Glauber amplitude for scattering of electrons by hydrogen atoms: $1s \rightarrow 2s, 2p$ excitations

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The method proposed by Yates [Chem. Phys. Lett. **25**, 480 (1974)] has been extended to the case of $1s \rightarrow 2s, 2p$ excitation of a hydrogen atom by electron impact. An alternative analytical procedure used by Berestetskii, Lifshitz, and Pitaevskii [*Quantum Electrodynamics* (Pergamon, Oxford, 1982), p. 539] has, however, been adopted for the evaluation of the third term of the Glauber series. In the process Yates's expression for the third Glauber term for electron–hydrogen-atom elastic scattering has been reproduced. The differential scattering cross section for the excitation process has been calculated as suggested by Yates. This cross section is found to be in reasonable agreement in the low-angle region with the exact Glauber results reported by Tai *et al.* [Phys. Rev. A **1**, 1819 (1970)] and by Gien [Phys. Rev. A **20**, 1457 (1979)].

I. INTRODUCTION

The Glauber[1] approximation has been shown to be a fairly successful procedure for analyzing electron-atom scattering at high incident energies and has been extensively applied to such processes.[2] In order to improve the accuracy of this method while still retaining some of its calculational simplicity, Yates[3] suggested that the first three terms in the Glauber series be calculated exactly while terms of order $1/k_i^4$ or higher in the incident momentum be neglected. He applied this technique to e-H elastic scattering with a fair amount of success.

Though Yates[3] pointed out the possibility of extending this procedure to inelastic electron–hydrogen-atom scattering, the only other (except the present work) practical application of this procedure is the work of Singh and Tripathi.[4] These authors investigated electron–helium-atom ($1s^2\,{}^1S$) elastic scattering as well as $1\,{}^1S \rightarrow 2\,{}^1S$ electron-impact excitation. We have now applied this procedure to electron-impact excitation of a hydrogen atom to the $n=2$ state (both $1s \rightarrow 2s$ and $1s \rightarrow 2p$ processes are considered), and discuss the results in the light of the "exact" Glauber results due to Tai *et al.*[5] and Gien.[6] We have adopted a different (as compared to Yates[3]) analytical procedure, earlier employed by Berestetskii, Lifshitz, and Pitaevskii,[7] to obtain a closed-form expression for the third term of the Glauber series. Yates's expression for the third Glauber term for electron-hydrogen (ground-state) elastic scattering is also reproduced.

Yates's procedure is summarized in Sec. II, while in Secs. III A and III B the closed-form expressions for the first three terms of the Glauber series are obtained for $1s \rightarrow 2s$ and $1s \rightarrow 2p$ excitations, respectively. In Sec. IV we tabulate our calculated differential scattering cross sections and compare our results with earlier exact Glauber results.

II. THEORY

The Glauber amplitude for the scattering of a structureless charged particle (with charge Z and initial momentum $\mathbf{k}_i$) by an N-electron atom which is excited from an initial state $\Psi_i(\mathbf{r}_1, \ldots, \mathbf{r}_N)$ to a final state $\Psi_f(\mathbf{r}_1, \ldots, \mathbf{r}_N)$ may be expanded in an infinite series

$$f_G(i \rightarrow f) = \sum_{n=1}^{\infty} (i)^{n-1} f_n(i \rightarrow f) , \quad (1)$$

where

$$f_n(i \rightarrow f) = \frac{k_i}{2\pi n!} \int d\mathbf{b}_0 \exp(i\mathbf{q}\cdot\mathbf{b}_0) \langle \Psi_f | \chi^n | \Psi_i \rangle , \quad (2)$$

with

$$\chi = -\frac{Z}{k_i} \int_{-\infty}^{\infty} dz_0 \sum_{j=1}^{N} \left[\frac{1}{r_0} - \frac{1}{|\mathbf{r}_0 - \mathbf{r}_j|} \right] . \quad (3)$$

(Note: atomic units are used throughout.) Here $\mathbf{r}_j = \mathbf{b}_j + z_j \hat{\mathbf{z}}$, $j = 1, \ldots, N$, are the position vectors of the bound electrons, and $\mathbf{r}_0 = \mathbf{b}_0 + z_0 \hat{\mathbf{z}}$ is the position vector of the incident particle, with respect to the nucleus (assumed to be infinitely heavy and situated at the origin of the coordinate system); $\mathbf{q} = \mathbf{k}_i - \mathbf{k}_f$ is the momentum-transfer vector, $\mathbf{k}_f$ being the momentum of the outgoing charged particle. It has been assumed that $\mathbf{q}$ is perpendicular to the axis of quantization of the atomic wave functions, which is specified by $\hat{\mathbf{z}} = \hat{\mathbf{k}}_i$. Consequently, $\mathbf{q}$, $\mathbf{b}_0$, and $\mathbf{b}_j$ lie in a plane perpendicular to the z axis. Yates[3] suggested that the expression within the parentheses in Eq. (3) be replaced by its Fourier transform. We may therefore write

$$\frac{1}{r_0}-\frac{1}{|\mathbf{r}_0-\mathbf{r}_j|}=\lim_{\nu\to 0+}\cdot\frac{1}{2\pi^2}\int d\mathbf{p}'\frac{\exp(-i\mathbf{p}'\cdot\mathbf{r}_0)[1-\exp(i\mathbf{p}'\cdot\mathbf{r}_j)]}{p'^2+\nu^2}\ . \tag{4}$$

We write $\mathbf{p}'=\mathbf{p}+p_z\hat{\mathbf{z}}$ so that p lies in the plane of $\mathbf{q}$, $\mathbf{b}_0$, and $\mathbf{b}_j$, and differential $d\mathbf{p}'$ becomes $d\mathbf{p}dp_z$. Using the definition of the Dirac δ function, first the integral over z_0, and then the integral over p_z, is performed to obtain the following integral representation for χ:

$$\chi=\lim_{\nu\to 0+}-\frac{Z}{\pi k_i}\int\frac{d\mathbf{p}}{p^2+\nu^2}\exp(-i\mathbf{p}\cdot\mathbf{b}_0)B(\mathbf{p},\mathbf{b}_1,\ldots,\mathbf{b}_N)\ , \tag{5}$$

with

$$B(\mathbf{p},\mathbf{b}_1,\ldots,\mathbf{b}_N)=\sum_{j=1}^{N}[1-\exp(i\mathbf{p}\cdot\mathbf{b}_j)]\ . \tag{6}$$

Substituting the expression for χ from Eq. (5) in Eq. (2) repeatedly with $\mathbf{p}_i$, $i=1,2,\ldots n$ as dummy variables, and changing the order of integrations freely, the integration over $\mathbf{b}_0$, and then the integration over $\mathbf{p}_n$, is performed to obtain an integral representation for $f_n(i\to f)$.

