Tunneling near the peaks of potential barriers: Consequences of higher-order Wentzel-Kramers-Brillouin corrections

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We derive the quantum-mechanical tunneling transmission coefficient for energies near the peak of a general potential barrier, to fourth order in the Wentzel-Kramers-Brillouin (WKB) approximation, using a method based on an eighth-order Taylor expansion of the potential near the peak. The result agrees with contour-integral formulations of WKB tunneling. For the Pöschl-Teller potential the WKB series converges rapidly to the known exact transmission coefficient. Higher-order WKB corrections may be important in attempts to determine the internuclear potential by inversion using low-energy fusion cross-section data.

I. INTRODUCTION

The Wentzel-Kramers-Brillouin (WKB) approximation is one of the central tools for obtaining approximate solutions in quantum mechanics for tunneling transmission and reflection coefficients, for bound-state energies, and for wave functions. As an asymptotic approximation technique, it finds use in many different situations where approximate solutions to certain kinds of differential equations are sought. For many applications, use of the first two orders in the WKB approximation is sufficient. Nevertheless, the WKB approximation is an asymptotic expansion in a small parameter and can be carried to higher order than the first two, in order to improve the accuracy, or at the very least, to estimate the errors in the lower-order approximation.

Formal development of higher-order WKB approximations has been carried out by a number of authors,[1] and several standard textbooks on quantum mechanics or mathematical physics treat higher-order effects, although usually only one order beyond the standard approximation.[2] Most of these higher-order WKB analyses focus either on bound states of a potential well, or in the case of potential-barrier tunneling, on situations with energies either well below or well above the peak of the potential. There has been rather less detailed discussion of tunneling with energies at or very near the peak.

At the first WKB order, by which we mean the eikonal approximation (zeroth order) together with the correction giving the multiplicative amplitude factor (first order), it has been known, since the work of Kemble in the 1930's,[3] that the transmission amplitude for tunneling with energy near the peak of a barrier approximated as a parabola is given by

$$T=(1+e^{2S})^{-1} , \tag{1}$$

$$S(E)\equiv\int_{x_1}^{x_2}\{(2m/\hbar^2)[V(x)-E]\}^{1/2}dx$$
$$=\pi(m/\kappa\hbar^2)^{1/2}(V_0-E) , \tag{2}$$

where $\hbar$ is Planck's reduced constant, m is the mass of the particle with energy E, and $V(x)\approx V_0-\frac{1}{2}\kappa x^2$ is the potential; x_1 and x_2 are the "turning points," where $V(x)-E=0$. When $E=V_0$, $S(E)=0$, and $T=\frac{1}{2}$.

It is the purpose of this paper to point out some of the formal and potentially important practical consequences of higher-order WKB corrections to the transmission coefficient T near the peak of a potential barrier. In Sec. II we use the results of previous papers[4,5] to write down the tunneling transmission coefficient to fourth order in the WKB approximation. In Sec. III we demonstrate that the result agrees with contour-integral formulations of WKB tunneling, and illustrate the accuracy gained in higher order using the Pöschl-Teller potential. We also demonstrate the effect of higher-order WKB corrections on attempts to determine the internuclear interaction potential from low-energy fusion cross-section data.

II. TRANSMISSION COEFFICIENT TO FOURTH WKB ORDER

This work was originally motivated by a program to analyze the normal modes of oscillation of black holes.[4,5] The connection between black-hole oscillations and quantum-mechanical tunneling is the mathematical parallel between the radial part of the decoupled and separated equations of black-hole perturbations and the one-dimensional Schrödinger equation on the infinite line. In both cases, the basic differential equation has the form $d^2\psi/dx^2+Q(x)\psi=0$; in quantum mechanics $-Q=(2m/\hbar^2)(V-E)$, while for black holes, Q depends on the nature of the perturbation, its frequency, and angular eigenvalues, and on the nature of the black hole being perturbed. In the black-hole case, x runs from $-\infty$ at the event horizon to $+\infty$ at radial infinity. The function $-Q(x)$ has the generic form shown in Fig. 1.

