Nonsymmetric four-dimensional volume-preserving maps: Universality classes of period doubling

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A *nonsymmetric* four-dimensional volume-preserving map (i.e., two nonsymmetrically-coupled two-dimensional area-preserving Hénon maps) is numerically studied. In this map it is found that the universality classes of period doubling are identical to those previously found in the *symmetric* four-dimensional volume-preserving maps, reported by J. M. Mao and R. H. G. Helleman [Phys. Rev. A **35**, 1847 (1987)]. This is the first time a global picture of period doubling in more general, nonsymmetric, four-dimensional volume-preserving maps has been given. Also, a simple analytic renormalization is presented for the new Feigenbaum constants.

I. INTRODUCTION

Four-dimensional (4D) volume-preserving mappings arise in many physical problems, including orbital stability of particles in storage rings of accelerators,[1,2] restricted three-body problems,[3,4] and other nonintegrable Hamiltonian systems. A large class of 4D volume-preserving maps is obtained by coupling two two-dimensional (2D) Hénon maps,

$$\begin{aligned} x' &= -y + f(x,u)\ , \\ y' &= x\ , \\ u' &= -v + g(x,u)\ , \\ v' &= u\ , \end{aligned} \tag{1}$$

where $f(x,u)$ and $g(x,u)$ are nonlinear functions of x and u with two parameters. In most of this article, f and g are quadratic polynomials, i.e., the generic ("typical") map, near a fixed point or periodic orbit. The map (1) is called symmetric if $f(x,u)=g(u,x)$ and called nonsymmetric otherwise.

We assume the reader has some familiarity with period doubling in conservative systems, cf. Refs. 5–7. For period doubling in the *symmetric* 4D volume-preserving maps, there exist three universality classes called the L, U, and E classes, see Ref. 8 for numerical work and Ref. 9 for renormalization calculations. We calculate pairs of parameter values (a 4D map has at least two parameters) at which period doublings occur. The rates δ_1 and δ_2 (i.e., the eigenvalues of a 2×2 matrix) at which those pairs converge are called "Feigenbaum constants." Each universality class is characterized by its own two Feigenbaum constants δ_1 and δ_2: $\delta_2=4,\ -2,\ -4.404\ldots$, respectively for the L, U, E classes (the δ_1 is, in all cases, the same as for 2D area-preserving maps, $\delta_1=8.721\ldots$). Bifurcation paths in the parameter plane, belonging to these universality classes, are called L, U, and E paths, respectively. In the numerical work, the evidence for the existence of the three classes is that infinities of different L (U, E) paths have the same parameter convergence rate $\delta_2=4$ (-2, $4.404\ldots$). There are a few exceptional L and U paths with $\delta_2=-15.1\ldots$. (The δ_2 for the exceptional E path does not exist.) In the renormalization calculation for the δ's, fixed points have indeed been found, cf. Ref. 9. The divergence rates from these fixed points under changes in parameters are $\delta_2=4$, -2, and $-4.4\ldots$, respectively ($\delta_1=8.721\ldots$ in all cases). Hence both the numerical and renormalization work indicate that there exist three universality classes (the L, U, and E classes) of period doubling in symmetric 4D volume-preserving maps.

For *nonsymmetric* 4D volume-preserving maps, however, a period-doubling sequence had previously[7] been found only for the exceptional L path ($\delta_2=-15.1\ldots$), even though the (numerically implemented) renormalization calculations of Ref. 9 suggest that the same L, U, and E universality classes exist also for the nonsymmetric case.

In this article we study period doubling numerically in nonsymmetric 4D volume-preserving maps. All paths we checked (including the exceptional ones) belonged, again, to the L, U, or E class. In the Appendix to this article, we give a derivation of $\delta_2=4$, -2 via a simple analytic renormalization. It therefore appears that period doubling in 4D volume-preserving maps of the form of Eq. (1) always has three universality classes, no matter whether the map is symmetric or nonsymmetric.