In particular, for scattering of electrons by a hydrogen atom and denoting the position vector of the bound electron by $\mathbf{r}=\mathbf{b}+z\hat{\mathbf{k}}_i$ we obtain

$$f_1(i\to f)=\frac{2}{q^2}[\delta_{if}-\langle\Psi_f(\mathbf{r})\,|\,\exp(i\mathbf{q}\cdot\mathbf{b})\,|\,\Psi_i(\mathbf{r})\rangle]\ , \tag{7}$$

$$f_2(i\to f)=\lim_{\nu\to 0+}\frac{1}{\pi k_i}\int\frac{d\mathbf{p}_1}{p_1^2+\nu^2}\frac{1}{|\mathbf{q}-\mathbf{p}_1|^2+\nu^2}\langle\Psi_f(\mathbf{r})\,|\,B(\mathbf{p}_1)B(\mathbf{q}-\mathbf{p}_1)\,|\,\Psi_i(r)\rangle\ , \tag{8}$$

and

$$f_3(i\to f)=\lim_{\nu\to 0+}\frac{1}{3\pi^2k_i^2}\int\frac{d\mathbf{p}_1}{p_1^2+\nu^2}\int\frac{d\mathbf{p}_2}{p_2^2+\nu^2}\frac{1}{|\mathbf{q}-\mathbf{p}_1-\mathbf{p}_2|^2+\nu^2}\langle\Psi_f(\mathbf{r})\,|\,B(\mathbf{p}_1)B(\mathbf{p}_2)B(\mathbf{q}-\mathbf{p}_1-\mathbf{p}_2)\,|\,\Psi_i(\mathbf{r})\rangle\ , \tag{9}$$

where $B(\mathbf{T})=1-\exp(i\mathbf{T}\cdot\mathbf{b})$. Retaining only the first three terms in the expansion in Eq. (1), the differential cross section as suggested by Yates[3] is given by

$$\frac{d\sigma}{d\Omega}=(k_f/k_i)[f_1^*f_1+f_2^*f_2-f_1^*f_3-f_3^*f_1+O(1/k_i^4)]\ . \tag{9a}$$

It is to be noted that for the case of inelastic scattering this expression [Eq. (9a)] is not quite correct, since the various terms in the Glauber series [Eq. (1)] have markedly different dependences on q (momentum transfer). At low angles of scattering, however, these deviations caused by these factors are quite small, and one expects that at least in such situations Eq. (9a) is a good approximation. We have, however, used Eq. (9a) for all angles of scattering, as this has been the usual practice in such calculations.[4]

III. APPLICATIONS

A. Application to $1s\to 2s$ excitation

The integrals over $\mathbf{r}$ in Eqs. (7)–(9) are easily performed to yield

$$f_1(1s\to 2s)=-\frac{2\sqrt{2}}{q^2(q^2+\lambda^2)^3}[q^2(1+2\lambda)-\lambda^2(3-2\lambda)]\ , \tag{10}$$

$$f_2(1s\to 2s)=-(\sqrt{2}\pi k_i)^{-1}\left[2+\frac{\partial}{\partial\lambda}\right]\frac{\partial}{\partial\lambda}\frac{D(q^2,\lambda^2)}{\lambda^2}\ , \tag{11}$$

$$f_3(1s\to 2s)=(3\sqrt{2}\pi^2k_i^2)^{-1}\left[2+\frac{\partial}{\partial\lambda}\right]\frac{\partial}{\partial\lambda}L(q^2,\lambda^2)\ , \tag{12}$$

where

$$D(q^2,\lambda^2)=\lim_{\nu\to 0+}\left[-\frac{q^2}{q^2+\lambda^2}I_0(q^2,\nu^2,\nu^2)+I_0(q^2,\lambda^2,\nu^2)+I_0(q^2,\nu^2,\lambda^2)\right]\ , \tag{13}$$

with, in general,

$$I_0(q^2,\gamma^2,\delta^2)\equiv\int\frac{d\mathbf{p}_1}{p_1^2+\gamma^2}\frac{1}{|\mathbf{q}-\mathbf{p}_1|^2+\delta^2} \tag{14}$$

and

$$L(q^2,\lambda^2)=\lim_{\nu\to 0+}\left[-\frac{q^2}{q^2+\lambda^2}I_{31}+3I_{32}-3I_{33}\right]\ , \tag{15}$$

with

$$I_{3k}\equiv\int\frac{d\mathbf{p}_1}{p_1^2+\nu^2}\frac{1}{\alpha_k}\int\frac{d\mathbf{p}_2}{p_2^2+\nu^2}\frac{1}{|\mathbf{q}-\mathbf{p}_1-\mathbf{p}_2|^2+\nu^2}\ ,\quad k=1,2,3 \tag{16}$$

and $\alpha_1=1$, $\alpha_2=p_1^2+\lambda^2$, and $\alpha_3=|\mathbf{q}-\mathbf{p}_1|^2+\lambda^2$. In Eqs. (10)–(16) and throughout this paper λ is taken as a parameter, and at the end of the simplification it is to be replaced by $\frac{3}{2}$. We wish to point out that each of these integrals I_{3k} and I_0 becomes absolutely divergent if ν is put equal to zero in any factor of the denominator. But it will be shown in what follows that the divergences actually get canceled and each of the combinations appearing in Eqs. (13) and (15) is finite.

Employing Feynman's technique[8] we obtain

$$I_0(q^2,\gamma^2,\delta^2)=\frac{2\pi}{\sqrt{c}}\ln\left[\frac{q^2+\gamma^2+\delta^2+\sqrt{c}}{2\gamma\delta}\right], \tag{17}$$

where

$$c(q^2,\gamma^2,\delta^2)\equiv q^4+\gamma^4+\delta^4+2q^2(\gamma^2+\delta^2)-2\gamma^2\delta^2 . \tag{18}$$

The expressions for the various integrals in Eq. (13) are obtained from Eqs. (17) and (18), and the following expression for $D(q^2,\lambda^2)$ is obtained:

$$D(q^2,\lambda^2)=\frac{4\pi}{q^2+\lambda^2}\ln\left[\frac{q^2+\lambda^2}{q\lambda}\right] \tag{19}$$

which is the same expression as has been reported by Byron and Latour,[9] but obtained by employing a different technique.

To evaluate the integrals I_{3k} we introduce a new variable $\mathbf{R}=\mathbf{q}-\mathbf{p}_1$ and rewrite I_{3k} as

$$I_{3k}=\int\frac{d\mathbf{R}}{|\mathbf{q}-\mathbf{R}|^2+\nu^2}\frac{1}{\xi_k}\int\frac{d\mathbf{p}_2}{p_2^2+\nu^2}\frac{1}{|\mathbf{R}-\mathbf{p}_2|^2+\nu^2}, \quad k=1,2,3 \tag{20}$$

where $\xi_1=1$, $\xi_2=|\mathbf{q}-\mathbf{R}|^2+\lambda^2$, and $\xi_3=R^2+\lambda^2$. The two factors in the denominator in the integral over $\mathbf{p}_2$ in Eq. (20) are combined by the use of the Feynman identity,[8] and then the integral over $\mathbf{p}_2$ is performed to obtain

$$I_{3k}=\pi\int_0^1\frac{dt}{t(1-t)}\int\frac{d\mathbf{R}}{|\mathbf{q}-\mathbf{R}|^2+\nu^2}\frac{1}{\xi_k}\times\frac{1}{R^2+[\nu^2/t(1-t)]} . \tag{21}$$

