included (solid line); (iii) simulation, $\langle S^+(\tau)S^-(0)\rangle$ (i.e., reflections excluded) (dot-dashed line).

correlations of the form $\langle \alpha_X^c(t)\alpha_Y^c(t')\rangle$ are zero has also been checked and confirmed.

The correlation function $\langle E_{\rm out}^{(-)}(t)E_{\rm out}^{(+)}(t')\rangle$, as we have seen above, is merely a sum of the above-mentioned correlations. Certain of these correlations are seen to fluctuate about zero after a certain time, and indications are that further averaging would in fact remove these fluctuations. A certain amount of "smoothing" is then justified; in particular, we set $\langle S_x(\tau)\alpha_x^c(0)\rangle=0$ for $\gamma\tau<-2$ and $\gamma\tau>6$, and $\langle S_y(\tau)\alpha_y^c(0)\rangle=0$ for $\gamma\tau<-2$. All of the correlations making up the last line of Eq. (5.13) were found to fluctuate about zero for all times and so can be set equal to zero, while we set $\langle S_x(\tau)S_x(0)\rangle=\langle S_x(\tau)S_y(0)\rangle=0$ for $\gamma\tau>6$. In Fig. 8 we plot $\langle E_{\rm out}^{(-)}(t)E_{\rm out}^{(+)}(t')\rangle$ with these modifications. We also display the corresponding theoretical result as computed using the corrected white-noise theory (CWN) of Ref. 3 and $\langle S^+(t)S^-(0)\rangle$ as computed by simulation. The CWN result has the form

$$\langle E_{\rm out}^{(-)}(t)E_{\rm out}^{(+)}(t')\rangle=\frac{(\lambda^2-\mu^2)}{4}\left[\frac{e^{-\mu|t-t'|}}{2\mu}-\frac{e^{-\lambda|t-t'|}}{2\lambda}\right]+\frac{\gamma}{2}\frac{N}{2N+1}(e^{-\gamma_x|t-t'|}+e^{-\gamma_y|t-t'|})$$
$$-\frac{\gamma}{4}\frac{(\lambda^2-\mu^2)}{2N+1}\frac{\gamma_x e^{-\mu|t-t'|}-\mu e^{-\gamma_x|t-t'|}}{\mu(\gamma_x^2-\mu^2)}+\frac{\gamma}{4}\frac{(\lambda^2-\mu^2)}{2N+1}\frac{\gamma_y e^{-\lambda|t-t'|}-\lambda e^{-\gamma_y|t-t'|}}{\lambda(\gamma_y^2-\lambda^2)}\,, \tag{5.14}$$

where we have omitted the factor $\hbar\Omega/2c$ (as we do in the graphs) and $N$ is defined in Eq. (3.4).

Spectra are computed using a fast-Fourier-transform (FFT) routine. The long-time behavior of the CWN correlation is known to be exponential, with a decay rate of 0.222. Examination of the other two correlation functions displayed in Fig. 8 also reveals exponential decays, with decay rates (least-squares fit, from $\gamma\tau=6\rightarrow12$) $0.21\pm0.01$ and $0.32\pm0.01$ for $\langle S^+(\tau)S^-(0)\rangle$ and $\langle E_{\rm out}^{(-)}(\tau)E_{\rm out}^{(+)}(0)\rangle$, respectively. An extrapolation of these functions to larger times than shown in the graph is then made to produce a suitable FFT—simulations to such times would be somewhat time consuming and, we feel, unnecessary since the long-time behavior of the correlations has been established, with sufficient information to accurately determine the width of the transform, although the overall shape may be less accurate.