In the black-hole problem, the sought-after resonant normal modes of low-harmonic order (fundamental, first harmonic, etc.) have frequencies such that the zeros of $Q(x)$ (corresponding to turning points in quantum mechanics) are close together, near the peak of $-Q$. This prevents the development of a valid WKB approximation in the "classically forbidden region" with which to match the exterior WKB solutions across the two turning points, at x_1 and x_2. We circumvented this problem[4,5] by

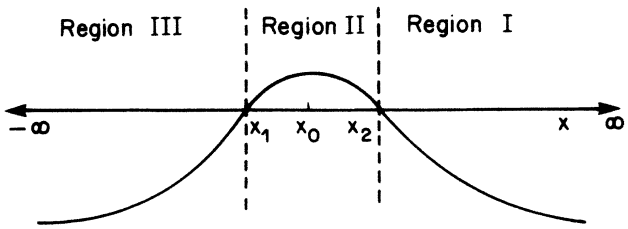

FIG. 1. The function $-Q(x)$. In quantum-mechanical applications, $-Q(x)=(2m/\hbar^2)[V(x)-E]$.

expanding $-Q$ near its peak at x_0 in a Taylor series of the form

$$Q=Q_0+\tfrac{1}{2}Q_0''z^2+\sum_{n=3}\frac{1}{n!}(d^nQ/dx^n)_0 z^n \equiv k\left[z^2-z_0^2+\sum_{n=3}b_n z^n\right] , \tag{3}$$

where

$$z=x-x_0,\quad z_0^2\equiv -2Q_0/Q_0'' ,$$

$$k\equiv \tfrac{1}{2}Q_0'',\quad b_n\equiv(2/n!Q_0'')(d^nQ/dx^n)_0 ,$$

primes denoting derivatives, and subscript zero denoting evaluation at $z=0$ ($x=x_0$). We kept terms up to and including order z^8. Using an approximate transformation to a new independent variable, we found an asymptotic solution to the differential equation for this polynomial potential in terms of parabolic cylinder functions D_ν, where ν is an index that depends on the Taylor coefficients of Q. The index ν is given implicitly by the equations

$$\nu+\tfrac{1}{2}=-ik^{1/2}z_0^2-\Lambda(\nu+\tfrac{1}{2})-\Omega(\nu+\tfrac{1}{2})-\Phi(\nu+\tfrac{1}{2}) , \tag{4}$$

$$\Lambda=ik^{-1/2}[(\tfrac{3}{16}b_4-\tfrac{7}{64}b_3^2)+(\nu+\tfrac{1}{2})^2(\tfrac{3}{4}b_4-\tfrac{15}{16}b_3^2)] , \tag{5}$$

$$\Omega=k^{-1}[(\nu+\tfrac{1}{2})(\tfrac{1155}{1024}b_3^4-\tfrac{459}{128}b_3^2b_4+\tfrac{67}{64}b_4^2+\tfrac{95}{21}b_3b_5-\tfrac{25}{16}b_6)+(\nu+\tfrac{1}{2})^3(\tfrac{705}{256}b_3^4-\tfrac{225}{32}b_3^2b_4+\tfrac{17}{16}b_4^2+\tfrac{35}{8}b_3b_5-\tfrac{5}{4}b_6)] , \tag{6}$$

$$\begin{aligned}\Phi=ik^{-3/2}[(&\tfrac{101\,479}{131\,072}b_3^6-\tfrac{131\,817}{32\,768}b_3^4b_4+\tfrac{40\,261}{8192}b_3^2b_4^2+\tfrac{14\,777}{4096}b_3^3b_5-\tfrac{5667}{1024}b_3b_4b_5\\&-\tfrac{6055}{2048}b_3^2b_6+\tfrac{1155}{512}b_3b_7-\tfrac{1539}{2048}b_4^3+\tfrac{945}{512}b_4b_6+\tfrac{1107}{1024}b_5^2-\tfrac{315}{256}b_8)\\&+(\nu+\tfrac{1}{2})^2(\tfrac{209\,055}{16\,384}b_3^6-\tfrac{239\,985}{4096}b_3^4b_4+\tfrac{62\,013}{1024}b_3^2b_4^2+\tfrac{23\,865}{512}b_3^3b_5-\tfrac{735}{128}b_3b_4b_5\\&\qquad-\tfrac{8535}{256}b_3^2b_6+\tfrac{1365}{64}b_3b_7-\tfrac{1707}{256}b_4^3+\tfrac{885}{64}b_4b_6+\tfrac{1085}{128}b_5^2-\tfrac{245}{32}b_8)\\&+(\nu+\tfrac{1}{2})^4(\tfrac{115\,755}{8192}b_3^6-\tfrac{116\,325}{2048}b_3^4b_4+\tfrac{2495}{512}b_3^2b_4^2+\tfrac{9765}{256}b_3^3b_5-\tfrac{2415}{64}b_3b_4b_5\\&\qquad-\tfrac{2715}{128}b_3^2b_6+\tfrac{315}{32}b_3b_7-\tfrac{375}{128}b_4^3+\tfrac{165}{32}b_4b_6+\tfrac{315}{64}b_5^2-\tfrac{35}{16}b_8)] .\end{aligned} \tag{7}$$