II. NONSYMMETRIC 4D VOLUME-PRESERVING MAPS

The nonsymmetric 4D volume-preserving map we numerically study is

$$\begin{aligned} x' &= -y + f(x)\ , \\ y' &= x\ , \\ u' &= -v + g(x,u)\ , \\ v' &= u\ , \end{aligned} \tag{2}$$

where f and g are nonlinear functions which will be chosen later in Eq. (12). Note that the nonlinear function $f(x)$ is a function of x only, and that the x' and y' equations are simply those of the 2D area-preserving map. This 4D map has a "dominant" symmetry surface[5,7,10]

$$y=(\tfrac{1}{2})f(x)\,,\qquad v=(\tfrac{1}{2})g(x,u)\,. \tag{3}$$

The Jacobian matrix of our 4D map has the form

$$\underline{J}=\begin{bmatrix}\underline{J}_1 & \underline{0}\\ \underline{J}_3 & \underline{J}_2\end{bmatrix}, \tag{4}$$

where $\underline{J}_1$, $\underline{J}_2$, and $\underline{J}_3$ are 2×2 matrices and $\underline{0}$ a 2×2 null matrix. A product of two 4×4 matrices in this form has the same form again. Thus the Jacobian matrix of T^N (composing T with itself N times) can also be written in the form of Eq. (4). Any 4×4 matrix has four invariants,

$$\begin{aligned}T_1&\equiv\lambda_1+\lambda_2+\lambda_3+\lambda_4\,,\\ T_2&\equiv\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3\lambda_4\,,\\ T_3&\equiv\lambda_1\lambda_2\lambda_3+\lambda_1\lambda_2\lambda_4+\lambda_1\lambda_3\lambda_4+\lambda_2\lambda_3\lambda_4\,,\\ T_4&\equiv\lambda_1\lambda_2\lambda_3\lambda_4\,,\end{aligned} \tag{5}$$

where λ_1, λ_2, λ_3, and λ_4 are eigenvalues of the matrix $\underline{J}$. For our map, $T_4=1$, and $\mathrm{Det}\underline{J}_1=\mathrm{Det}\underline{J}_2=1$. Hence, after some trivial algebra,

$$T_1=\mathrm{Tr}\underline{J}_1+\mathrm{Tr}\underline{J}_2,\quad T_2=(\mathrm{Tr}\underline{J}_1)(\mathrm{Tr}\underline{J}_2)+2\,,\qquad T_3=T_1,\quad T_4=1\,. \tag{6}$$

The characteristic equation (for the eigenvalues λ) of the Jacobian matrix can therefore be reduced to the quadratic equation,

$$\rho^2-(\mathrm{Tr}\underline{J}_1+\mathrm{Tr}\underline{J}_2)\rho+(\mathrm{Tr}\underline{J}_1)(\mathrm{Tr}\underline{J}_2)=0\,, \tag{7}$$

where

$$\rho\equiv\lambda+1/\lambda \tag{8}$$

is called the stability index.[11] The solutions of Eq. (7) for ρ are

$$\rho_1=\mathrm{Tr}\underline{J}_1,\quad \rho_2=\mathrm{Tr}\underline{J}_2\,. \tag{9}$$

Furthermore, the eigenvalues of $\underline{J}_i$, $i=1,2$, are a reciprocal pair $(\lambda_i,1/\lambda_i)$ since $\mathrm{Det}\underline{J}_i=1$. Thus the eigenvalue configuration of $\underline{J}$ is

$$\lambda_1,1/\lambda_1,\lambda_2,1/\lambda_2\,. \tag{10}$$

Hence a periodic orbit is stable if each of its stability indices (ρ_1 and ρ_2) has absolute value less than or equal to 2. Period-doubling bifurcation occurs when one of the stability indices (to be called ρ_1 in our case) is equal to -2, i.e.,

$$\rho_1=-2,\quad |\rho_2|\le 2\,. \tag{11}$$

III. NUMERICAL RESULTS

In our numerical study, the nonlinear functions $f(x)$ and $g(x,u)$ used in the 4D map (2) are

$$f(x)=2(Cx+x^2)\,,\qquad g(x)=2(Cu+u^2-Exu)\,, \tag{12}$$

where C and E are parameters [we need at least two parameters because there are two independent invariants T_1 and T_2, cf. Eq. (6)]. The period-doubling bifurcation values of C (and the initial values of x and y) for $\rho_1=-2$ can be determined from just two of the mapping equations [i.e., the x' and y' equations of Eq. (2)]. These values are the same as in the 2D area-preserving map.[6] We denote them by C_n, x_n, and y_n for the nth bifurcation, i.e., from period 2^n to period 2^{n+1}. Also note that $y_n=(\frac{1}{2})f(x_n)$, i.e., the initial orbital point (x_n,y_n) is on the symmetry surface, cf. Eq. (3). In the CE parameter plane, the stable regions of period 2^n are quadrangles enclosed by the following four curves

$$C=C_n,\quad C=C_{n-1}\,,\qquad \rho_2(C,E)=2,\quad \rho_2(C,E)=-2\,. \tag{13}$$

