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Ground-state phase transitions of the degenerate parametric amplifier

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Ground-state phase transitions of the degenerate parametric amplifier assuming the pump mode in an ordinary coherent state and the signal mode in an SU(1,1) coherent state are discussed. A second-order phase transition with the coupling constant playing the role of a control parameter is found. The lower-energy state can be a squeezed state in the signal mode. An additional anharmonic term related to optical bistability is shown to enhance the effect.

For some time now there has been considerable interest in the subject of phase transitions in the realm of quantum mechanics. Perhaps the best known example is the phenomenon of the decay of the vacuum that occurs in the vicinity of a nuclear charge for which $Z > Z_0$ where $Z_0 = 173$ for an extended nucleus.[1] Other interesting examples are those associated with cooperative behavior in many-particle systems. For example, Gilmore and Feng[2] have studied the ground-state properties of the Lipkin-Meshkov-Glick nuclear model[3] using the SU(2) atomic coherent states.[4] A second-order phase transition occurs whereby there is competition between all nucleons being in the ground state or each nucleon being in a linear combination of ground and excited states. The interaction strength acts as the control parameter. A similar study was performed by Bhaumik, Choudhury, Dutta Roy, and Dutta Roy,[5] who extended the model to include bosons, and using the atomic and ordinary coherent states obtained a description of boson condensation. We also mention that similar ground-state phase transitions have been described for the Dicke model of N two-level atoms interacting cooperatively with the quantized electromagnetic field.[6]

In this paper we wish to discuss the ground-state properties of another system of great importance in quantum optics, namely the degenerate parametric amplifier (DPA). The fully quantized Hamiltonian for this system in the absence of damping is

$$H = \omega a_s^\dagger a_s + 2\omega a_p^\dagger a_p + \gamma(a_s^{\dagger 2} a_p + a_s^2 a_p^\dagger) , \tag{1}$$

where $(a_s, a_s^\dagger)$ and $(a_p, a_p^\dagger)$ are the usual Bose operators associated with the signal and pump modes, respectively. This Hamiltonian has been studied by a number of authors as it gives rise to nonclassical effects such as squeezing antibunching.[7] The quantum theory of Eq. (1) has been recently discussed by Hillery and Zubairy[8] using the path-integral approach in the coherent-state representation.

Usually the DPA is studied in the semiclassical approximation in which one makes the replacement $a_p \to -i\Gamma(t)$, where $\Gamma(t) = \Gamma_0 e^{-2i\omega t}$. In this approximation, the Hamiltonian acting on the ground state (the vacuum) gives rise to the squeezed vacuum, which has previously been shown to be a special case of an SU(1,1) coherent state.[9,10] This comes about because the Lie algebra of SU(1,1) may be realized in terms of the signal mode operators as

$$K_0 = \tfrac{1}{4}(a_s^\dagger a_s + a_s a_s^\dagger) = \tfrac{1}{2}(N_s + \tfrac{1}{2}), N_s = a_s^\dagger a_s ,$$
$$K_+ = \tfrac{1}{2} a_s^\dagger a_s^\dagger ,$$
$$K_- = \tfrac{1}{2} a_s a_s . \tag{2}$$

Thus, in terms of these operators, the Hamiltonian of Eq. (1) may be written

$$H = 2\omega K_0 + 2\omega a_p^\dagger a_p + 2\gamma(a_p K_+ + a_p^\dagger K_-) , \tag{3}$$

where an additive constant has been dropped.

Our aim is this paper is to study the ground-state phase transitions of Eq. (3) assuming the signal mode is in an SU(1,1) coherent state $|\xi,k\rangle$ while the pump mode is in an ordinary coherent state $|\alpha\rangle$. The coupling constant will play the roll of a control parameter.