The spectra corresponding to the three correlation functions shown in Fig. 8 are given in Fig. 9. The curves that incorporate reflections clearly exhibit narrowing from a Lorentzian of halfwidth $\gamma/2$ (i.e., the spectrum of spontaneous emission in a normal "unsqueezed" vacuum). However, the spectrum that results from transforming only $\langle S^+(t)S^-(0)\rangle$, and which thus excludes reflections, does not display this effect and possesses a weaker signal strength. We note, though, that all of the curves exhibit non-Lorentzian features. The decay rates quoted above (which might be interpreted as linewidths) are not obvious in the fluorescent spectra due to the presence of other decay rates and features in the correlation functions. They can, however, be "isolated" to some extent via homodyne measurements which are now stan-

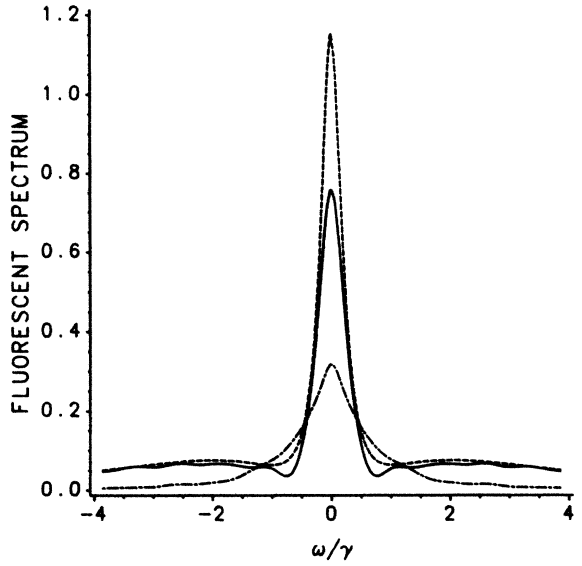


FIG. 9. Fluorescent spectra corresponding to correlation functions shown in Fig. 8.

dard in squeezing experiments.

In Fig. 10 we graph the normally ordered homodyne spectra for the same data collected and displayed in the previous figures. Such measurements separate the two major distinct time scales involved, resulting in spectra with widely different characteristics. Line narrowing and line broadening are now evident for the case which omits reflections.

## VI. CONCLUSION

This paper has shown that the initial simple calculations of Ref. 2 are basically correct, but has added refinements to take account of the kind of squeezed light expected from a degenerate parametric amplifier or, indeed, any device which produces squeezed light near an instability. For in all such situations, there will be a component which becomes large and develops a long correlation time, and it is this that causes most of the effects seen.

The major problem in these experiments is the production of a situation in which all the modes coupling to the atom are squeezed. This might be achieved by producing incoming electric dipole waves, or perhaps by microengineering an appropriate one-dimensional situation.

Apart from the intrinsic interest in the properties of squeezed light, this paper has also shown how the use of the adjoint equation enables simulations and approximate treatments of a quantum-mechanical problem in a way which has not been done before. These methods clearly have other applications and should prove a useful addition to the collection of methods available in quantum optics.

