Relations among effective Fokker-Planck equations for systems driven by colored noise

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A number of different effective Fokker-Planck equations have been proposed to describe systems driven by colored noise. Herein we show that the effective Fokker-Planck equation obtained from the path-integral technique is identical to that obtained from other techniques at the same level of approximation. Recently reported differences thus arise not from the use of different formalisms but merely as a consequence of different approximations.

I. INTRODUCTION

The importance of systems driven by colored noise has recently been recognized in a number of physical situations, e.g., statistical properties of dye lasers,[1,2] fluctuation-induced phase transitions in chemical reactions,[3] and optical bistability.[4] A number of formalisms have been developed to address the problems associated with non-Markovian systems, and the apparent differences in these formalisms have elicited a number of controversies and discussions over the proper interpretation of physical phenomena.[5–16] It is our contention that these differences are in fact often associated with the implementation of some approximation scheme and not with the general results to which the approximations are applied. In an earlier paper we demonstrated that seemingly different approximations to the problem of a bistable system driven by colored noise all give the same results within experimental accuracy[10] (except for the decoupling method[11]). Herein our purpose is different: we show that the *same* approximations consistently applied within different formalisms in fact lead to the *same* formal results.

At least three formalisms have been developed to deal with nonlinear systems driven by fluctuations that can assume a continuous range of values: the cumulant-expansion technique,[8,9] the functional-calculus approach,[5–7,15,16] and the projection-operator method.[12–14] Each of these methods leads to a formally exact equation of evolution for the probability density of the driven process. At this level the different descriptions are therefore equivalent. The exact formal results do not lend themselves to calculations and therefore require that approximations be made.

Colored Gaussian fluctuations are characterized by two parameters: the correlation time τ_c and the intensity of the fluctuations D_c. The approximations made within the various formalisms involve truncations and/or partial resummations of doubly infinite series in the parameters τ_c and D_c. The standard approximation made in the cumulant-expansion technique is the retention of the second-cumulant contribution and the neglect of the higher cumulants in the evolution operator for the probability density.[8,9] This term is of order D_c and contains contributions to all orders in τ_c. The result is the so-called "best Fokker-Planck equation" (BFPE). Within the functional calculus approach, several approximations have been implemented, all of them leading to an "effective Fokker-Planck equation" (EFPE), i.e., an evolution operator that is of first order in D_c. The approximation of Sancho and co-workers[6,7] leads to the BFPE. That of Fox[15,16] involves a small-τ_c approximation and leads to an equation that differs from the BFPE at $O(\tau_c^2)$. Grigolini and co-workers[12–14] have used a projection-operator formalism to elucidate the properties of both of these approximate results and to discuss contributions to the evolution operator from higher-derivative terms, i.e., the breakdown of the Fokker-Planck structure.

The BFPE and Fox's EFPE both have their advocates (the latter has also been used[15] as a rationale to justify the decoupling approach of Hänggi *et al.*[11]). Claims as to the superiority of one or another of these equations have been made even though the relation between them remains obscure. It would therefore be of value to establish the precise conditions under which one or the other is obtained within the same formalism. We do so here using the exact evolution equation in the functional-calculus formalism as a starting point. In Sec. II we show that to lowest order in D_c and with no other approximations *all* of the above formalisms leads to the BFPE.

A criticism of the BFPE is that the resulting probability density is often restricted to a finite region of phase space, even though in the limit $\tau_c \to 0$ the density extends over all of phase space. This criticism will be discussed in Sec. III.

II. EQUATIONS OF EVOLUTION

The processes to be considered are described by a single variable $X(t)$ that satisfies a stochastic differential equation of the form

$$\dot{X}(t)=G(X)+g(X)f(t) . \tag{2.1}$$

Here $G(X)$ and $g(X)$ are in general nonlinear functions of X, and $f(t)$ is a Gaussian zero-centered stochastic process with correlation function

$$\langle f(t)f(\tau)\rangle=\frac{D_c}{\tau_c}e^{-|t-\tau|/\tau_c} , \tag{2.2}$$

where the angle brackets denote an average over an ensemble of realizations of the fluctuations $f(t)$. The probability density $P(x,t)$ for $X(t)$ to lie in the interval $(x,x+dx)$ evolves in the phase space of the system. If the fluctuations are δ correlated ($\tau_c\to 0$), then the evolution of $P(x,t)$ is described by the Fokker-Planck equation[17]

$$\frac{\partial}{\partial t}P(x,t)=-\frac{\partial}{\partial x}G(x)P(x,t) +D_c\frac{\partial}{\partial x}g(x)\frac{\partial}{\partial x}g(x)P(x,t) . \tag{2.3}$$