Before we go further, we briefly review the SU(1,1) coherent states and the connection to the notion of squeezing. (For details see Refs. 9 and 10.) We use the Perelomov definition[11] which is

$$|\xi,k\rangle = \exp(zK_+ - z^* K_-)|0k\rangle , \tag{4}$$

where $z = -\frac{1}{2}\theta e^{-i\phi}$, $-\infty < \theta < \infty$, $0 \le \phi \le 2\pi$, $\xi = -\tanh(\frac{1}{2}\theta)e^{-i\phi}$. The state $|0,k\rangle$ for $k = \frac{1}{4}$ is the usual vacuum state while for $k = \frac{3}{4}$ it is a state containing one photon. Squeezing is defined in terms of the quadrature operatures

$$X_1 = \tfrac{1}{2}(a_s + a_s^\dagger), \; X_2 = \frac{1}{2i}(a_s - a_s^\dagger) , \tag{5}$$

for which

$$[X_1, X_2] = \tfrac{1}{2} i \tag{6}$$

leads to

$$V(X_1)V(X_2) \ge \tfrac{1}{16} , \tag{7}$$

where $V(X_i) = \langle X_i^2 \rangle - \langle X_i \rangle^2$. Squeezing exists if $V(X_i) < \frac{1}{4}$. In terms of the SU(1,1) algebra,

$$V(X_{1,2}) = \langle K_0 \rangle \pm \tfrac{1}{2}\langle K_+ + K_- \rangle . \tag{8}$$

For the SU(1,1) coherent state of Eq. (4),

$$V(X_{1,2}) = k[\cosh\theta \mp \cos\phi \sinh\theta] . \tag{9}$$

For $\phi=0$ and $k=\frac{1}{4}$ we get the usual result

$$V(X_{1,2})=\tfrac{1}{4}e^{\mp\theta}\ . \qquad (10)$$

Such a state (for $k=\frac{1}{4}$) is also a minimum uncertainty state.

Now using the fact that for an SU(1,1) coherent state

$$\langle K_0\rangle=k\cosh\theta,\ \langle K_\pm\rangle=-k\sinh\theta e^{\pm i\phi}\ , \qquad (11)$$

the expectation value of Eq. (3) for the state $|\xi,k\rangle|\alpha\rangle$ is

$$\mathcal{H}=\langle H\rangle=2\omega k\cosh\theta+2\omega\alpha^*\alpha -2\gamma k\sinh\theta(e^{i\phi}\alpha+e^{-i\phi}\alpha^*)\ . \qquad (12)$$

We note that this model exhibits "gauge invariance" under the transformation $\phi\rightarrow\phi+\psi$, $\alpha\rightarrow\alpha e^{-i\psi}$. Minimizing with respect to α, we obtain

$$\frac{\partial\mathcal{H}}{\partial\alpha}=2\omega\alpha^*-2\gamma k\sinh\theta e^{i\phi}=0\ , \qquad (13)$$

or

$$\alpha^*=\frac{\gamma k}{\omega}\sinh\theta e^{i\phi}\ . \qquad (14)$$

Substituting this into Eq. (12), we obtain

$$\mathcal{H}'=2\omega k\cosh\theta-\frac{2(\gamma k)^2}{\omega^2}\sinh\theta\ . \qquad (15)$$

Minimizing with respect to θ, we obtain

$$\frac{\partial\mathcal{H}'}{\partial\theta}=2\omega k\sinh\theta-\frac{4(\gamma k)^2}{\omega}\sinh\theta\cosh\theta=0\ , \qquad (16)$$

from which we obtain the solutions

$$\text{(i) }\theta_m=0\ ,$$
$$\text{(ii) }\theta_m=\cosh^{-1}\left[\frac{\gamma_c^2}{\gamma^2}\right]\ , \qquad (17)$$

where $\gamma_c^2=\omega^2/2k$. If $\gamma>\gamma_c$ only the case $\theta_m=0$ is possible, which corresponds to the vacuum in both pump and signal modes. On the other hand if $\gamma<\gamma_c$ solution (ii) is possible. The energy of this state is

$$E(\theta_m)=2\omega k\cosh\theta_m-\frac{2\gamma_c^2k^2}{\omega}\sinh\theta_m\tanh\theta_m\ . \qquad (18)$$