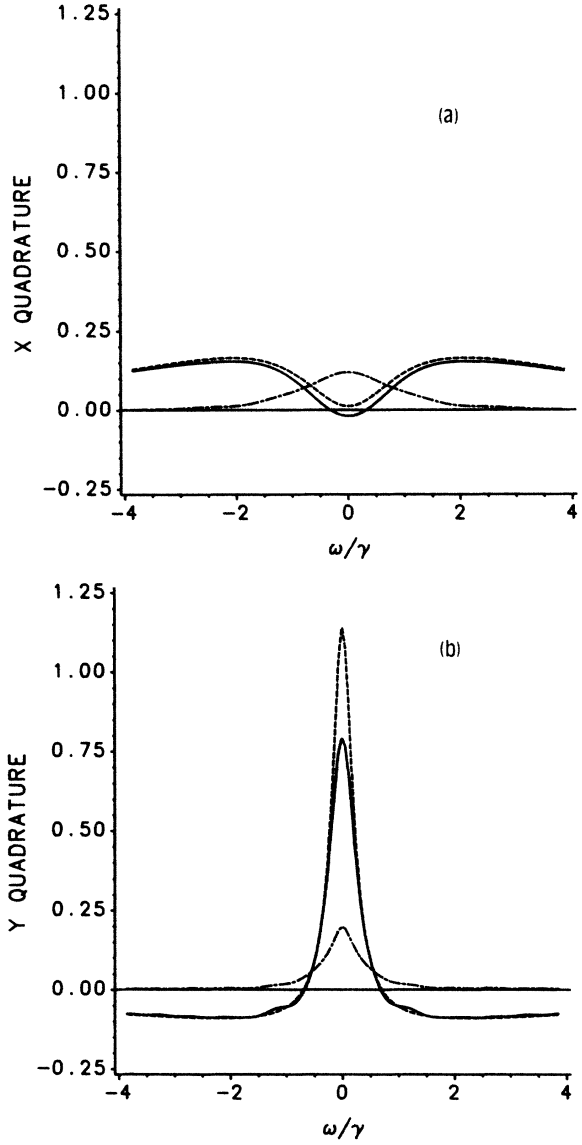


FIG. 10. Normally ordered homodyne spectra: (a) $X$ quadrature, (b) $Y$ quadrature. For key, see Fig. 8.

## ACKNOWLEDGMENTS

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## APPENDIX

### 1. Computation of atomic correlation functions using the adjoint equation

We assume that the quantum Langevin equation can be written in the form

$$\dot{Y}=\frac{i}{\hbar}[H_{\rm sys},Y]-\frac{i}{2\hbar}[[X,Y],\xi(t)-\frac{\gamma}{2}\dot{X}]_{+}\ , \qquad \text{(A1)}$$

which follows from Eq. (2.20) of Ref. 3 in the case that we assume

$$\kappa(x)=\sqrt{2\gamma c}\,\delta(x)\ . \qquad \text{(A2)}$$

The commutator of $\xi(t)$ with an arbitrary system opera-

tor then follows from Eq. (2.19) of Ref. 3 and is

$$[Y(t),\xi(s)]=\frac{d}{ds}\{\gamma u(t-s)[Y(t),X(s)]\} . \quad \text{(A3)}$$

The derivation of time correlation functions in terms of the adjoint equation requires a slightly more explicit derivation of the adjoint equation than given previously, as follows.

### 2. Derivation of the adjoint equation

We consider that the density operator factorizes in the Heisenberg picture, which corresponds to the system and bath being initially uncorrelated at some time in the remote past. Let $Y(t)$ be an arbitrary operator in the Heisenberg picture and $Y$ a Schrödinger picture version of the same operator. We can consistently define a quantity $\mu(t)$ by

$$\mathrm{Tr}_s[Y(t)\rho_s]=\mathrm{Tr}_s[Y\mu(t)] \quad \text{(A4)}$$

if the equality is true for all system operators $Y$ and $Y(t)$. In fact, this implicit definition of $\mu(t)$ can be made explicit as follows. Suppose $e_i$ is a complete set of Schrödinger picture operators, in the sense that any operator can be expressed as a linear combination of them, and suppose that they are also orthogonal with respect to the trace,

$$\mathrm{Tr}_s(e_i^+ e_j)=\delta_{ij} . \quad \text{(A5)}$$

From this and the assumed completeness it follows that any operator $A$ can be written

$$A=\sum_i \mathrm{Tr}_s(e_i A)e_i^+ . \quad \text{(A6)}$$

In particular, if $G$ is any operator, then

$$Ge_i=\sum_j G_{ij}e_j , \quad \text{(A7)}$$

where

$$G_{ij}=\mathrm{Tr}_s(e_j^+ Ge_i) . \quad \text{(A8)}$$

Let us denote the corresponding Heisenberg operators by $e_i(t)$. Since the time evolution is unitary, all algebraic relations will be preserved. In particular, if $G(t)$ is the Heisenberg operator corresponding to $G$, then the Heisenberg version of (A7) is

$$G(t)e_i(t)=\sum_j G_{ij}e_j(t) , \quad \text{(A9)}$$

where the coefficients $G_{ij}$ are exactly the same as in the Schrödinger version (A7).