For a nonvanishing correlation time, (2.3) no longer describes the evolution. The form of the exact evolution equation for $P(x,t)$ depends on the formalism that one chooses. In the functional-calculus approach the evolution is expressed as an average over an ensemble of the fluctuations $f(t)$,[5–7,18]

$$\frac{\partial}{\partial t}P(x,t)=-\frac{\partial}{\partial x}G(x)P(x,t) +\frac{\partial}{\partial x}g(x)\frac{\partial}{\partial x}\int_0^t d\tau\frac{D_c}{\tau_c}e^{-|t-\tau|/\tau_c}\left\langle\delta[x-X(t)]\exp\left[\int_\tau^t du[G'(X(u))+g'(X(u))f(u)]\right]g(X(\tau))\right\rangle . \tag{2.4}$$

Similarly complicated exact formal expressions have been given within the other formalisms mentioned earlier.

To obtain a useful description of the evolution, i.e., calculationally tractable, methods have been introduced to systematically proceed from (2.4) and its counterparts to an EFPE of the form

$$\frac{\partial}{\partial t}P(x,t)=-\frac{\partial}{\partial x}G(x)P(x,t) +\frac{\partial}{\partial x}g(x)\frac{\partial}{\partial x}g(x)D(x)P(x,t) , \tag{2.5}$$

where the forms of the effective diffusion function $D(x)$ distinguish one procedure from another. All of these techniques yield a diffusion function whose explicit dependence on D_c is linear (this discussion excludes decoupling procedures that introduce self-consistency requirements[11]). Differences among the approximate forms arise via the treatment of the correlation time (τ_c) dependence of $D(x)$. In particular, Sancho and co-workers[6,7] use the Furutsu-Novikov method to do a τ_c expansion of (2.3) in terms of functional derivatives to obtain

$$D(x)=D_c\frac{G(x)}{g(x)}\left[1+\tau_c G(x)\frac{d}{dx}\right]^{-1}\frac{g(x)}{G(x)} \tag{2.6}$$

[note that $D(x)$ is a function and not an operator]. The form (2.5) has also been obtained from the cumulant-expansion method by Lindenberg *et al.*[8,9] and from the projection-operator approach by Grigolini and co-workers.[12–14] It arises from the full retention of the second-cumulant contribution and the neglect of higher cumulants. Equation (2.4) with (2.5) has been called the BFPE.

Starting from a path-integral definition of the probability density, Fox[15,16] uses the functional calculus to express the exact evolution equation for exponentially correlated fluctuations [cf. (2.2)] as follows:

$$\frac{\partial}{\partial t}P(x,t)=-\frac{\partial}{\partial x}G(x)P(x,t) +\frac{\partial}{\partial x}g(x)\frac{\partial}{\partial x}\int_0^t d\tau\frac{D_c}{\tau_c}e^{-|t-\tau|/\tau_c}\int \underline{D}f\,\rho(f)\delta[x-X(t)]\exp\left[\int_\tau^t du[G'(X(u))+g'(X(u))f(u)]\right]g(X(\tau)). \tag{2.7}$$

Here $\int \underline{D}f$ denotes a functional path integral over the fluctuations whose distribution is specified by $\rho(f)$ (here taken to be Gaussian). Note that (2.4) and (2.7) are identical if one identifies the angle brackets $\langle\ \rangle$ in the former with the path integral in the latter. Fox proceeds to obtain an EFPE to approximate (2.7) in two steps. First he uses the equation of motion (2.1) to exactly eliminate the explicit f-dependence in the exponent of (2.7),

$$\frac{d}{dt}g(X(t))=g'(X(t))\dot{X}(t)=g'(X(t))[G(X(t))+g(X(t))f(t)] , \tag{2.8}$$

so that upon formal integration

$$g(X(\tau))=g(X(t))\exp\left[\int_t^\tau du\frac{g'(X(u))}{g(X(u))}[G(X(u))+g(X(u))f(u)]\right] . \tag{2.9}$$