This interior solution was matched asymptotically in both directions to higher-order WKB solutions in the exterior to obtain connection coefficients between the incoming and outgoing amplitudes $Z_{\rm in}$ and $Z_{\rm out}$ of ψ on either side of the barrier. The ratio between the required order of the WKB approximation and the power of z retained is 1:2, so that a fourth-order analysis requires terms through z^8. The connection coefficients were given by the matrix representation

$$\begin{bmatrix} Z_{\rm out}^{\rm III} \\ Z_{\rm in}^{\rm III}\end{bmatrix}=\begin{bmatrix} e^{i\pi\nu} & iR^2e^{i\pi\nu}(2\pi)^{1/2}/\Gamma(\nu+1) \\ R^{-2}(2\pi)^{1/2}/\Gamma(-\nu) & -e^{i\pi\nu}\end{bmatrix}\begin{bmatrix} Z_{\rm out}^{\rm I} \\ Z_{\rm in}^{\rm I}\end{bmatrix} , \tag{8}$$

where Γ denotes the gamma function, and R is given by

$$R=(\nu+\tfrac{1}{2})^{(\nu+1/2)/2}\exp[-\tfrac{1}{2}(\nu+\tfrac{1}{2})+O(\nu+\tfrac{1}{2})^{-1}] . \tag{9}$$

For quantum-mechanical tunneling with real potential and real energy, Q is real, and thus, from Eqs. (4)–(7), $\nu+\frac{1}{2}$ is imaginary. As a consequence,

$$(e^{i\pi\nu})^*=-e^{i\pi\nu},\quad R^*=e^{-i\pi(\nu+1/2)/2}R^{-1} , \tag{10}$$

and the fourth-order WKB tunneling transmission coefficient $T\equiv|Z_{\rm out}^{\rm III}|^2/|Z_{\rm in}^{\rm I}|^2$ is given by Eq. (1) with

$$S(E)=i\pi(\nu+\tfrac{1}{2}) . \tag{11}$$

Since for smooth functions Q, the coefficients Λ, Ω, and Φ are progressively smaller corrections, Eq. (4) can be solved for $\nu+\frac{1}{2}$ by iteration. The result is

$$
\begin{aligned}
S(E)=\pi k^{1/2}\Big[&\tfrac{1}{2}z_0^2+\left(\tfrac{15}{64}b_3^2-\tfrac{3}{16}b_4\right)z_0^4+\left(\tfrac{1155}{2048}b_3^4-\tfrac{315}{256}b_3^2b_4+\tfrac{35}{128}b_4^2+\tfrac{35}{64}b_3b_5-\tfrac{5}{32}b_6\right)z_0^6\\
&+\left(\tfrac{255\,255}{131\,072}b_3^6-\tfrac{225\,225}{32\,768}b_3^4b_4+\tfrac{45\,045}{8192}b_3^2b_4^2+\tfrac{15\,015}{4096}b_3^3b_5-\tfrac{3465}{1024}b_3b_4b_5-\tfrac{3465}{2048}b_3^2b_6+\tfrac{315}{512}b_3b_7-\tfrac{1155}{2048}b_4^3\right.\\
&\left.+\tfrac{315}{512}b_4b_6+\tfrac{315}{1024}b_5^2-\tfrac{35}{256}b_8\right)z_0^8\Big]\\
+\pi k^{-1/2}\Big[&\left(\tfrac{3}{16}b_4-\tfrac{7}{64}b_3^2\right)-\left(\tfrac{1365}{2048}b_3^4-\tfrac{525}{256}b_3^2b_4+\tfrac{85}{128}b_4^2+\tfrac{95}{64}b_3b_5-\tfrac{25}{32}b_6\right)z_0^2\\
&-\left(\tfrac{285\,285}{65\,536}b_3^6-\tfrac{315\,315}{16\,384}b_3^4b_4+\tfrac{79\,695}{4096}b_3^2b_4^2+\tfrac{28\,875}{2048}b_3^3b_5-\tfrac{8505}{512}b_3b_4b_5\right.\\
&\left.-\tfrac{9765}{1024}b_3^2b_6+\tfrac{1365}{256}b_3b_7-\tfrac{2625}{1024}b_4^3+\tfrac{1155}{256}b_4b_6+\tfrac{1085}{512}b_5^2-\tfrac{245}{128}b_8\right)z_0^4\Big]\\
+\pi k^{-3/2}\Big(&\tfrac{119\,119}{131\,072}b_3^6-\tfrac{153\,153}{32\,768}b_3^4b_4+\tfrac{47\,229}{8192}b_3^2b_4^2+\tfrac{16\,107}{4096}b_3^3b_5-\tfrac{6237}{1024}b_3b_4b_5-\tfrac{6405}{2048}b_3^2b_6+\tfrac{1155}{512}b_3b_7-\tfrac{1995}{2048}b_4^3\\
&+\tfrac{1095}{512}b_4b_6+\tfrac{1107}{1024}b_5^2-\tfrac{315}{256}b_8\Big)\;. \qquad (12)
\end{aligned}
$$