Employing the method of partial fractions, the integrals over the variable $\mathbf{R}$ are reduced to the form of $I_0(q^2,\gamma^2,\delta^2)$ as given by Eq. (17) and are evaluated accordingly. After suitable rearrangement we can express $L(q^2,\lambda^2)$ as

$$L(q^2,\lambda^2)=\lim_{\nu\to 0+}2\pi^2\left[\left[\frac{3}{\lambda^2}-\frac{q^2}{\lambda^2(q^2+\lambda^2)}\right]K_2-\frac{3}{\lambda^2}K_1-3K_3+\frac{6}{q^2+\lambda^2}\ln\left[\frac{q^2+\lambda^2}{\nu\lambda}\right]K_4\right], \tag{22}$$

where

$$K_1=\int_{1/2}^1 dt\, T(q^2,\lambda^2,\nu^2,t) , \tag{23}$$

$$K_2=\int_{1/2}^1 dt\, T(q^2,\nu^2,\nu^2,t) , \tag{24}$$

$$K_3=\int_{1/2}^1 dt\frac{t(1-t)}{\lambda^2 t(1-t)-\nu^2}T(q^2,\nu^2,\nu^2,t) , \tag{25}$$

and

$$K_4=\int_{1/2}^1\frac{dt}{\lambda^2 t(1-t)-\nu^2} . \tag{26}$$

Here, in general,

$$T(q^2,\mu^2,\nu^2,t)\equiv\frac{F(q^2,\mu^2,\nu^2,t)}{\sqrt{E}(q^2,\mu^2,\nu^2,t)} , \tag{27}$$

where

$$E(q^2,\mu^2,\nu^2,t)\equiv[(q^2+\mu^2)^2t^2(1-t)^2+\nu^4+2\nu^2(q^2-\mu^2)t(1-t)] \tag{28}$$

and

$$F(q^2,\mu^2,\nu^2,t)\equiv\ln\left[\frac{[(q^2+\mu^2)t(1-t)+\nu^2+\sqrt{E}(q^2,\mu^2,\nu^2,t)]^2}{4\mu^2\nu^2 t(1-t)}\right] . \tag{29}$$

We observe that $E>0$; if $\nu^2=0$, $E\to 0$ as $t\to 1$. Also, in the integrals K_3 and K_4 the factor $[\lambda^2 t(1-t)-\nu^2]$ in the denominator becomes zero for values of $t=t_\pm$,

$$t_\pm=\frac{1}{2}\left[1\pm\left[1-\frac{4\nu^2}{\lambda^2}\right]^{1/2}\right] . \tag{30}$$

Obviously, in the limit $\nu\to 0+$, $1>t_+>\frac{1}{2}$ and $\frac{1}{2}>t_->0$. Hence the integrals K_3 and K_4 are to be evaluated in the sense of the Cauchy principal value.

To evaluate the integrals K_1, K_2, K_3, and K_4 in the limit $\nu\to 0+$, we have adopted an analytical procedure used by Berestetskii, Lifshitz, and Pitaevskii.[7] In Appendix A we have outlined this method while obtaining the expression for K_1 in the limit $\nu\to 0+$. The limiting expressions obtained by us for the various integrals are listed below:

$$K_1=\frac{1}{q^2+\lambda^2}\left[A+2\ln\left[\frac{q^2+\lambda^2}{q\nu}\right]\ln\left[\frac{q(q^2+\lambda^2)}{\nu\lambda^2}\right]\right], \tag{31}$$

$$K_2=\frac{6}{q^2}\left[\ln\frac{q}{\nu}\right]^2, \tag{32}$$

$$K_3=\frac{6}{q^2\lambda^2}\left[\ln\frac{q}{\nu}\right]^2 -\frac{1}{\lambda^2(q^2+\lambda^2)}\left[\frac{\pi^2}{6}-2\sum_{n=1}^{\infty}\frac{1}{n^2}\left[\frac{q^2}{q^2+\lambda^2}\right]^n+G\right], \tag{33}$$

and

$$K_4=\frac{2}{\lambda^2}\left[\ln\frac{\lambda}{\nu}\right], \tag{34}$$

where

$$\begin{aligned} A&=-\sum_{n=1}^{\infty}\frac{1}{n^2}\left[-\frac{\lambda^2}{q^2}\right]^n \quad \text{if } q>\lambda\,,\\ &=\frac{\pi^2}{6}+2\left[\ln\frac{\lambda}{q}\right]^2+\sum_{n=1}^{\infty}\frac{1}{n^2}\left[-\frac{q^2}{\lambda^2}\right]^n \quad \text{if } q\le\lambda \end{aligned} \tag{35}$$

and

$$G=2\left[\ln\frac{q}{\lambda}\right]^2-4\ln q\,\ln\lambda-\ln(q^2+\lambda^2)\ln\left[\frac{q^2+\lambda^2}{q^4}\right] -4\ln\nu\ln\left[\frac{q}{\lambda}\right]. \tag{36}$$

Using Eqs. (31)–(34) and (22) we obtain the following expression for $L(q^2,\lambda^2)$:

$$L(q^2,\lambda^2)=-\frac{3\pi^2}{\lambda^2(q^2+\lambda^2)}\times\left[-\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{1}{n^2}\left[\frac{q^2}{q^2+\lambda^2}\right]^n +2A+2H\right], \tag{37}$$

where

$$H=\ln(q^2+\lambda^2)\ln\left[\frac{(q^2+\lambda^2)^3}{(\lambda^8q^4)}\right]+2\ln(\lambda q)\ln\left[\frac{\lambda}{q}\right] +12\ln\lambda\ln q \tag{38}$$

and A is given by Eq. (35).

In order to show that the expression for $L(q^2,\lambda^2)$ given by Eq. (37) is identical with the expression reported by Yates[3] we have derived the following identity in Appendix B:

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\left[\frac{q^2}{q^2+\lambda^2}\right]^n \equiv\frac{\pi^2}{6}-A-\frac{1}{2}\ln\left[\frac{q^2+\lambda^2}{q^2}\right]\ln\left[\frac{q^2(q^2+\lambda^2)}{\lambda^4}\right]. \tag{39}$$

Using Eqs. (37) and (39) we obtained the following expression for $L(q^2,\lambda^2)$ exactly as reported by Yates:[3]

$$L(q^2,\lambda^2)=-\frac{3\pi^2}{\lambda^2(q^2+\lambda^2)}\left\{4\left[\ln\left[\frac{q^2+\lambda^2}{q\lambda}\right]\right]^2 +\frac{\pi^2}{3}-2A\right\}. \tag{40}$$

Because of the inconvenient dependence of A on $q \gtrless \lambda$, we have eliminated A from Eq. (40) using Eq. (39), and obtained the following simple expression for f_3 from Eq. (12):

$$f_3=-\frac{1}{\sqrt{2}}\frac{1}{k_i^2}\left[2+\frac{\partial}{\partial\lambda}\right]\frac{\partial}{\partial\lambda} \times\left\{\frac{1}{\lambda^2(q^2+\lambda^2)}\left[2\sum_{n=1}^{\infty}\frac{1}{n^2}\left[\frac{q^2}{q^2+\lambda^2}\right]^n+B'\right]\right\}, \tag{41}$$

where

$$B'=4\ln\lambda\,\ln(\lambda q^4)+\ln(q^2+\lambda^2)\ln\left[\frac{(q^2+\lambda^2)^5}{q^8\lambda^{12}}\right]. \tag{42}$$

