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Stochastically forced Hopf bifurcation: Approximate Fokker-Planck equation in the limit of short correlation times

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We derive an approximate Fokker-Planck equation up to first order in the correlation time for a stochastically forced Hopf bifurcation. The general results are illustrated on the Brusselator model. In the range of validity of our approximation, it is shown that the noise always postpones the bifurcation point (appearance of new extrema of the probability distribution). This shift is a decreasing function of the correlation time. Qualitative agreement is found with recent experimental observations [L. Fronzoni, R. Mannella, P. McClintock, and F. Moss, Phys. Rev. A **36**, 834 (1987)].

In recent years extensive investigations have been devoted to nonlinear systems driven by colored noise. Particular attention has been focused on the stationary[1-7] and dynamical[6-10] properties of one variable system. In this case there seems to be agreement between different theories in the literature.[10,11] On the other hand, the understanding of higher-dimensional systems is much less complete:[12-16] Here the stationary probability density is not, in general, known, even in the white-noise limit.

In this paper we are concerned with two-variable systems forced by colored noise and operating in the vicinity of a Hopf bifurcation. In the absence of fluctuations, the normal form theory[17] provides a powerful tool for the study of dynamical systems. This permits the reduction of the dynamical flow to a local universally valid form. When external noise is incorporated into the normal form approach, the situation becomes more delicate. One way to tackle the problem is simply to let fluctuate the parameter of the normal form.[18,19] However, because of the complexity of the nonlinear variable change inherent in the passage to the normal form, the above description does not contain all terms describing the system-noise coupling as they arise when the parameters fluctuations are studied in the original system.[15,21,22] A more satisfactory approach should therefore start by including the noise in the original variables and then perform the reduction to the normal form, keeping consistent track of the stochastic terms in the successive transformations.[20-22]

Following this last point of view, we consider the normal form of a Hopf bifurcation with arbitrary coupling between normal form variables and noise. The expressions of the multiplicative functions in the stochastic terms depend on the particular system under consideration. By use of cumulant-projector techniques we derive an approximate Fokker-Planck equation (FPE) up to first order in the correlation time of the noise. To this order our equation turns out to be identical with the best FPE type[7,10] which would be obtained by a small-D approximation.

We apply our general result to the Brusselator model[23] perturbed by multiplicative colored noise. This system has been the subject of both recent analytical[15] and experimental[24] works. Our results which extend those of Lefever and Turner explain qualitatively well the decrease of the postponement of the transition on increasing the correlation time of the noise observed by Fronzoni, Mannella, McClintock, and Moss.[24]

We study the normal form of a Hopf bifurcation[17] in a system in which some parameters are fluctuating:

$$\dot{r}=f(r)+g_1(r,\phi)z_t \ , \qquad \dot{\phi}=\omega(r)+g_2(r,\phi)z_t \ . \tag{1}$$

Here $f(r)=\alpha r-\beta r^3+O(r^5)$, $\omega(r)=\Omega+\delta r^2+O(r^4)$ are the standard deterministic parts of the normal form; the functions $g_1(r,\phi)$, $g_2(r,\phi)$, which depend on which parameters are varying, account for the coupling of the original system with the colored noise z_t. The noise is assumed to be a Ornstein-Uhlenbeck process with a correlation function given by[5]

$$\langle z_t z_{t'}\rangle=\tfrac{1}{2}\epsilon\gamma e^{-\gamma|t-t'|} \ . \tag{2}$$

We adopt here an approach based on cumulants and on projection operator methods. Our procedure is similar to the cumulant expansion scheme of Terwiel[26] applied recently by Sancho, Sagues, and San Miguel[9] in the context of mean first passage times. We indicate briefly the main steps in the derivation, referring to the abundant literature for details.[25-28]

For each realization z_t, we consider an ensemble of systems obeying to Eq. (1) with different initial conditions. The probability density $\rho(r,\phi,t;z_t)$ of finding one system at a given point r,ϕ of the phase space at time t satisfies a stochastic Liouville equation:[25,27]

$$\partial_t\rho(r,\phi,t;z_t)=-[\partial_r(f+g_1z_t)+\partial_\phi(\omega+g_2z_t)]\rho(r,\phi,t;z_t) \ . \tag{3}$$

The probability density is obtained by averaging $\rho(r,\phi,t;z)$ over the different realizations of the noise.[27] In terms of operators we have

$$P(r,\phi,t)=\mathbf{P}\rho(r,\phi,t;z_t) \ , \tag{4}$$

where the projection operator averages any function depending on the noise over all realizations of z:

$$\mathbf{P}\psi(r,\phi,t)=\langle\psi\rangle(r,\phi,t)\ ,\qquad \mathbf{P}^2\psi=\mathbf{P}\psi\ . \tag{5}$$

In the interaction picture,

$$\Phi(r,\phi,t;z_t)=e^{(\partial_r f+\partial_\phi \omega)t}\rho(r,\phi,t;z_t)\ , \tag{6}$$