The first term $\pi k^{1/2}z_0^2/2$ is equivalent to the formula of Kemble Eq. (2). For details of the derivation of these results, the reader is referred to Ref. 5. We wish to point out three important features of this result.

III. CONSEQUENCES OF HIGHER-ORDER CORRECTIONS

A formal higher-order WKB expansion for the quantity S in Eq. (1) was proposed by Fröman and Fröman:[6]

$$S(E)=\tfrac{1}{2}\oint\sum_{n=0}^{\infty}S'_{2n}(x)dx\ , \qquad (13)$$

where the contour is taken in the complex x plane surrounding the two turning points, these being real for sub-barrier tunneling and complex for superbarrier tunneling. The quantities S'_{2n} are the terms that appear in the WKB sequence of equations that occurs when the trial solution $\psi\sim e^{(S/\varepsilon)}$, $S=\sum_{n=0}^{\infty}\varepsilon^nS_n$ is substituted into the equation

$$\varepsilon^{-2}d^2\psi/dx^2+Q\psi=0\ ,$$

and the equation is solved at each order in ε:

$$
\begin{aligned}
&S'_0=(-Q)^{1/2},\quad S'_1=-\tfrac{1}{4}Q'/Q\ ,\\
&S'_2=-\tfrac{1}{8}\left[\frac{Q''}{(-Q)^{3/2}}+\frac{5}{4}\frac{Q'^2}{(-Q)^{5/2}}\right],
\end{aligned}
\qquad (14)
$$

and so on. It is implicit in the derivations of Eq. (13) that the energy not be near the peak of the barrier, so the question arises whether this equation gives a valid approximation for $T(E)$ for energies at or near the peak.

We have verified this in the affirmative by direct calculation of the contour integrals in Eq. (13) for energies near the peak of a general barrier. The first step is to find approximate roots of $Q(z)$; to the necessary order these are given schematically by

$$z_1\simeq\pm z_0+\sum_{n=1}^{3}\left(c_{2n}z_0^{2n}\pm d_{2n+1}z_0^{2n+1}\right)\ , \qquad (15)$$

where the coefficients c_{2n} and d_{2n+1} are functions of the b_n (one can introduce a small parameter ε or use z_0 as the small parameter in carrying out these approximate solutions). Then, by defining

$$y\equiv z-\sum_{n=1}^{3}c_{2n}z_0^{2n}\ , \qquad (16)$$

$$y_0\equiv z_0+\sum_{n=1}^{3}d_{2n+1}z_0^{2n+1}\ , \qquad (17)$$

one can write Q in the form

$$Q\simeq k(y^2-y_0^2)\left[1+\sum_{n=1}^{6}\sum_{m=0}^{n}f_{nm}y^my_0^{n-m}\right]\ , \qquad (18)$$

where, for example, the first few coefficients f_{nm} are given by $f_{10}=0$, $f_{11}=b_3$, $f_{20}=b_4-\tfrac{3}{2}b_3^2$, $f_{21}=0$, $f_{22}=b_4$, and so on.