Substitution of (2.9) into (2.7) then yields the still exact expression

$$\frac{\partial}{\partial t}P(x,t)=-\frac{\partial}{\partial x}G(x)P(x,t)$$
$$+\frac{\partial}{\partial x}g(x)\frac{\partial}{\partial x}\frac{D_c}{\tau_c}\int_0^t d\tau e^{-|t-\tau|/t_c}\int \underline{D}f\rho(f)\delta[x-X(t)]$$
$$\times\exp\left[\int_\tau^t du\, G'(X(u))-\frac{g'(X(u))}{g(X(u))}G(X(u))\right]g(X(t)) . \tag{2.10}$$

The difficulty with this formal expression is the actual implementation of the path integral due to the implicit dependence of the trajectory $X(u)$ on the fluctuations. Fox's approximation is to eliminate this dependence by replacing the integrand of the exponent with its value at time t. The justification for this replacement is the rapid decay of the correlation function for small values of τ_c. This is therefore a short-correlation-time approximation,

$$\int_\tau^t du\left[G'(X(u))-\frac{g'(X(u))}{g(X(u))}G(X(u))\right]$$
$$\approx(t-\tau)\left[G'(X(t))-\frac{g'(X(t))}{g(X(t))}G(X(t))\right] . \tag{2.11}$$

Since the δ function in (2.10) sets $X(t)=x$, the exponent (2.11) is then independent of the fluctuations and the path integral can be trivially performed to yield $P(x,t)$. The remaining integrals are easy to perform and lead to the diffusion function

$$D_F(x)=\frac{D_c}{1-\tau_c\left[G'(x)-\frac{g'(x)}{g(x)}G(x)\right]} . \tag{2.12}$$

Fox argues for the relative superiority of this result by noting that the time integration in (2.7) is valid *uniformly* in x provided that $D_F^{-1}(x)$ is positive for all x. This places a constraint on the correlation time τ_c.

We wish to show that the implementation of the same approximation to (2.10) as was made using the other techniques mentioned earlier leads to the BFPE diffusion function (2.6) rather than the EFPE function (2.12). Thus the differences arise *not* from the different techniques but from the different approximations made within the techniques. Let us return to Eq. (2.10) with the aim of retaining all contributions of $O(D_c)$ to the integral term. There is one explicit factor D_c in front of the integral, so that our goal is to evaluate the integral to zeroth order in D_c. In practice this is achieved by considering only the *deterministic* contribution to the evolution $X(u)$, i.e., $X(u)$ in the exponent is replaced by the solution of the deterministic equation

$$\dot{X}_d(t)=G[X_d(t)] , \tag{2.13}$$

with $X_d(0)=x_0$. Since $X_d(u)$ is independent of the fluctuations, the path integral in (2.10) can be carried out explicitly. We carry out the integration in the exponent with the replacement of $X(u)$ with $X_d(u)$ and the observation that for any function $F[X(u)]$

$$F'[X_d(u)]=\frac{1}{G[X_d(u)]}\frac{d}{du}F[X_d(u)] , \tag{2.14}$$

where we have used (2.13). The integral in the exponent then becomes

$$\int_\tau^t du\frac{d}{du}\ln\left|\frac{G[X_d(u)]}{g[X_d(u)]}\right|=\ln|\hat{h}(x,t-\tau)| , \tag{2.15}$$

where

$$\hat{h}(x,t-\tau)\equiv\frac{G(x)}{g(x)}\frac{g[X_d(\tau)]}{G[X_d(\tau)]} \tag{2.16}$$

and where the explicit τ dependence of $\hat{h}(x,t-\tau)$ arises from the fact that the deterministic solution $X_d(\tau)$ is subject to the end-point condition $X_d(t)=x$. We can now perform the path integration

$$P_t\equiv P(x,t)=\int \underline{D}\eta\rho(\eta)\delta[x-X_\eta(t)] \tag{2.17}$$

to obtain

$$\frac{\partial}{\partial t}P_t=L_0P_t+\frac{\partial}{\partial x}g(x)\frac{\partial}{\partial x}g(x)$$
$$\times\int_0^t d\tau Q(t-\tau)|\hat{h}(x,t-\tau)|P_t . \tag{2.18}$$