B. Application to $1s\to 2p$ excitation

The integration over $\mathbf{r}$ in Eqs. (7) and (8) is performed easily to obtain the following expressions for the vector amplitudes $\mathbf{f}_1$ and $\mathbf{f}_2$ for $1s\to 2p$ excitation:

$$\mathbf{f}_1(1s\to 2p)=-\frac{12\sqrt{2}i\mathbf{q}}{q^2(q^2+\lambda^2)^3} \tag{43}$$

and

$$\mathbf{f}_2(1s\to 2p)=-\frac{3\sqrt{2}i}{\pi k_i}\left[\frac{\partial}{\partial\lambda^2}\right]^2\frac{1}{\lambda^2}\mathbf{J}_0(\mathbf{q},q^2,\lambda^2)\,, \tag{44}$$

where

$$J_0(\mathbf{q},q^2,\lambda^2) =\lim_{\nu\to 0+}\int d\mathbf{p}_1\left[-\frac{2\mathbf{p}_1}{(p_1^2+\lambda^2)(|\mathbf{q}-\mathbf{p}_1|^2+\nu^2)} +\frac{q^2}{\lambda^2(q^2+\lambda^2)} \times\frac{\mathbf{q}}{(p_1^2+\nu^2)(|\mathbf{q}-\mathbf{p}_1|^2+\nu^2)}\right]. \tag{45}$$

Employing Feynman's technique[8] we obtain, in general,

$$\int d\mathbf{p}_1 \frac{\mathbf{p}_1}{(p_1^2+\gamma^2)(|\mathbf{q}-\mathbf{p}_1|^2+\delta^2)} = \frac{\pi\mathbf{q}}{q^2}\left[\frac{q^2+\delta^2-\gamma^2}{\sqrt{c}}\ln\left[\frac{q^2+\delta^2+\gamma^2+\sqrt{c}}{2\gamma\delta}\right] + \ln\left[\frac{\gamma}{\delta}\right]\right], \quad (46)$$

where $c(q^2,\gamma^2,\delta^2)$ is given by Eq. (18). Using Eqs. (44)–(46) we obtain the following final expression for $\mathbf{f}_2$:

$$\mathbf{f}_2(1s\to 2p) = -\frac{3\sqrt{2}i}{\pi k_i}\left[\frac{\partial}{\partial\lambda^2}\right]^2 \frac{2\pi\mathbf{q}}{\lambda^2 q^2(q^2+\lambda^2)} \times [q^2\ln(q^2)-\lambda^2\ln(\lambda^2) -(q^2-\lambda^2)\ln(q^2+\lambda^2)] . \quad (47)$$

In order to obtain a closed-form expression for the vector amplitude $\mathbf{f}_3$ for $1s\to 2p$ excitation, first the integral over $\mathbf{r}$ in Eq. (9) is performed to reduce $\mathbf{f}_3$ to the following form:

$$\mathbf{f}_3(1s\to 2p) = \lim_{\nu\to 0+} -\frac{\sqrt{2}i}{\pi^2 k_i^2}\left[\frac{\partial}{\partial\lambda^2}\right]^2 (3\mathbf{I}_1-3\mathbf{I}_2+\mathbf{I}_3) , \quad (48)$$

where, in general,

$$\mathbf{I}_k = \int \frac{d\mathbf{p}_1}{p_1^2+\nu^2}\int \frac{d\mathbf{p}_2}{p_2^2+\nu^2}\frac{1}{(|\mathbf{q}-\mathbf{p}_1-\mathbf{p}_2|^2+\nu^2)}\mathbf{J}_k , \quad k=1,2,3 \quad (49)$$

with

$$\mathbf{J}_1 = \frac{\mathbf{p}_1}{p_1^2+\lambda^2}, \quad \mathbf{J}_2 = \frac{\mathbf{q}-\mathbf{p}_1}{|\mathbf{q}-\mathbf{p}_1|^2+\lambda^2}, \quad \mathbf{J}_3 = \frac{\mathbf{q}}{q^2+\lambda^2} . \quad (50)$$

Making a change of variable through the relation $\mathbf{R}=\mathbf{q}-\mathbf{p}_1$, and employing Feynman's technique,[8] the integral over $\mathbf{p}_2$ is replaced by a one-dimensional integral over the Feynman parameter t. Further using the method of partial fractions wherever necessary, the integrals over $\mathbf{R}$ can be reduced to either of the two forms given by Eqs. (17) and (46), and hence evaluated easily. We have therefore to contend with a one-dimensional integral over the variable t only. After some rearrangement and simple algebra we obtain

$$\mathbf{I}_1 = \frac{\pi^2\mathbf{q}}{\lambda^2 q^2}\left\{(\lambda^2-q^2)K_1+q^2K_2 + \left[\ln\left[\frac{\nu^2}{\lambda^2}\right]K_5-\nu^2K_6+\nu^2K_7\right]\right\}, \quad (51)$$

$$\mathbf{I}_2 = -\frac{\pi^2\mathbf{q}}{q^2}\left\{\frac{1}{\lambda^2}\ln\left[\frac{\lambda^2}{\nu^2}\right] \times \left[\ln\left[\frac{\lambda^2}{\nu^2}\right]+2\frac{q^2-\lambda^2}{q^2+\lambda^2}\ln\left[\frac{q^2+\lambda^2}{\lambda\nu}\right]\right] -q^2K_3+K_8+\nu^2K_9\right\}, \quad (52)$$

and

$$\mathbf{I}_3 = \frac{2\pi^2\mathbf{q}}{q^2+\lambda^2}K_2 , \quad (53)$$

where K_1, K_2, and K_3 are given by Eqs. (23), (24), and (25), respectively, and

$$K_5 = \int_{1/2}^{1}\frac{dt}{t(1-t)} , \quad (54)$$

$$K_6 = \int_{1/2}^{1}\frac{dt}{t(1-t)}T(q^2,\lambda^2,\nu^2,t) , \quad (55)$$

$$K_7 = \int_{1/2}^{1}\frac{dt}{t(1-t)}T(q^2,\nu^2,\nu^2,t) , \quad (56)$$

$$K_8 = \int_{1/2}^{1}\frac{dt\,\ln[t(1-t)]}{\lambda^2 t(1-t)-\nu^2} , \quad (57)$$

and

$$K_9 = \int_{1/2}^{1}\frac{dt}{\lambda^2 t(1-t)-\nu^2}T(q^2,\nu^2,\nu^2,t) , \quad (58)$$

with $T(q^2,\mu^2,\nu^2,t)$ as defined by Eqs. (27)–(29).