Then, in $(-Q)^{1/2}$, we expand the first factor in Eq. (18) according to

$$(y^2-y_0^2)^{1/2}\simeq y(1-\tfrac{1}{2}y_0^2/y^2+\cdots)\ , \qquad (19)$$

which converges for $y>y_0$. On the other hand, the second factor in Eq. (18), arising from the Taylor expansion, converges for $y<y^*$, where we expect $y^*>y_0$. Thus we can deform the contour to a circle surrounding the turning points with a radius $y_0<y<y^*$ (assuming that there are no unforeseen poles of the integrands in the vicinity); the above expansion provides a valid Laurent series for the integrands from which the residues can be obtained. Using this method, we verified that Eq. (13) agrees with Eq. (12) term by term to the appropriate order; in Eq. (12) the terms proportional to $k^{1/2}$, $k^{-1/2}$, and $k^{-3/2}$ come, respectively, from the contour integrals of S'_0, S'_2, and S'_4.

It is useful to study the accuracy gained by the use of higher-order WKB approximations. In the case of black-hole normal modes, we found that the agreement of the resulting complex frequencies with those obtained by other methods improved dramatically with higher-order corrections. For example, for the fundamental gravitational normal mode of the Schwarzschild black hole, the agreement with accurate analytic-numerical techniques[7] improved in the real and imaginary parts of the frequency from (6.7%, 0.79%) at the first WKB order[4] to (0.13%, 0.22%) at third order.[8]

Another test of accuracy is to apply the WKB approximation to a potential for which an exact solution is known; an example is the Pöschl-Teller potential[9]

$V(x)=V_0\cosh^{-2}(\alpha x)$, where V_0 and α are constants. The wave function ψ consists of hypergeometric functions of $\frac{1}{2}[1-\tanh(\alpha x)]$, and the exact transmission coefficient is given by

$$T=\left[1+\frac{\cosh^2[\pi(J^2-\frac{1}{4})^{1/2}]}{\sinh^2[\pi J(1-\eta)^{1/2}]}\right]^{-1}, \tag{20}$$

where $J^2=2mV_0/\hbar^2\alpha^2$ and $\eta=1-E/V_0$.

The WKB approximation is valid provided the functions S_n [Eq. (14)] satisfy the conditions $S_{n+1}/S_n \ll 1$ and $S_{n+1} \ll 1$; this gives the restriction $J \gg 1$. Evaluating the derivatives of the Pöschl-Teller potential at the peak to the eighth order, and substituting into Eq. (12), we obtain

$$S_{\mathrm{WKB}}=\tfrac{1}{2}\pi J(\eta+\tfrac{1}{4}\eta^2+\tfrac{1}{8}\eta^3+\tfrac{5}{64}\eta^4) -\tfrac{1}{8}\pi J^{-1}-\tfrac{1}{128}\pi J^{-3}, \tag{21}$$

where the three terms in Eq. (21) correspond respectively to S_0, S_2, and S_4. Expanding the exact transmission coefficient in powers of J^{-1}, we get

$$T \approx [1+e^{2S_{\mathrm{WKB}}}+O(e^{-\pi J})]^{-1},$$

agreement to the appropriate order. For the case $E=V_0$, Table I shows values for T for the exact solution and for the WKB approximation to lowest, second, and fourth orders, respectively. Notice that even for values of J as low as unity for which the WKB approximation need not be accurate *a priori,* the fourth-order WKB approximation agrees to better than two parts in a thousand.

Table I illustrates an important aspect of tunneling that is often overlooked in standard WKB treatments of the subject, namely that in general, the transmission coefficient *at* the peak ($E=V_0$) is not equal to one-half. The rule of "half transmitted, half reflected at the peak" is only valid at the lowest WKB order. At second WKB order, for instance, T depends on the shape of the potential at the peak: evaluating S in Eq. (12) to first and second order at the peak ($z_0=0$), we obtain

$$S(V_0)=\tfrac{1}{64}\pi k^{-1/2}(12b_4-7b_3^2)+O(k^{-3/2}). \tag{22}$$

Thus $S(V_0)$ can be positive or negative, in other words $T(V_0)$ can be less than or greater than 0.5, depending on the values of the third and fourth derivatives of the potential.

The preceding observation may lead to practical uses for higher-order WKB approximations in specific applications. We shall illustrate this with an example from sub-barrier nuclear fusion.