It is easy to show that $E(\theta_m)\leq E(\theta=0)$, for $0\leq\theta_m<\infty$. Thus at $\gamma=\gamma_c$ we have a second-order phase transition. It appears that we have a kind of boson condensation in both the pump and signal modes. This is illustrated in Fig. 1 as a bifurcation diagram where $|\xi|=\tanh(\theta_m/2)$ is taken as the order parameter. We also note that for the case $\theta_m=0$ the ground state, as expected, is not squeezed. However, for the case $\gamma<\gamma_c$, the ground state may be squeezed with an appropriate phase ϕ. ϕ is not determined by the minimization due to the gauge invariance of H.

Finally let us add to our Hamiltonian of Eq. (3) an anharmonic term of the form

$$H_{\rm an}=\tfrac{1}{2}\lambda a_s^{\dagger 2}a_s^2=2\lambda K_+K_- \qquad (19)$$

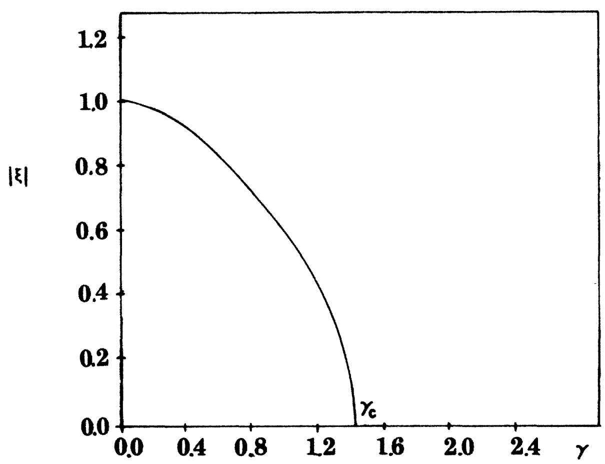

FIG. 1. Phase diagram with γ as the control parameter and $|\xi|=\tanh\frac{1}{2}\theta_m$ as the order parameter, for $\omega=1$, $k=\frac{1}{4}$.

Such a photon conserving term is known to lead to optical bistability.[12] In combination with Eq. (1) in the semiclassical pump approximation one obtains enhanced squeezing.[13,14] The interaction of a squeezed vacuum with such a system has been discussed elsewhere.[15] Here we obtain the effect of this term on the ground-state phase transition. Using the SU(1,1) coherent-state expectation value of K_+K_-,

$$\langle K_+K_-\rangle=\tfrac{1}{2}k(2k+1)\sinh^2\theta+2k\ , \qquad (20)$$

we obtain

$$\mathcal{H}=2\omega k\cosh\theta+2\omega\alpha^*\alpha-2\gamma k\sinh\theta(\alpha e^{i\phi}+\alpha^*e^{-i\phi}) +\lambda k(2k+1)\sinh^2\theta+4\lambda k\ . \qquad (21)$$

Following through the same procedure as before we arrive at the solutions

$$\text{(i) }\theta_m=0\ ,$$
$$\text{(ii) }\theta_m=\cosh^{-1}\left[\frac{\gamma_c^2}{\gamma^2-\omega\lambda(2k+1)/2k}\right]\ .$$

Thus for $\lambda>0$, the anharmonic term appears to have the effect of raising the critical value of γ and also of enhancing the squeezing (if any).

In summary, using the SU(1,1) and ordinary coherent states we have shown that there exists a second-order phase transition in the ground state of the degenerate parametric amplifier. In the region $\gamma<\gamma_c$ there is competition between the vacuum and a lower-energy state in which photons apparently condense and where the noise of one quadrature may be less than that of the vacuum. The addition of the anharmonic term enhances the effect. Finally, we simply point out that the Hamiltonian of Eq. (3) is essentially a noncompact form of the Dicke model with SU(2) being replaced by SU(1,1).

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