Eq. (3) becomes

$$\partial_t\Phi=\mathbf{L}\Phi\ , \tag{7}$$

with

$$\mathbf{L}=-e^{(\partial_r f+\partial_\phi \omega)t}(\partial_r g_1+\partial_\phi g_2)z_t e^{-(\partial_r f+\partial_\phi \omega)t}\ . \tag{8}$$

Applying $\mathbf{P}$ and $(1-\mathbf{P})$ successively to Eq. (7), eliminating $(1-\mathbf{P})\Phi$, and assuming uncorrelated initial conditions between the system and the external noise source yields a closed equation for $\mathbf{P}\Phi$:

$$\partial_t\mathbf{P}\Phi=\mathbf{P}\mathbf{L}\int_0^t dt'\mathbf{K}(t\,|\,t')\mathbf{L}(t')\mathbf{P}\Phi(t')\ . \tag{9}$$

The operator $K(t\,|\,t')$ is given by the following expansion:

$$\mathbf{K}(t\,|\,t')=1+\int_{t'}^t dt_1(1-\mathbf{P})\mathbf{L}(t_1)+\int_{t'}^t dt_1\int_{t'}^{t_1}dt_2(1-\mathbf{P})\mathbf{L}(t_1)(1-\mathbf{P})\mathbf{L}(t_2)+\cdots\ . \tag{10}$$

With the strength ϵ and the correlation time $1/\alpha$ of the noise being small, this formal expression may be seen as an expansion in powers of ϵ/γ. All terms higher than the first three in Eq. (10) give contributions which are, at most, of order ϵ^2/γ^2 [note that owing to the Gaussian property of the noise and $\langle z\rangle=0$, the second, fourth, etc., terms of Eq. (10) do not contribute to Eq. (9)]. In order to derive an approximate FPE which contains the effects of the noise up to order $1/\gamma$, we need to keep the first and third terms of Eq. (10).

Neglecting all corrections of order higher than $1/\gamma$ and transforming back to original variables yields

$$\partial_t P(r,\phi,t)=-(\partial_r f+\partial_\phi\omega)P+\int_0^t ds\langle z_t z_{t-s}\rangle(\partial_r g_1+\partial_\phi g_2)[1-s(\partial_r g_1+\partial_\phi g_2)]\ \{1+s[\partial_r f+\partial_\phi\omega-\tfrac{1}{2}\epsilon(\partial_r g_1+\partial_\phi g_2)^2]\}P$$
$$+\int_0^t dt'\int_{t'}^t dt_1\int_{t'}^{t_1}dt_2[\langle z_t z_{t_2}\rangle\langle z_{t_1}z_{t'}\rangle+\langle z_t z_{t'}\rangle\langle z_{t_1}z_{t_2}\rangle](\partial_r g_1+\partial_\phi g_2)^2P\ . \tag{11}$$

After straightforward algebra, Eq. (11) becomes, for $t\gg 1/\gamma$,

$$\partial_t P=-\partial_r\left[f+\frac{1}{2}\epsilon(g_1g_{1,r}g_{1,\phi}g_2)+\frac{1}{2}\frac{\epsilon}{\gamma}(f_rg_1g_{1,r}-fg_{1,r}^2-fg_{1,\phi}g_{2,r}-\omega g_{1,\phi}g_2+\omega_rg_1g_{1,\phi})\right]P$$
$$-\partial_\phi\left[\omega+\frac{1}{2}\epsilon(g_1g_{2,r}+g_{1,\phi}g_2)+\frac{1}{2}\frac{\epsilon}{\gamma}(f_rg_1g_{2,r}-fg_{2,r}g_{2,\phi}-\omega g_{1,\phi}g_{2,r}-\omega g_{1,\phi}^2-\omega_rg_1g_{2,\phi})\right]P$$
$$+\tfrac{1}{2}\epsilon\partial_r^2\left[g_1^2+\frac{1}{\gamma}(f_rg_1^2-fg_1g_{1,r}-\omega g_1g_{1,\phi})\right]P$$
$$+\tfrac{1}{2}\epsilon\partial_r\partial_\phi\left[2g_1g_2+\frac{1}{\gamma}[f_rg_1g_2-f(g_1g_2)_r-\omega g_{1,\phi}g_2-\omega g_2g_{2,\phi}+\omega_rg_1^2]\right]P$$
$$+\tfrac{1}{2}\epsilon\partial_\phi^2\left[g_2^2+\frac{1}{\gamma}(-fg_2g_{2,r}-\omega g_2g_{2,\phi}+\omega_rg_1g_2)\right]P\ , \tag{12}$$