Two nuclei collide with energy just below the peak of the internuclear potential made up of an attractive nuclear well and a repulsive Coulomb barrier. When the nuclei tunnel through this barrier, they undergo fusion and the release of new nuclear species. One of the problems in this subject is to use cross-section data for fusion to study the shape of this internuclear potential. A method for carrying out this inversion using the WKB approximation was pioneered by Balantekin, Koonin, and Negele (BKN).[10] By making certain approximations, it is possible to relate the total fusion cross section σ to the tunneling coefficient $T(E)$ through the barrier in the s-wave ($l=0$) channel. The problem is then reduced to the equivalent one-dimensional Schrödinger tunneling problem (the behavior of the wave function at the origin is not important, since fusion is assumed to occur upon tunneling through the barrier; thus there is no incident wave from region III in Fig. 1). From Eq. (1), one can then infer the amplitude $S(E)$ from the cross-section data, using $S(E)=\frac{1}{2}\ln(T^{-1}-1)$.

Now, if one knows V_0, the peak of the barrier, and if one knows $S_0(E)$, the lowest-order WKB function as a function of energy, given by the integral in Eq. (2), then the quantity

$$t(E)=-\frac{2}{\pi}\left[\frac{\hbar^2}{2m}\right]^{1/2}\int_E^{V_0}\frac{dS_0/dE'}{(E'-E)^{1/2}}dE' \tag{23}$$

is an *exact* expression for the width of the barrier at the energy E.[11] (In Newtonian mechanics for a particle in a potential well, the analogous formula solves the problem of determining the shape of the well given a knowledge of the period of oscillation as a function of energy.[12])

BKN's procedure was to insert the empirically-determined $S(E)$ into Eq. (23), and identify the upper limit of integration as the energy at which $S(E)=0$ [cf. Eq. (2)]. They then tested their procedure by calculating $t(E)$ from cross-section values $\sigma(E)$ generated numerically using two different analytic internuclear potentials. One was the Woods-Saxon potential[13] with Coulomb barrier

$$V(r)=-V_N[1+e^{(r-R)/a}]^{-1}+Z_1Z_2e^2/r,$$

where a, R, V_N, and Z_i are determined by the nuclei undergoing fusion, carbon-12 nuclei in this example. The second potential was the Krappe-Nix-Sierk (KNS) potential [Ref. 14, Eqs. (17)–(20)], applied to nickel-64. As a measure of the success of the inversion method, they then computed $\Delta t \equiv t(E)-t_0(E)$, where $t_0(E)$ is the exact width of the analytic potential at energy E. They obtained the solid curves shown in Fig. 2. The "error" Δt is small away from the peak of the barrier, but grows significantly as the energy approaches the peak. They attributed this difference to approximations that were made in obtaining the s-wave transmission coefficient $T(E)$ [thence $S(E)$] from the *total* cross section $\sigma(E)$.

However, the empirical data do *not* exactly determine the variables that enter Eq. (23). In the first place, Eq.

TABLE I. Transmission coefficients for the Pöschl-Teller potential.

		WKB		
J	Exact	Lowest order	2nd order	4th order
1	0.696 228 2	0.5	0.6868	0.697 302 5
2	0.598 451 5	0.5	0.5969	0.598 407 7
3	0.565 531 6	0.5	0.5651	0.565 525 3
4	0.549 121 7	0.5	0.5489	0.549 120 2
5	0.539 287 4	0.5	0.5392	0.539 286 9

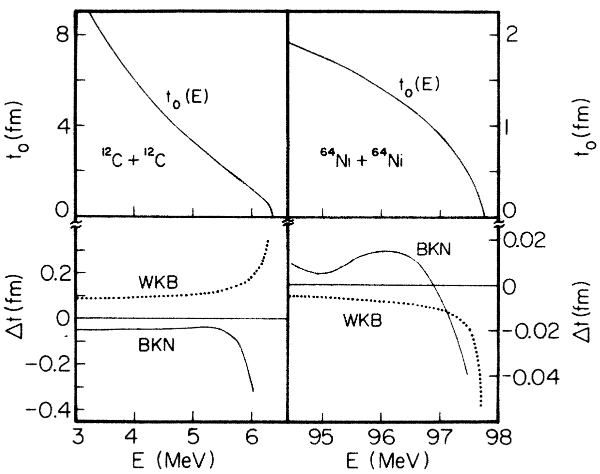

FIG. 2. Estimates for the width of a potential barrier from cross-section data for $^{12}C+^{12}C$ (left-hand panel) and for $^{64}Ni+^{64}Ni$ (right-hand panel). Upper portion: Barrier width $t_0(E)$ for two phenomenological potentials, the Woods-Saxon potential (Ref. 13) for ^{12}C and the KNS potential (Ref. 14) for ^{64}Ni. Lower portion: Error in the thickness estimate $\Delta t=\tilde{t}-t_0$ determined using a WKB inversion procedure. Solid curve shows results of BKN (Ref. 10) using numerically generated cross-section data for the two potentials. Dotted curve shows effects of higher-order WKB corrections in the definition of $\tilde{t}$.