Before attempting to evaluate the integrals K_5, K_6, K_7, K_8, and K_9 we make the following observations. It is seen that $E>0$, and so if $\nu=0$ then as $t\to 1$, $E\to 0$. Therefore the factor T which appears in K_6, K_7, and K_9 introduces a denominator which goes to zero as $t\to 1$ for $\nu=0$. In addition to this, K_6, K_7, and K_9 contain other factors which may lead to a zero in the denominator for some value (values) of t. There are two types of such factors. Integrals K_6 and K_7, and also K_5 (which, however, does not contain the factor T), have integrands whose denominator becomes zero for $t\to 1$, whatever the value of ν may be. On the other hand, K_9 and also K_8 (which does not contain the T factor) have a denominator which may become zero not only at $t=1$ when $\nu=0$, but also at some other value of t provided $\lambda^2>4\nu^2$, which is true in the present case. In Appendix A a procedure has been outlined for evaluating integrals of this type, and the integrals K_8 and K_9 are easily evaluated using this procedure. For evaluating the remaining integrals K_5, K_6, and K_7 we have to exercise some further caution. Thus in order to evaluate K_5 we integrate over the limits $\frac{1}{2}$ and $1-\epsilon$, while in evaluating K_6 and K_7 we divide the range $\frac{1}{2}$ to $1-\epsilon$ into two parts, from $\frac{1}{2}$ to $1-\delta$ and from $1-\delta$ to $1-\epsilon$, with the requirement that $1\gg\delta\gg\nu\gg\epsilon$ while all three tend to 0^+ in the final step. The same procedure is to be used while we obtain the limit of the expression within large curly brackets in Eq. (51). The Eulerian substitution given in Appendix A and the auxiliary integrals S and $R(\beta)$ of Appendix A and Q of Appendix B are used to obtain the final expressions for the various integrals. After lengthy and tedious but straightforward algebra it can be shown that the expression within the large curly brackets in Eq. (51) becomes independent of $\ln\epsilon$ and $(\ln\epsilon)^2$ both, because the coefficients of $\ln\epsilon$ and $(\ln\epsilon)^2$ vanish separately. In addition, the coefficients of $\ln\nu$ and $(\ln\nu)^2$ in the final expression for the combination $3\mathbf{I}_1-3\mathbf{I}_2+\mathbf{I}_3$ vanish separately. Thus the logarithmic divergences stemming from various divergent integrals mutually cancel and we obtain a finite expression for the combination $3\mathbf{I}_1-3\mathbf{I}_2+\mathbf{I}_3$. Denoting this combination by $\mathbf{I}$, we can write

$$\mathbf{I}=3\pi^2\mathbf{q}\left\{\frac{1}{q^2+\lambda^2}\left[\frac{1}{q^2}-\frac{1}{\lambda^2}\right]\left[A+H+2\sum_{n=1}^{\infty}\frac{1}{n^2}\left[\frac{q^2}{q^2+\lambda^2}\right]^n\right]\right.$$
$$-\frac{1}{q^2\lambda^2}\left[4\ln(q)\ln(\lambda)+\frac{1}{2}\ln\left[\frac{q^2+\lambda^2}{q^2\lambda^2}\right]\ln\left[\frac{\lambda^2(q^2+\lambda^2)}{q^2}\right]+\sum_{n=1}^{\infty}\frac{1}{n^2}\left[\frac{\lambda^2}{q^2+\lambda^2}\right]^n\right]$$
$$\left.+\frac{4}{q^2(q^2+\lambda^2)}[\ln(q)]^2+\frac{1}{\lambda^2(q^2+\lambda^2)}\frac{\pi^2}{3}\right\}, \tag{59}$$

where A is given by Eq. (35) and H by Eq. (38).

We can simplify the expression in Eq. (59) by using the identity given in Eq. (39). By interchanging q and λ in the identity (39) and some rearrangement we get

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\left[\frac{\lambda^2}{q^2+\lambda^2}\right]^n\equiv A-\frac{1}{2}\left[\ln\left[\frac{q^2+\lambda^2}{q^2}\right]\right]^2 . \tag{60}$$

We make use of the identity

$$\frac{1}{\lambda^2(q^2+\lambda^2)}\equiv\frac{1}{q^2\lambda^2}-\frac{1}{q^2(q^2+\lambda^2)} , \tag{61}$$

as well as placing (39) and (60) in Eq. (59) to obtain the following expression for $\mathbf{I}$:

$$\mathbf{I}=\frac{6\pi^2\mathbf{q}}{q^2(q^2+\lambda^2)}(P-A) , \tag{62}$$

where

$$P=\frac{\pi^2}{6}+2\left[\ln\left[\frac{q}{\lambda}\right]\right]^2+\left[\frac{\lambda^2-q^2}{\lambda^2}\right]\ln\left[\frac{q^2+\lambda^2}{\lambda^2}\right]\ln\left[\frac{q^2+\lambda^2}{q^2}\right] \tag{63}$$

and A is given by Eq. (35). Due to the inconvenient dependence of A on the condition $q\lessgtr\lambda$, we eliminate it from Eq. (62) using Eq. (39) and refer back to Eq. (48) to get

$$\mathbf{f}_3(1s\rightarrow2p)=-\frac{\sqrt{2}i}{\pi^2k_i^2}\left[\frac{\partial}{\partial\lambda^2}\right]^2\mathbf{I} , \tag{64}$$

where

$$\mathbf{I}=\frac{6\pi^2\mathbf{q}}{q^2(q^2+\lambda^2)}\left\{\sum_{n=1}^{\infty}\frac{1}{n^2}\left[\frac{q^2}{q^2+\lambda^2}\right]^n+\frac{1}{2}\left[\ln\left[\frac{q^2+\lambda^2}{\lambda^2}\right]\right]^2+\left[\frac{\lambda^2-q^2}{\lambda^2}\right]\ln\left[\frac{q^2+\lambda^2}{\lambda^2}\right]\times\ln\left[\frac{q^2+\lambda^2}{q^2}\right]\right\} . \tag{65}$$

The expressions for $f_n(1s\rightarrow2p_0,2p_{\pm1})$, $n=1,2,3$, can be obtained from the corresponding vector amplitudes given by Eqs. (43), (47), and (64) by using the following relations:

$$f_n(1s\rightarrow2p_0)=f_n^z \tag{66a}$$

and

$$f_n(1s\rightarrow2p_{\pm1})=\mp\frac{1}{\sqrt{2}}(f_n^x\mp if_n^y) . \tag{66b}$$

The superscript of f_n signifies the particular component of the vector amplitude $\mathbf{f}_n(1s\rightarrow2p)$.

We wish to point out that instead of Eq. (66b), Byron and Latour[9] have used the relation

$$f_n(1s\rightarrow2p_{\pm1})=\mp\frac{1}{\sqrt{2}}(f_n^x\pm if_n^y) . \tag{66c}$$

This, however, does not affect the cross sections though the individual scattering amplitudes for excitation to $m=\pm1$ magnetic sublevels do change.

$f_n(1s \to 2p_0)$, $n=1,2,3$ and the corresponding differential cross section become zero because of our particular choice of the z axis. Also $f_n(1s \to 2p_{\pm 1})$, $n=1,2,3$ may be expressed as a product of $\pm i\exp(\mp i\phi_q)$ and a real factor. Hence the differential cross sections for excitation to these magnetic sublevels become identical. It is the value of the coefficient of $\pm i\exp(\mp i\phi_q)$ in the case of $1s \to 2p_{\pm 1}$ excitation that has actually been calculated and tabulated as f_1^p, f_2^p, and f_3^p in this work.