Thus one can define the time-dependent diffusion function

$$D(x,t)=\int_0^t d\tau Q(\tau)h(x,\tau) , \tag{2.19}$$

where

$$h(x,\tau)\equiv|\hat{h}(x,\tau)| \ . \tag{2.20}$$

We show below that $h(x,\tau)$ is not an explicit function of t because of the end-point condition $X_d(t)=x$. To make contact with the diffusion function obtained earlier, we again consider the limit $t\gg\tau_c$ and thus replace the diffusion function by its long-time limit

$$D(x)=\lim_{t\to\infty} D(x,t)=\int_0^\infty d\tau\, Q(\tau)h(x,\tau) \ . \tag{2.21}$$

The functional-calculus method has thus led at $O(D_c)$ to the EFPE (2.5) with the diffusion function (2.21).

To relate the result (2.21) to the BFPE diffusion function given in (2.6), we note that the latter function can be written as[8–10]

$$D(x)=\int_0^\infty d\tau\, Q(\tau)H(x,\tau) \ , \tag{2.22}$$

where $H(x,\tau)$ satisfies the partial differential equation

$$\begin{aligned}\frac{\partial}{\partial\tau}H(x,\tau)&=G'(x)H(x,\tau)-G(x)\frac{\partial}{\partial x}H(x,\tau)\\&\quad-G(x)H(x,\tau)\frac{g'(x)}{g(x)}\\&\equiv\hat{O}H(x,\tau) \ .\end{aligned} \tag{2.23}$$

The initial condition for (2.23) is $H(x,0)=1$.[10] Comparison of (2.21) and (2.23) indicates that equality of the two diffusion functions is obtained if $h(x,\tau)=H(x,\tau)$. The equality is trivially satisfied at $\tau=0$. This equality can of course only be satisfied in the x regime for which $H(x,\tau)\geq 0$. In the nonphysical regime $H(x,\tau)<0$ (discussed in Sec. III) the equality cannot hold because $h(x,\tau)$ is an absolute value. In this range, equality (were it desirable) could be achieved by analytic continuation of the integral in (2.15), i.e., by setting $H(x,\tau)=\hat{h}(x,\tau)$. It remains to be shown that $h(x,\tau)$ satisfies the differential equation (2.23). Using (2.20) we have, for $\hat{h}(x,t-\tau)>0$,

$$\begin{aligned}\frac{\partial}{\partial t}h(x,\tau)&=\frac{G(x)}{g(x)}\left[\frac{g_d}{G_d}\right]'\frac{d}{d\tau}X_d(t-\tau)\\&=-\frac{G(x)}{g(x)}\left[\frac{g_d}{G_d}\right]'G_d \ ,\end{aligned} \tag{2.24}$$

where $g_d\equiv g[X_d(t-\tau)]$, similarly for G_d, and where we have used (2.13) to obtain the last equality. For the right-hand side of the differential equation (2.23) we obtain [again noting the x dependence of $X_d(t-\tau)$ through the end-point condition]

$$\begin{aligned}\hat{O}h(x,\tau)&=G'(x)\frac{G(x)}{g(x)}\frac{g_d}{G_d}-\frac{G^2(x)}{g(x)}\frac{g_d}{G_d}\frac{g'(x)}{g(x)}\\&\quad-G(x)\left[\frac{G(x)}{g(x)}\right]'\frac{g_d}{G_d}\\&\quad-\frac{G^2(x)}{g(x)}\left[\frac{g_d}{G_d}\right]'\frac{dX_d(t-\tau)}{dx} \ .\end{aligned} \tag{2.25}$$