Let us now construct $\mu(t)$ explicitly. We can write

$$\mu(t)=\sum_i \mathrm{Tr}_s[e_i(t)\rho_s]e_i^+ , \quad \text{(A10)}$$

since it is clear from (A10) that

$$\mathrm{Tr}_s[e_i\mu(t)]=\mathrm{Tr}_s[e_i(t)\rho_s] , \quad \text{(A11)}$$

and the general form (A4) follows by completeness; i.e., if

$$Y=\sum_i Y_i e_i ,$$

then

$$Y(t)=\sum_i Y_i e_i(t) . \quad \text{(A12)}$$

Let us now take the quantum Langevin equation in the form (A1). The equation of motion for $e_i(t)$ is then

$$\dot{e}_i(t)=\frac{i}{\hbar}[H_{\mathrm{sys}},e_i(t)] - \frac{i}{2\hbar}[[X(t),e_i(t)],\xi(t)-(\gamma/2)\dot{X}(t)]_+ . \quad \text{(A13)}$$

To evaluate $\dot{\mu}(t)$, we need to evaluate $\mathrm{Tr}_s[\dot{e}_i(t)\rho_s]$, and this will contain terms of the form [as well as similar terms with $\xi(t)$ on the left]

$$\mathrm{Tr}_s[A(t)e_i(t)B(t)\xi(t)\rho_s]=\mathrm{Tr}_s\left[\sum_j G_{ij}e_j(t)\rho_s\right]\xi(t) , \quad \text{(A14)}$$

where

$$G_{ij}=\mathrm{Tr}_s[A(t)e_i(t)B(t)e_j^+(t)] \quad \text{(A15)}$$

$$=\mathrm{Tr}_s[Ae_iBe_j^+] \quad \text{(A16)}$$

$$=\mathrm{Tr}_s[Be_j^+ Ae_i] , \quad \text{(A17)}$$

and $A$ and $B$ are such operators as turn up in the individual terms. The corresponding term in $\dot{\mu}(t)$ is then obtained by multiplying (A14) by $e_i^+$ and summing over $i$

$$\sum_{i,j}\mathrm{Tr}_s[e_j(t)\rho_s]\mathrm{Tr}_s[Be_j^+ Ae_i]e_i^+\xi(t) = \sum_j \mathrm{Tr}_s[e_j(t)\rho_s]Be_j^+ A\xi(t)=B\mu(t)A\xi(t) . \quad \text{(A18)}$$

Using this result it is then straightforward to derive the *adjoint equation* for $\mu(t)$,

$$\frac{d\mu}{dt}=[A_0+\alpha(t)A_1]\mu , \quad \text{(A19)}$$

in which

$$A_0\mu=-\frac{i}{\hbar}[H_{\mathrm{sys}},\mu]+\frac{i}{2\hbar}[[(\gamma/2)\dot{X},\mu(t)]_+,X] , \quad \text{(A19a)}$$

$$A_1\mu=-\frac{i}{\hbar}[\mu,X] , \quad \text{(A19b)}$$

$$\alpha(t)\mu=\tfrac{1}{2}[\xi(t),\mu]_+ . \quad \text{(A19c)}$$

The operator $\alpha(t)$ is in fact Abelian, i.e.,

$$[\alpha(t),\alpha(t')]=0 \text{ for all } t,t' ,$$

which follows from the fact that the commutator $[\xi(t),\xi(t')]$ is merely a $c$ number. This means that it will often be possible to simulate the solutions of the adjoint

equation by choosing a random function with the same statistics as $\alpha(t)$.

### 3. Correlation functions using the adjoint equation

We wish to compute the quantity

$$\mathrm{Tr}_s[e_i(t)e_j(t')\rho_s]\equiv\mu_{ij}(t,t') , \tag{A20}$$

from which

$$\langle e_i(t)e_j(t')\rangle=\mathrm{Tr}_B[\mu_{ij}(t,t')\rho_B] . \tag{A21}$$

Using the completeness of the operators $e_i(t)$, any two-time correlation can be computed. What is the equation of motion for $\mu_{ij}(t,t')$? We consider the case $t\geq t'$, so that an initial condition at $t=t'$ can be obtained from the fact that

$$\begin{aligned}\mu_{ij}(t',t')&=\mathrm{Tr}_s[e_i(t')e_j(t')\rho_s]\\&=\mathrm{Tr}_s[e_ie_j\mu(t')] .\end{aligned} \tag{A22}$$