IV. RESULTS AND DISCUSSIONS

The values of f_1, f_2, and f_3 calculated for $1s \to 2s$ and $1s \to 2p$ excitation are shown for a typical impact energy of 200 eV in Table I. The differential cross sections $(d\sigma/d\Omega)$ for the two excitations are also separately given, as calculated from Eq. (9a). The differential cross sections $(d\sigma/d\Omega)_c$ calculated by us for $1 \to 2$ excitation is also given. Exact Glauber results for these differential cross sections have been given by Tai *et al.*[5] and Gien,[6] and these latter values $(d\sigma/d\Omega)_G$ for $1 \to 2$ excitation are also included for the sake of comparison. As mentioned in connection with Eq. (9a), our results give a fairly good approximation to the exact results at low scattering angles, but at higher angles the exact results deviate appreciably from those calculated using Eq. (9a). The variation of f_1, f_2, and f_3 with the size of the angle discussed below in Secs. IV A and IV B for the $1s \to 2s$ and $1s \to 2p$ excitations, respectively.

A. $1s \to 2s$ excitation

For case of $1s \to 2s$ excitation all three terms (f_1, f_2, f_3) are negative in the entire range of energies and at all angles of scattering. We shall discuss the behavior of their magnitudes only. It is observed that at any particular impact energy E_i, f_1, and f_2 go on decreasing monotonically with the increase in the scattering angle. At very high impact energies, of the order of 10 keV, $f_1 > f_2$ for scattering angles in a narrow cone around the forward direction, while for all other angles $f_2 > f_1$. As the incident energy decreases this cone gradually opens up, acquiring a semiapex angle $\sim 26°$ for incident energy $\simeq 230$ eV. At still lower incident energies the cone continues to grow, but inside the cone a smaller cone appears inside which $f_2 > f_1$. Thus for an incident energy ~ 15 eV there is an annular region corresponding to scattering angles between 23° and 72° where $f_1 > f_2$, while $f_2 > f_1$ for all other scattering angles. This is in contrast to the case of elastic scattering, where for incident energies less than 22 eV, f_2 exceeds f_1 for all scattering angles.

On the other hand, it is clear from formula (41) that $f_3(1s \to 2s)$ is a function of k_i and q since $\lambda = \frac{3}{2}$ and varies according to the relation $f_3 = \tilde{f}_3(q)/k_i^2$, where $\tilde{f}_3(q)$ is a function only of q. Hence a plot of f_3 versus q shows that f_3 becomes maximum for a momentum transfer $q_m = 0.69$ a.u., irrespective of the incident electron energy. For a particular incident electron energy, the angle of scattering θ_{3m}^s at which f_3 becomes maximum can be calculated from the relation $q_m^2 = k_i^2 + k_f^2 - 2k_i k_f \cos\theta_{3m}^s$. This is a more convenient way of obtaining q_m, which may of course also be obtained, in principle, from the analytic expression for $f_3(1s \to 2s)$ as given by Eq. (41).

From our calculated results we have observed that only for high energies ($\gtrsim 300$ eV) at all angles of scattering is f_3 smaller than f_2. At lower energies f_3 is comparable with, and may even become larger than, f_2 for sufficiently large scattering angles. At any impact energy the decrease of f_1 with the scattering angle is very rapid and for large scattering angles f_1 is very much smaller than both f_2 and f_3. This is true even for energies as low as 27.2 eV. For small scattering angles, however, f_3 is smaller than f_1 and f_2. Hence we conclude that retaining only the first three terms in the Glauber series is justified only for high impact energies and at low scattering angles.

B. $1s \to 2p$ excitation

The first three terms of the Glauber series for the $1s \to 2p_\pm$ excitation of a hydrogen atom by electron impact have been calculated using Eqs. (43), (47), (64), and (66b). It is observed that for all impact energies E_i, f_1^p is maximum in the forward direction and goes on decreasing monotonically with the scattering angle, but always remains positive. From Eq. (47) we observe that f_2^p is a function of k_i and q and varies according to the relation $f_2^p = \tilde{f}_2^p(q)/k_i$, where $\tilde{f}_2^p(q)$ is a function only of q. Similarly, Eqs. (64) and (65) indicate the variation of f_3^p with q and k_i, and that $f_3^p = \tilde{f}_3^p(q)/k_i^2$ where $\tilde{f}_3^p(q)$ is a function only of q. A plot of f_2^p against q shows a minimum for the momentum transfer 0.218 a.u. and a maximum for the momentum transfer 1.37 a.u., irrespective of the impact energy. A plot of f_3^p versus q shows a maximum for the momentum transfer 0.334 a.u., irrespective of the incident energy. As in the $1s \to 2s$ case the corresponding scattering angles at which these maxima or minima occur can be calculated for different impact energies.

From the calculated values we have observed that for a particular value of the impact energy the decrease of f_1^p with the scattering angle is very rapid as compared to that of f_2^p and f_3^p. For low scattering angles the contribution of the first Glauber term to the differential cross section is larger than the second or the third term. For large scattering angles, however, f_1^p becomes smaller than both f_2^p and f_3^p. Particularly for high impact energies and large scattering angles the major contribution towards the differential cross section comes from the second and the third term only.[10]

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TABLE I. Scattering amplitude (in atomic units) and differential cross section (in units of a_0^2/sr) for $1s \rightarrow 2s,2p$ excitation of hydrogen atoms by electron impact at 200 eV.