with the notation $\psi_r=\partial_r\psi$, $\psi_\phi=\partial_\phi\psi$. The remarkable fact is that as a consequence of the Gaussian character of the noise, the contribution of order ϵ^2/γ (which could possibly break the FPE structure) cancels. This implies that Eq. (12) is identical, up to order $1/\gamma$ to "the best FPE approximation."[7,10] In our approach, the best FPE would be recovered by truncating expansion (10) to the first term (Bourret approximation[9,27]), discarding second-order terms in ϵ in Eq. (11) (small-D approximation;[7,27] linear response theory[29]), and summing up all the orders in $1/\gamma$.[4,7] As pointed out recently,[11] care has to be taken when using the best FPE beyond the first order in $1/\gamma$. In fact, terms of order ϵ^2/γ^2 neglected in the best FPE give rise to third-order derivatives which, in the small noise limit, could be of diffusive nature to order ϵ/γ^2. Here we bypass these difficulties by invoking the vicinity of the white-noise limit and adopting $1/\gamma$ as our expansion parameter in the derivation of Eq. (12). Note that, as expected, in the white-noise case, Eq. (12) gives the Stratanovitch version of the FPE associated with Eqs. (1).[1,2]

A perturbative analysis[19] of Eq. (12) shows that in the limit $\epsilon\rightarrow 0$, and in view of the circular symmetry of the normal form, the stationary probability density $P_s(r,\phi)$ factorizes as

$$P_s(r,\phi)=P(\phi\,|\,r)P_s(r)=(2\pi)^{-1}P_s(r)\ . \tag{13}$$

The radial part $P(r)$ obeys a closed-form equation obtained by integrating Eq. (12) over the angular variable.

We now investigate the effects of colored noise on the two-variable chemical system known as the Brusselator[23]

$$\dot{X}=A-(1+B)X+X^2Y\ ,\qquad \dot{Y}=BX-X^2Y\ , \tag{14}$$

where X and Y are reactant concentrations, A and B control parameters. This system undergoes a Hopf bifurcation for the critical value $B_c=1+A^2$ of the control parameter B. Following the ideas of previous works on this model,[15,24] we consider here the bifurcation parameter as a fluctuating quantity

$$B \to B_c(1+z_t) \ . \tag{15}$$

Our next step is to transform Eq. (14) together with (15) to the normal form [Eqs. (1)]. This is a very classical calculation.[17] Putting $A=1$ for the sake of simplicity, we finally arrive at the following expressions of the functions appearing in Eqs. (1):

$$f(r)=\tfrac{1}{2}\beta r-\tfrac{3}{2}r^3,\quad \omega(r)=1-\tfrac{1}{6}r^2,$$

$$g_1(r,\phi)=-[\cos\phi+2r\cos^2\phi+\tfrac{8}{3}r^2\cos\phi(2\cos^2\phi-1)] \ ,$$

$$g_2(r,\phi)=-\frac{1}{r}[-\sin\phi-2r\cos\phi\sin\phi+\tfrac{8}{3}r^2\sin\phi(2\sin^2\phi-1)] \ , \tag{16}$$

where $\beta=\langle B\rangle-B_c$ is the unfolding parameter assumed to be small, $\beta\ll 1$. Recall that the normal form (1), (16) is a local Taylor expansion of the dynamical flow valid in a neighborhood $r\sim\beta^{1/2}$. In this sense, terms higher than the cubic one are neglected in the expression of $f(r)$.

We are now in the position to apply our preceding results. Focusing our attention on the most probable values of the radial variable, we introduce the stochastic potential $U(r)$:

$$P(r)=Nre^{-U(r)} \ . \tag{17}$$

Here N is a normalization constant and the factor r accounts for the passage to polar coordinates.[19,22] As mentioned above the equation for $P(r)$ is obtained by integrating Eq. (12) combined with Eqs. (16) over ϕ. After some lengthy calculation, the expression of the stochastic potential reduces to

$$U(r)=-\frac{2}{\epsilon}\left[\tfrac{1}{2}\beta r^2-\tfrac{3}{4}r^4+2\epsilon\left(-\frac{7}{6}+\frac{3}{2\gamma}\right)r^2\right] \ . \tag{18}$$

The extrema of the potential, which determine the most probable values satisfy the equation

$$\tfrac{1}{2}\beta r_m-\tfrac{3}{2}r_m^3+2\epsilon\left(-\frac{7}{6}+\frac{3}{2\gamma}\right)r_m=0 \ . \tag{19}$$

This relation gives us explicitly the role of the correlation time of the noise in the transition leading to new extrema in the probability density. We see that the postponement found in the white-noise limit tends to be weakened by the finiteness of the correlation time. We believe that here we have a qualitative explanation of the decrease of the postponement of the transition on increasing the correlation time observed experimentally.[24]

Noteworthy is the fact that in order to keep $\langle r^2\rangle$ finite, ϵ should scale as[19]

$$\epsilon=\beta^\delta,\quad \delta\geq 2 \ . \tag{20}$$

This implies that the corrections introduced by the noise in Eq. (19) are at most of the same order of magnitude as the deterministic term ($\sim r^5$) neglected in the normal form. However, this term would only give an additive contribution to the shift of the bifurcation point in the absence of noise. It will not affect the stochastic terms to dominant order and will not modify our previous conclusions.

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