We use the quantum Langevin equation (A1) and define

$$\mu_j(t,t')=\sum_i e_i^+\mu_{ij}(t,t') , \tag{A23}$$

and proceed as for the derivation of the adjoint equation. However, we meet a term, corresponding to (A14),

$$\mathrm{Tr}_s[A(t)e_i(t)B(t)\xi(t)e_j(t')\rho_s] . \tag{A24}$$

In this term we can use our procedure to get a term in $\mu_j(t,t')$,

$$\sum_k\mathrm{Tr}_s[e_k(t)\xi(t)e_j(t')\rho_s]Be_k^+A , \tag{A25}$$

and the operator $\xi(t)$ is still in the middle of the trace. However, we can bring it fully to the right by using the commutation relation (A3), by means of which (A25) becomes

$$B\mu_j(t,t')A\xi(t)+\sum_k\mathrm{Tr}_s\left[e_k(t)\frac{d}{dt}\{\gamma u(t'-t)[X(t),e_j(t')]\}\rho_s\right]Be_k^+A . \tag{A26}$$

Since we are considering only $t\geq t'$, we can write

$$B\mu_j(t,t')A\xi(t)+\sum_k\mathrm{Tr}_s\left[e_k(t)\frac{d}{dt}\{\gamma u(t'-t)[X(t),e_j(t')]\}\rho_s\right]Be_k^+A$$

$$=B\mu_j(t,t')A\xi(t)-\sum_k\mathrm{Tr}_s\{e_k(t')\gamma\delta(t-t')[X(t'),e_j(t')]\rho_s\}Be_k^+A \tag{A27}$$

$$=B\mu_j(t,t')A\xi(t)-\gamma\delta(t-t')\sum_k\mathrm{Tr}_s\{e_k[X,e_j]\mu(t')\}Be_R^+A . \tag{A28}$$

The conclusion is that $\mu_j(t,t')$ obeys the adjoint equation in $t$ for $t>t'$, but there is an initial $\delta$-function transient. This is equivalent to modifying the initial condition by an additional term

$$\frac{i\gamma}{2\hbar}[X,[X,e_j]\mu(t')] . \tag{A29}$$

In our case,

$$X=(\hbar/\Omega)^{1/2}(S^++S^-) , \tag{A30}$$

from which it is obvious that the correction is of order $\gamma/\Omega$ and therefore negligible. Therefore the procedure to compute the correlation function is in this degree of approximation.

(i) Define the quantity $\mu_j(t,t')$ as a solution of the adjoint equation with the initial condition

$$\mu_j(t',t')=\sum_i e_i^+\mathrm{Tr}_s[e_ie_j\mu(t')]=e_j\mu(t') . \tag{A31}$$

(ii) Then

$$\begin{aligned}\langle e_i(t)e_j(t')\rangle&=\mathrm{Tr}_B\{\mathrm{Tr}_s[e_i\mu_j(t,t')]\rho_B\}\\&=\langle\mathrm{Tr}_s[e_i\mu_j(t,t')]\rangle .\end{aligned} \tag{A32}$$

The result is obviously true for arbitrary operators.

[1]R. E. Slusher, L. W. Hollberg, B. Yurke, J. C. Mertz, and J. F. Valley, Phys. Rev. Lett. **55**, 2409 (1985); R. M. Shelby, M. D. Levenson, S. H. Perlmutter, R. G. DeVoe, and D. F. Walls, *ibid.* **57**, 691 (1986); Ling-An Wu, H. J. Kimble, J. L. Hall, and Huifa Wu, *ibid.* **57**, 2520 (1986); W. Maeda, P. Kumar, and J. H. Shapiro, Opt. Lett. **12**, 161 (1987); S. Machida, Y. Yamamoto, and Y. Itaya, Phys. Rev. Lett. **58**, 1000 (1987); M. G. Raizen, L. A. Orozco, Min Xiao, T. L. Boyd, and H. J. Kimble, *ibid.* **59**, 198 (1987).

[2]C. W. Gardiner, Phys. Rev. Lett. **56**, 1917 (1986).

[3]C. W. Gardiner, A. S. Parkins, and M. J. Collett, J. Opt. Soc. Am. B **4**, 1683 (1987).

[4]L.-A. Wu *et al.*, see Ref. 1.

[5]M. J. Collett and C. W. Gardiner, Phys. Rev. A **30**, 1386 (1984).

[6]C. W. Gardiner, *Handbook of Stochastic Methods* (Springer, Berlin, 1983).