θ	$1s \rightarrow 2s$ excitation				$1s \rightarrow 2p$ excitation				$1 \rightarrow 2$ excitation	
(deg)	f_1	f_2	f_3	$(d\sigma/d\Omega)$	f_1^p	f_2^p	f_3^p	$(d\sigma/d\Omega)$	$(d\sigma/d\Omega)_c$	$(d\sigma/d\Omega)_G$
0	-9.804^{-1}	-1.001	-8.915^{-3}	1.914	1.050^{+1}	-1.300^{-1}	5.200^{-2}	2.127^{2}	2.147^{2}	
3	-9.310^{-1}	-5.858^{-1}	-2.921^{-2}	1.126	4.458	-1.590^{-1}	7.709^{-2}	3.744^{1}	3.856^{1}	
4	-8.949^{-1}	-4.628^{-1}	-3.955^{-2}	9.198^{-1}	3.365	-1.539^{-1}	8.180^{-2}	2.103^{1}	2.195^{1}	
5	-8.512^{-1}	-3.665^{-1}	-4.928^{-2}	7.549^{-1}	2.619	-1.409^{-1}	8.288^{-2}	1.256^{1}	1.331^{1}	
10	-5.780^{-1}	-1.352^{-1}	-7.318^{-2}	2.609^{-1}	9.191^{-1}	-3.951^{-2}	5.794^{-2}	1.442	1.703	
15	-3.340^{-1}	-8.920^{-2}	-6.405^{-2}	7.472^{-2}	3.568^{-1}	2.569^{-2}	2.883^{-2}	2.092^{-1}	2.839^{-1}	
20	-1.785^{-1}	-8.124^{-2}	-4.865^{-2}	2.055^{-2}	1.436^{-1}	4.301^{-2}	1.410^{-2}	3.591^{-2}	5.646^{-2}	5.868^{-2}
25	-9.366^{-2}	-7.340^{-2}	-3.745^{-2}	6.961^{-3}	6.053^{-2}	3.932^{-2}	8.404^{-3}	8.169^{-3}	1.513^{-2}	1.529^{-2}
30	-4.992^{-2}	-6.300^{-2}	-3.040^{-2}	3.336^{-3}	2.699^{-2}	3.070^{-2}	6.215^{-3}	2.613^{-3}	5.949^{-3}	5.633^{-3}
40	-1.573^{-2}	-4.341^{-2}	-2.251^{-2}	1.387^{-3}	6.438^{-3}	1.708^{-2}	4.431^{-3}	5.380^{-4}	1.925^{-3}	1.646^{-3}
50	-5.805^{-3}	-3.009^{-2}	-1.789^{-2}	7.125^{-4}	1.924^{-3}	9.780^{-3}	3.353^{-3}	1.684^{-4}	8.809^{-4}	7.302^{-4}
60	-2.482^{-3}	-2.179^{-2}	-1.471^{-2}	3.975^{-4}	6.954^{-4}	6.039^{-3}	2.565^{-3}	6.504^{-5}	4.626^{-4}	3.809^{-4}
70	-1.203^{-3}	-1.655^{-2}	-1.239^{-2}	2.392^{-4}	2.940^{-4}	4.015^{-3}	1.999^{-3}	2.928^{-5}	2.685^{-4}	2.212^{-4}
80	-6.491^{-4}	-1.311^{-2}	-1.066^{-2}	1.544^{-4}	1.415^{-4}	2.845^{-3}	1.597^{-3}	1.493^{-5}	1.694^{-4}	1.399^{-4}
90	-3.834^{-4}	-1.077^{-2}	-9.350^{-3}	1.062^{-4}	7.598^{-5}	2.128^{-3}	1.310^{-3}	8.447^{-6}	1.147^{-4}	9.501^{-5}
100	-2.450^{-4}	-9.132^{-3}	-8.351^{-3}	7.731^{-5}	4.482^{-5}	1.666^{-3}	1.102^{-3}	5.221^{-6}	8.253^{-5}	6.855^{-5}
110	-1.679^{-4}	-7.951^{-3}	-7.582^{-3}	5.913^{-5}	2.871^{-5}	1.357^{-3}	9.501^{-4}	3.485^{-6}	6.262^{-5}	5.213^{-5}
120	-1.223^{-4}	-7.088^{-3}	-6.989^{-3}	4.728^{-5}	1.979^{-5}	1.145^{-3}	8.379^{-4}	2.490^{-6}	4.977^{-5}	4.152^{-5}
130	-9.436^{-5}	-6.453^{-3}	-6.534^{-3}	3.937^{-5}	1.459^{-5}	9.963^{-4}	7.551^{-4}	1.892^{-6}	4.126^{-5}	3.448^{-5}
140	-7.668^{-5}	-5.989^{-3}	-6.191^{-3}	3.402^{-5}	1.143^{-5}	8.920^{-4}	6.944^{-4}	1.519^{-6}	3.554^{-5}	2.973^{-5}
160	-5.854^{-5}	-5.438^{-3}	-5.768^{-3}	2.815^{-5}	8.330^{-6}	7.730^{-4}	6.225^{-4}	1.144^{-6}	2.929^{-5}	
180	-5.360^{-5}	-5.269^{-3}	-5.636^{-3}	2.646^{-5}	7.510^{-6}	7.376^{-4}	6.003^{-4}	1.042^{-6}	2.750^{-5}	

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APPENDIX A

In this appendix we outline the method of the evaluation of the various integrals occuring in Sec. III A in the limit $\nu \to 0+$. In particular, we shall discuss about the integral K_1,

$$K_1=\int_{1/2}^{1}\frac{dt}{\sqrt{E}}\ln\left[\frac{[(q^2+\lambda^2)t(1-t)+\nu^2+\sqrt{E}\,]^2}{4\lambda^2\nu^2 t(1-t)}\right], \tag{A1}$$

where

$$E=(q^2+\lambda^2)^2t^2(1-t)^2+\nu^4+2\nu^2(q^2-\lambda^2)t(1-t) .$$

It is observed that if $\nu^2=0$, $E\to 0$ as $t\to 1$. Hence we divide the range of integration and write $K_1=K_{11}+K_{12}$, where

$$K_{11}=\int_{1/2}^{1-\delta}\frac{dt}{\sqrt{E}}\ln\left[\frac{[(q^2+\lambda^2)t(1-t)+\nu^2+\sqrt{E}\,]^2}{4\lambda^2\nu^2 t(1-t)}\right] \tag{A2}$$

and

$$K_{12}=\int_{1-\delta}^{1}\frac{dt}{\sqrt{E}}\ln\left[\frac{[(q^2+\lambda^2)t(1-t)+\nu^2+\sqrt{E}\,]^2}{4\lambda^2\nu^2 t(1-t)}\right] \tag{A3}$$

with the assumption that $1\gg\delta\gg\nu\to 0+$. In the first integral, K_{11}, the integral is analytic throughout the range of integration and hence we can put $\nu^2=0$ everywhere except in the denominator of the argument of logarithm and obtain

$$K_{11}=\int_{1/2}^{1-\delta}\frac{dt}{(q^2+\lambda^2)t(1-t)}\ln\left[\frac{(q^2+\lambda^2)^2}{\lambda^2\nu^2}t(1-t)\right] . \tag{A4}$$

This integration is straightforward and we obtain

$$K_{11}=-\frac{1}{q^2+\lambda^2}\left[\frac{\pi^2}{6}+\frac{1}{2}(\ln\delta)^2+2\ln\delta\ln\left[\frac{q^2+\lambda^2}{\nu\lambda}\right]\right] . \tag{A5}$$

In the second integral, K_{12}, $t\simeq 1$ throughout the range of integration, and hence we can put $t=1$ everywhere except in the term $(1-t)$, but we cannot ignore ν^2. Therefore K_{12} can be written as

$$K_{12}=\int_{1-\delta}^{1}\frac{dt}{\sqrt{E'}}\ln\left[\frac{[(q^2+\lambda^2)(1-t)+\nu^2+\sqrt{E'}]^2}{4\lambda^2\nu^2(1-t)}\right] , \tag{A6}$$

where

$$E'=(q^2+\lambda^2)^2(1-t)^2+2\nu^4+2\nu^2(q^2-\lambda^2)(1-t) .$$

We now make a change of variable $x=1-t$ to obtain

$$K_{12}=\int_{0}^{\delta}\frac{dx}{\sqrt{E'}}\ln\left[\frac{[(q^2+\lambda^2)x+\nu^2+\sqrt{E'}]^2}{4\lambda^2\nu^2 x}\right] , \tag{A7}$$

where

$$E'=(q^2+\lambda^2)^2x^2+\nu^4+2\nu^2(q^2-\lambda^2)x .$$

To rationalize the integrand in (A7), we use the following Euler substitution:

$$[(q^2+\lambda^2)^2x^2+\nu^4+2\nu^2(q^2-\lambda^2)x]^{1/2}=Y-(q^2+\lambda^2)x .$$