The nth bifurcation occurs on the line $C=C_n$ (where $\rho_1=-2$). A point on this line can be specified by the value of ρ_2. *Self-similar* period-doubling bifurcations successively occur at those points where ρ_2 always has the same value as $n\to\infty$. A sequence of such points (C_n,E_n) in the parameter plane forms a "bifurcation path." At the nth order we define a 2×2 matrix $\underline{D}_n$ by

$$\begin{bmatrix}C_{n-1}-C_{n-2}\\ E_{n-1}-E_{n-2}\end{bmatrix}=\underline{D}_n\begin{bmatrix}C_n-C_{n-1}\\ E_n-E_{n-1}\end{bmatrix}, \tag{14}$$

and calculate its two eigenvalues. In the limit as $n\to\infty$, these eigenvalues are called the Feigenbaum constants δ_1,δ_2 of this path. We have numerically determined a large number of bifurcation paths, each corresponding to one definite value of ρ_2, for our nonsymmetric map (2) with the nonlinear functions of Eq. (12). We found that all these bifurcation paths can be classified into just three classes (L, U, and E classes), each characterized by its own Feigenbaum constants δ_1 and δ_2. All bifurcation paths (with $-2\le\rho_2<2$) in the L class have $\delta_2=4$. The (few exceptional) paths with $\rho_2=2$ in the L class, however, have an exceptional δ_2: $\delta_2=-15.1\ldots$, see Table I. For the U class, we obtain $\delta_2=-2$ for all regular U paths [i.e., $-2\le\rho_2\le2$ ($\rho_2\neq-1$)], but again $\delta_2=-15.1\ldots$ for the exceptional U paths (at $\rho_2=-1$). For the regular E paths (all E paths have $E_\infty=0$) with $-2<\rho_2\le2$, we found $\delta_2=-4.404\ldots$. For the exceptional E path (at $\rho_2=-2$), however, δ_2 cannot be determined because the path coincides with one of the axes $E=0$. Hence the scaling matrices $\underline{D}_n$ are just scalars. In all cases, of course, $\delta_1=8.721\ldots$, the well-known Feigenbaum constant in two-dimensional area-preserving maps. These L, U, and E classes are identical to the three universality classes for the symmetric 4D volume-preserving maps, reported in Ref. 8.

In order to determine the orbital scaling factors, we introduce new coordinates

TABLE I. Numerical results: Feigenbaum constants δ_1 and δ_2 for the L, U, and E universality classes.

Class	Path	δ_1	δ_2
L	Regular ($-2 \le \rho_2 < 2$)	8.721 096. . .	4.000. . .
	Exceptional ($\rho_2 = 2$)		-15.1. . .
U	Regular ($-2 \le \rho_2 \le 2$, $\rho \neq -1$)	8.721 096. . .	-2.000. . .
	Exceptional ($\rho_2 = -1$)		-15.1. . .
E	Regular ($-2 < \rho_2 \le 2$, $E_\infty = 0$)	8.721 096. . .	-4.404. . .
	Exceptional ($\rho_2 = -2$, $E_n = 0$)		(does not exist)

$$X \equiv x, \quad Y \equiv y - (\tfrac{1}{2}) f(x) ,$$
$$U \equiv u, \quad V \equiv v - (\tfrac{1}{2}) g(x,u) . \tag{15}$$