Integration of the dynamical equation (2.13) gives

$$\tau=\int_{X_d(t-\tau)}^{x} dX_d G(X_d) \ , \tag{2.26}$$

and taking the derivative of this solution with respect to x immediately yields

$$\frac{d}{dx}X_d(t-\tau)=\frac{G_d}{G(x)} \ . \tag{2.27}$$

Substitution of (2.27) and (2.25) then leads to the equality of (2.25) and (2.24), i.e.,

$$\frac{\partial}{\partial t}h(x,\tau)=\hat{O}h(x,\tau) \ . \tag{2.28}$$

The identity of $h(x,\tau)$ in (2.20) with the function $H(x,\tau)$ that appears in the BFPE [and the consequent explicit t independence of $h(x,\tau)$] is clear. We have thus obtained the BFPE using the *functional path integral technique.*

III. EFPE's: LIMITATIONS AND ADVANTAGES

The diffusion function $D(x)$ of the BFPE is not necessarily positive definite. There are a number of examples for which $D(x)\to 0$ for some finite value of x_s of x. Beyond this value the sign of $D(x)$ is ambiguous and depends on the technique used [if $D(x)$ is analytic, then beyond this value it can be negative]. It is important to note that typically $x_s=O(\tau_c^{-\alpha})$, where $\alpha>0$, and hence $x_s\to\infty$ as $\tau_c\to 0$. If one blindly applies the BFPE theory to values of $x>x_s$, then one often finds a nonphysical buildup of probability that is an artifact of the method. To circumvent this artifact it is customary to restrict the region of support of the distribution to $x<x_s$. It is useful to note that the EFPE (2.5) implies that the dynamical system (2.1) with colored noise can be approximated by an "equivalent" dynamical system driven by Gaussian white multiplicative fluctuations,

$$\dot{X}=G(X)-g^2(X)[D^{1/2}(X)]'+\left[\frac{D(X)}{D_c}\right]^{1/2}g(X)f^W(t) \ , \tag{3.1}$$

where

$$\langle f^W(t)f^W(\tau)\rangle=2D_c\delta(t-\tau) \ . \tag{3.2}$$

The effective "restoring force" $G-g^2(D^{1/2})'$ is negative provided x is below x_s by an amount of $O(D_c\tau_c^\beta)$ with $\beta>0$.

The appearance of a finite limit to the region of support when $D(x)\to 0$ for a finite value of x has elicited a certain amount of criticism of the BFPE. This has prompted a number of investigators to seek EFPE's that do not exhibit this behavior,[6,7,11,15,16] i.e., whose diffusion functions are positive definite for all x. We emphasize that the quality of EFPE's depends on the physical property of interest. In particular, those properties that do not specifically involve P_t for extremely large x (e.g., for $x\geq x_s$) may be well represented by any of the EFPE's. For instance, in Fig. 1 we show three steady-state distri-

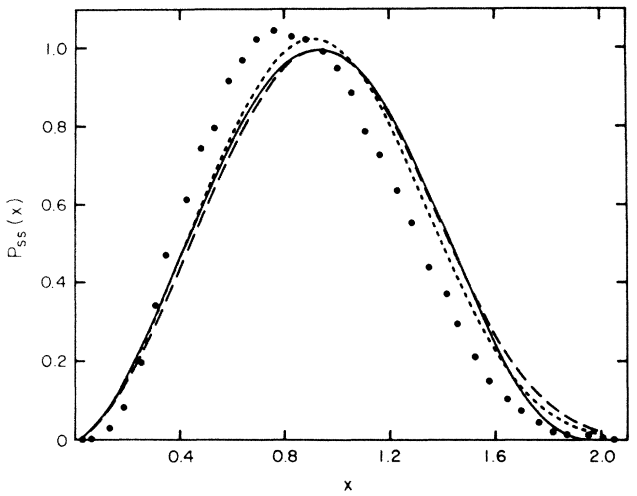

FIG. 1. Steady-state distributions for the system $\dot{X}=a(X-X^3)+Xf(t)$ with $\alpha\tau_c=0.167$ and $D_c\tau_c=0.0635$. Solid curve, BFPE; dashed curve, exponentiation of Sancho *et al.* (Ref. 7); dotted curve, EFPE of Fox (Ref. 16); points, analog simulation of Sancho *et al.* (Ref. 6).