In order to simplify further, we make another change of variable,

$$Z=Y+\frac{q^2-\lambda^2}{q^2+\lambda^2}\nu^2 ,$$

and finally obtain

$$K_{12}=\frac{1}{q^2+\lambda^2}\int_{z_-}^{z_+}\frac{dz}{z}\ln\left[\frac{q^2+\lambda^2}{2\lambda^2\nu^2}\frac{\left[z+\dfrac{2\nu^2\lambda^2}{q^2+\lambda^2}\right]}{\left[1-\dfrac{z_-}{z}\right]}\right] , \tag{A8}$$

where

$$z_+=(q^2+\lambda^2)\delta+[(q^2+\lambda^2)^2\delta^2+\nu^4+2\nu^2(q^2-\lambda^2)\delta]^{1/2}+\frac{q^2-\lambda^2}{q^2+\lambda^2}\nu^2$$

and

$$z_-=\frac{2q^2\nu^2}{q^2+\lambda^2} .$$

Using the following results:

$$R(\beta)\equiv\int_{u_-}^{u_+}\frac{du}{u}\ln(u+\beta),\quad \beta>0,\quad u_+>u_->0$$
$$=\tfrac{1}{2}\ln(u_+u_-)\ln\left[\frac{u_+}{u_-}\right]-\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\left[\left[\frac{\beta}{u_+}\right]^n-\left[\frac{\beta}{u_-}\right]^n\right]\quad\text{if } u_-\geq\beta\,,$$
$$=\tfrac{1}{2}\left[\ln\left[\frac{u_+}{\beta}\right]\right]^2+\ln\beta\ln\left[\frac{u_+}{u_-}\right]+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\left[2-\left[\frac{u_-}{\beta}\right]^n-\left[\frac{\beta}{u_+}\right]^n\right]\quad\text{if } u_+>\beta>u_-\,,\tag{A9}$$
$$=\ln\beta\ln\left[\frac{u_+}{u_-}\right]+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\left[\left[\frac{u_+}{\beta}\right]^n-\left[\frac{u_-}{\beta}\right]^n\right]\quad\text{if } \beta\geq u_+\,,$$

and

$$S\equiv\int_{u_-}^{u_+}\frac{du}{u}\ln\left[1-\frac{u_-}{u}\right],\quad u_+>u_->0$$
$$=\ln\left[\frac{u_+}{u_-}\right]\ln\left[1-\frac{u_-}{u_+}\right]-\sum_{n=1}^{\infty}\frac{1}{n^2}\left[1-\frac{u_-}{u_+}\right]^n,\tag{A10}$$

we get

$$K_{12}=\frac{1}{q^2+\lambda^2}\left[\frac{\pi^2}{6}+A+\frac{1}{2}\ln\left[\frac{(q^2+\lambda^2)^2\delta}{q^2\nu^2}\right]\times\ln\left[\frac{q^2(q^2+\lambda^2)^2\delta}{\lambda^4\nu^2}\right]\right],\tag{A11}$$

with A given by Eq. (35). Addition of the expression in (A5) and (A11) leads to the expression given by Eq. (31).

The expression for K_2 in the limit $\nu\to 0+$ can be obtained from the expression for K_1 by replacing λ^2 by ν^2 throughout. Applying exactly the same procedure as for K_1, the limiting expression for K_3 can be obtained after slightly lengthy but straightforward algebra, while K_4 is easily evaluated.

APPENDIX B

In this appendix we shall prove Eq. (39). Consider the integral

$$M=\mathrm{P}\int_0^1\frac{dt}{t-t_+}\ln(\alpha-t)\,,\tag{B1}$$

where

$$t_+=\frac{1}{2}\left[1+\left[1-\frac{4\nu^2}{\lambda^2}\right]^{1/2}\right],\tag{B2}$$

$$\alpha=\frac{1}{2}\left[1+\left[1+\frac{4\nu^2}{q^2}\right]^{1/2}\right],\tag{B3}$$

and P stands for the Cauchy principal value. We observe that in the limit $\nu\to 0+$,

$$t_+\to 1-\frac{\nu^2}{\lambda^2}\,,$$
$$\alpha\to 1+\frac{\nu^2}{q^2}\,,$$

and

$$(\alpha-t_+)\to\nu^2\left[\frac{q^2+\lambda^2}{q^2\lambda^2}\right].$$

By a change of variable $z=t-t_+$, M can be written as

$$M=\mathrm{P}\int_{-t_+}^{1-t_+}\frac{dz}{z}\ln(\alpha-t_+-z)$$
$$=\lim_{\epsilon\to 0+}\left[-\int_{\epsilon}^{t_+}\frac{dz}{z}\ln(\alpha-t_++z)+\int_{\epsilon}^{1-t_+}\frac{dz}{z}\ln(\alpha-t_+-z)\right],\tag{B4}$$

where $\nu\gg\epsilon\to 0+$. In the limit $\nu\to 0+$, $t_+>(\alpha-t_+)>\epsilon$ and $(\alpha-t_+)>(1-t_+)>\epsilon$. After evaluating both integrals above we obtain the following expression for M in the limit $\nu\to 0+$:

$$M=-\frac{\pi^2}{6}-\sum_{n=1}^{\infty}\frac{1}{n^2}\left[\frac{q^2}{q^2+\lambda^2}\right]^n+\frac{1}{2}\ln\left[\frac{q^2+\lambda^2}{q^2\lambda^2}\nu^2\right]\ln\left[\frac{q^2\nu^2}{\lambda^2(q^2+\lambda^2)}\right].\tag{B5}$$

The integral M can also be written as

$$M=\lim_{\epsilon\to 0+}\left[\int_0^{t_+-\epsilon}\frac{dt}{t-t_+}\ln(\alpha-t)+\int_{t_++\epsilon}^{1}\frac{dt}{t-t_+}\ln(\alpha-t)\right].\tag{B6}$$

Integrating both integrals in (B6) by parts, making use of the integral in (A10) and the following:

$$Q=\int_{u_-}^{u_+}\frac{du}{u}\ln\left[\frac{u_+}{u}-1\right],\quad u_+>u_->0$$

$$=-\frac{1}{2}\left[\ln\left[\frac{u_+}{u_-}-1\right]\right]^2+T'-\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\left[2-\left[\frac{u_-}{u_+-u_-}\right]^n\right]\quad \text{if } \frac{u_-}{u_+-u_-}<1\,, \tag{B7}$$

$$=T'-\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\left[\frac{u_+}{u_-}-1\right]^n\quad \text{if } \frac{u_-}{u_+-u_-}\geq 1\,,$$

with

$$T'=\ln\left[\frac{u_+}{u_-}\right]\ln\left[\frac{u_+}{u_-}-1\right],$$

and after slight rearrangement we obtain, in the limit $\nu\rightarrow 0+$,

$$M=-\frac{\pi^2}{3}+A+\frac{1}{2}\ln\left[\frac{\nu^2}{\lambda^2}\right]\ln\left[\frac{\lambda^2\nu^2}{q^4}\right]+\ln\left[\frac{q^2}{\lambda^2}\right]\ln\left[\frac{q^2+\lambda^2}{q^2\lambda^2}\nu^2\right], \tag{B8}$$

where A is given by Eq. (35). Equating the expressions (B5) and (B8), we obtain the relation given by Eq. (39).

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