Equation (15) is called a DeVogelaere transformation.[10] Defining (X_n, Y_n, U_n, V_n) as the coordinates of the orbit point of the 2^n cycle on the symmetry surface $Y = V = 0$, we introduce $X_n(\frac{1}{2})$, $Y_n(\frac{1}{2})$, $U_n(\frac{1}{2})$, $V_n(\frac{1}{2})$, and $X_n(\frac{1}{4})$, $Y_n(\frac{1}{4})$, $U_n(\frac{1}{4})$, $V_n(\frac{1}{4})$ as coordinates of the orbit points $(\frac{1}{2})2^n$, $(\frac{1}{4})2^n$ iterations thereafter. The orbital scaling factors are now defined as

$$\alpha_1 \equiv \lim_{n\to\infty} \frac{X_{n-1} - X_{n-1}(\frac{1}{2})}{X_n - X_n(\frac{1}{2})} ,$$
$$\beta_1 \equiv \lim_{n\to\infty} \frac{Y_{n-1}(\frac{1}{4})}{Y_n(\frac{1}{4})} ,$$
$$\alpha_2 \equiv \lim_{n\to\infty} \frac{U_{n-1} - U_{n-1}(\frac{1}{2})}{U_n - U_n(\frac{1}{2})} ,$$
$$\beta_2 \equiv \lim_{n\to\infty} \frac{V_{n-1}(\frac{1}{4})}{V_n(\frac{1}{4})} . \tag{16}$$

It is well known that $\alpha_1 = -4.018\,076\,704\ldots$ and $\beta_1 = 16.363\,896\,879\ldots$, the orbital scaling factors in two-dimensional area-preserving maps.[5,6] For our 4D map we found that in all cases (i.e., for all the regular and exceptional paths in all L, U, and E classes),

$$\alpha_2 = \alpha_1, \quad \beta_2 = \beta_1 . \tag{17}$$

This is, again, the same result as in the symmetric case.

IV. CONCLUSIONS

One conclusion is that the three universality classes of period doubling we found for the symmetric 4D volume-preserving maps are also valid for the nonsymmetric 4D volume-preserving maps. This was already suggested by the renormalization calculations of Ref. 9. There, three maps, which are fixed under the renormalization operator, have been determined to quadratic terms for nonsymmetric as well as symmetric 4D volume-preserving maps. The relevant eigenvalues, under linearization about these fixed maps, are $\delta_2 = 4$, -2, $-4.4\ldots$, respectively ($\delta_1 = 8.721\ldots$ in all cases), agreeing with the numerical results reported here. A simpler, analytical derivation of $\delta_2 = 4$, -2 is given in the Appendix. Finally, we note that the nonsymmetric 4D volume-preserving map of Eq. (2), numerically studied here, is not yet the most general 4D volume-preserving map since the x' and y' equations are just the regular 2D Hénon map.

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APPENDIX: DERIVATION OF $\delta_2 = -2$ AND 4

In this appendix we derive $\delta_2 = -2$ and 4 from an extremely simple renormalization procedure. As is well known, period-doubling bifurcations in 4D volume-preserving maps occur when the Jacobian matrix of map evaluated at the periodic orbit has the following eigenvalues;

$$-1, -1, e^{i\theta}, e^{-i\theta} . \tag{A1}$$

The sum of the second pair of eigenvalues, i.e., the second stability index defined by Eq. (8), is

$$\rho = 2\cos\theta . \tag{A2}$$

When the period doubles, the eigenvalues of new periodic orbit are

$$1, 1, e^{i2\theta}, e^{-i2\theta} , \tag{A3}$$

and the second stability index becomes

$$\rho' = 2\cos(2\theta) . \tag{A4}$$

Eliminating θ between Eqs. (A2) and (A4), we get a recursion relation for ρ:

$$\rho' = \rho^2 - 2 . \tag{A5}$$

Fixed points of Eq. (A5), ρ_∞, are solutions of this equation with $\rho'=\rho=\rho_\infty$. The resulting two roots of the quadratic equation are

$$\rho_\infty=2,-1\ , \tag{A6}$$

i.e.,

$$\theta_\infty=0,\pm 2\pi/3\ . \tag{A7}$$

Note that the pair of eigenvalues $(e^{i\theta},e^{-i\theta})$, with either $\theta=0$ or $\theta=\pm 2\pi/3$, remains the same after period doubling. Hence, the Feigenbaum constants, i.e., the corresponding rates of divergence away from the fixed points ρ_∞, are, respectively,

$$\delta_2=\left.\frac{\partial\rho'}{\partial\rho}\right|_{\rho_\infty}=2\rho_\infty=4,-2\ . \tag{A8}$$

These values are identical to the numerical δ_2 results for the regular L and U paths, cf. Table I.

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