(23) requires the lowest-order WKB function $S_0(E)$, whereas the data determine the full amplitude $S(E)=S_0+S_2+S_4+\cdots$. Secondly, the integral requires knowledge of the potential barrier peak V_0, whereas the most that can be determined from cross-section data is the energy at which $T(E)=\frac{1}{2}$ or $S(E)=0$. Call this energy B. Because of the higher-order WKB contributions to $S(E)$, $B\neq V_0$ in general. In actuality, then, the data determine a "phenomenological barrier width" $\tilde{t}(E)$, given by

$$\tilde{t}(E)=-\frac{2}{\pi}\left[\frac{\hbar^2}{2m}\right]^{1/2}\int_E^B\frac{dS/dE'}{(E'-E)^{1/2}}dE' , \tag{24}$$

which may thus differ from the true width $t_0(E)$. The difference $\Delta t=\tilde{t}-t_0$ may be expected to be greatest near the peak of the barrier, where $S_0(E)\to 0$, while the higher-order terms are nonzero. In addition, if $B<(>)V_0$, then $\tilde{t}\to 0$ below (above) the true peak.

This behavior can be studied using our formula for $S(E)$ near the peak Eq. (12). Confining ourselves to second WKB order with $S(E)=S_0(E)+S_2(E)$ and with B determined from $S_0(B)+S_2(B)=0$, we find

$$B=V_0+(\hbar^2/m)(\tfrac{3}{16}b_4-\tfrac{7}{64}b_3^2) ,$$

and

$$\tilde{t}(E)=2(2/\kappa)^{1/2}[V_0-E+(\hbar^2/m)(\tfrac{3}{16}b_4-\tfrac{7}{64}b_3^2)]^{1/2}$$
$$\times[1+\tfrac{16}{3}\kappa^{-1}(E-V_0)(\tfrac{3}{16}b_4-\tfrac{15}{64}b_3^2)] , \tag{25}$$

where $\kappa=-V_0''$, and in the definitions of b_n, V can be substituted for Q. From the Taylor expansion of the potential near the peak, the true width is given by

$$t_0(E)=2(2/\kappa)^{1/2}(V_0-E)^{1/2}$$
$$\times[1+\tfrac{16}{3}\kappa^{-1}(E-V_0)(\tfrac{3}{16}b_4-\tfrac{15}{64}b_3^2)] . \tag{26}$$

Calculating the needed derivatives of the two analytic nuclear potentials at their peaks, substituting the results into Eqs. (25) and (26), and evaluating using the relevant nuclear parameters for ^{12}C and ^{64}Ni, we obtain values for Δt shown as dashed curves in Fig. 2. In the case of ^{64}Ni, it turns out that $S(V_0)\approx-0.018<0$, so (since $dS/dE<0$) $B<V_0$, the phenomenological width vanishes before the true width, and Δt becomes large and negative near the peak of the barrier, in agreement with the behavior found by BKN. In this case, the "error" Δt can be understood, at least in part, as a result of the failure of Eq. (23) to give the true width because of higher-order WKB effects. By contrast for ^{12}C, $S(V_0)\approx 0.080>0$, so $B>V_0$, and Δt becomes large and positive near the peak, a trend opposite to the BKN result, although comparable in magnitude. In this case, the other approximations that are made in the BKN method may be the dominant source of the behavior of Δt, offsetting the WKB effects.

The lesson is that the inclusion of higher-order WKB corrections may in some cases lead to better agreement between theory and experiment in such potential inversion procedures. Unfortunately, we have been unable to derive an equation to replace Eq. (23) that gives the true width $t_0(E)$ (or a better approximation thereto) yet that uses exclusively observable functions, such as $S(E)$.

Whether there may be other problems in physics for which these higher-order WKB effects in tunneling near the peaks of potential barriers may be important is an open question. Perhaps this paper will motivate investigation of this question.

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