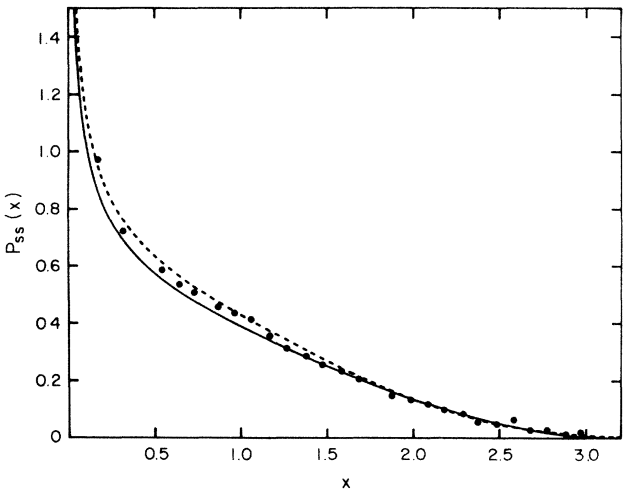

FIG. 2. Steady-state distributions for the system $\dot{X}=a(X-X^3)+Xf(t)$ with $\alpha\tau_c=\frac{1}{18}$ and $D_c\tau_c=\frac{1}{12}$. Solid curve, BFPE; dashed curve, Sancho *et al.*'s exponentiation (Ref. 7) and Fox's EFPE (Ref. 16) (indistinguishable); points, digital simulation of Sancho *et al.* (Ref. 7).

butions $P_{SS}(x)$ for the system $G(X)=a(X-X^3)$ and $g(X)=X$. The solid curve is the steady-state solution of the BFPE and has support in the region (0,2), while the other curves are solutions of related EFPE's that have infinite support. One of these is the result of Fox's EFPE,[16] while the other is obtained from the exponentiation procedure of Sancho *et al.*[7] The data points are the analog simulations of Sancho *et al.*[6] (only qualitative accuracy is claimed for these simulations). Any property dependent on $P_{SS}(x)$ for $0\leq x\leq 2$ will not vary significantly with the choice of distributions. In Fig. 2 the steady-state distributions for the same system and results of a digital simulation[7] are shown for a different set of parameter values (these data are accurate to within the size of the points). Clearly, the differences among the theories are no larger than those between any one theory and the data. The additional success of the BFPE or EFPE approach in predicting experimental results has recently been detailed in the context of a particular bistable system.[10]

IV. CONCLUSION

This paper has been motivated by the different diffusion functions that have been proposed in effective Fokker-Planck equations used to model systems driven by colored fluctuations. We have shown that the different forms do not arise as a result of the differences in techniques but rather as a consequence of different approximation schemes implemented within these techniques. It is of course true that different methods suggest different approximations at various stages; however, one should focus on the approximations and not the techniques as the source of the differences. In fact, we demonstrate that the same approximation within different methods leads to the same EFPE. This point has already been recognized by Grigolini *et al.*[12–14] who discussed the different EFPE's within the context of one technique (projection operators). Grigolini *et al.* also recognized that the entire Fokker-Planck structure (2.5) for any value of D_c and/or $\tau_c>0$ may break down because the contributions of neglected terms may not be negligible. They show that these terms yield corrections to $D(x)$ of $O(D_c\tau_c^2)$ for Gaussian noise and of $O(D_c\tau_c)$ for non-Gaussian noise. This breakdown often influences the asymptotic region of low-probability more than the high probability regions. Grigolini *et al.* do point out one important result that should be recognized: the EFPE equation of Fox provides an *exact* equilibrium distribution in the limit of very long correlation times $\tau_c\rightarrow\infty$; the BFPE does not share this virtue.

Finally, we note that a big motivating factor for the pursuit of the EFPE structure (2.5) is that until recently[19,20] no methods other than the Fokker-Planck equation route were available to relate a dynamical stochastic equation such as (2.1) to first-passage time properties and other physical observables. Thus extension of the evolution equation for the probability P_t beyond the Sturm-Liouville form (2.5) does in itself not provide a formalism for the determination of such observables. In the absence of the EFPE structure different approaches to the first-passage time problem must be developed.

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