Power-law mass distribution of aggregation systems with injection

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We introduce a new family of aggregation models with constant interjection. In our models, the asymptotic distribution of particle mass, s, always follows a power law, $P(\geq s) \propto s^{-\alpha}$, where $\frac{1}{3} \leq \alpha \leq \frac{1}{2}$. It is clarified that this power law is realized by a balance of two effects, injection and aggregation.

I. INTRODUCTION

The aggregation phenomenon is one of the most typical irreversible processes, and has a long and steadily continuous history in statistical physics going back to the pioneering works of Smoluchowski.[1] In recent years, it has attracted much interest along with the extensive studies carried out on fractals.[2] Aided by powerful computers, our knowledge about aggregation is now increasing very rapidly.

We can find two interesting aspects of the aggregation phenomenon, considered from the viewpoint of fractals. One is the geometrical structure. The structures of diffusion-limited aggregates[3] and cluster-cluster aggregates[4] have been studied intensively and found to be well described by fractal geometry. The other is a scale-invariant statistical quantity, that is, the power-law size distribution. In the aggregation problem we often encounter a distribution which follows a power law. The best-known example is the distribution of cluster size at a sol-gel transition point.[5] The power-law size distributions are experimentally found not only at a critical point but also in many other aggregation systems, such as aerosols.[6] There may exist a wide class of aggregation systems which give the power law.

Recently, a general statement relating to this problem was proposed by one of the authors (H.Takayasu) and collaborators that "constant injection of small particles makes an aggregation system asymptotically follow a power-law mass distribution." This statement has been confirmed theoretically under a mean-field approximation[7] and numerically in a one-dimensional system of shock-wave aggregation.[8] Such a statement was first pointed out by White[9] who studied steady-state solutions to coagulation equations of the mean-field type.

In this paper, we introduce some random aggregation models with injection, and show that, as expected, all of them asymptotically approach statistical stationary states where the particle mass distributions obey power laws. In Sec. II, we introduce two basic models. The definition of the models and the numerical results are given in Sec. II A. Theoretical analyses of both models are performed in Secs. II B and II C. In Sec. III we discuss some generalization of the model. Section IV is devoted to a short summary.

II. BASIC MODELS

A. The basic models and results of the computer simulation

We consider simple models of random aggregation. For convenience, in computer and theoretical analyses, every quantity in our models is discretized. That is, particles with integer mass are placed on each site of the one-dimensional lattice, and they aggregate by random-walk processes with discrete time steps.

The time evolution of our models is defined by the following procedure: At the beginning of each time step, there are particles on every site of the lattice. All of them independently jump to randomly chosen sites according to a given probability. If two or more particles come together at a site after the jump, they are combined to form a new particle with a conserved mass. Then, a particle with unit mass is added to every site. Thus the evolution of one time step is completed and we repeat this process of aggregation without considering any fragmentation. As a result, there is nonzero integer mass on every site at every time.

The above dynamics can be represented by the following stochastic equation for $S_i(n)$, the mass of the particle on ith site at time n:

$$S_i(n+1)=\sum_j W_{ij}(n)S_j(n)+1 , \tag{1}$$

where $W_{ij}(n)$ is a random variable given by

$$W_{ij}(n)=\begin{cases}1, & \text{with probability } q(i-j)\\ 0, & \text{with probability } 1-q(i-j) .\end{cases} \tag{2}$$

Here $W_{ij}(n)$ denotes the realization that the particle on the jth site jumps to the ith site at time n and that

$q(i-j)$ is its probability of occurrence. Hence we need the following two constraints:

$$\sum_i q(i)=1 , \tag{3}$$

$$\sum_i W_{ij}(n)=1 . \tag{4}$$

The system is translationally invariant, and all the particles jump in the same way, independent of their masses. Note that we neglect the geometrical structure of the particles.

In this section we analyze the two simplest cases: case A,

$$q(i,j)=\begin{cases}\frac{1}{2}, & \text{for } i,j=0 \text{ or } 1\\ 0, & \text{otherwise ;}\end{cases}$$

case B,

$$q(i-j)=1/N ,$$

where N is the total number of sites. The boundary condition is periodic in both cases. We choose the initial condition as $S_i(0)$ equals 1 for all i.

In case A, particles jump, at farthest, to their nearest-neighbor sites as shown in Fig. 1. Figure 2 shows the corresponding trajectories in $(1+1)$-dimensional space-time. There, we can find interesting dendritic structures such as rivers. Actually, this model has been shown to be equivalent to Scheidegger's model of rivers.[10]

This model also has a profound relation to the "voter" model.[11] The votor model is a model composed of randomly interacting "frickle voters." Each votor has a favorite candidate but he easily changes his mind if he is told by a neighbor that another candidate is better. If we place all the voters in a line, the propagation of an opinion can be illustrated by a diagram exactly identical to Fig. 2 with reversed time axis.

Another obvious connection can be found with directed percolation and stochastic cellular automata. Our model of case A is equivalent to the special case of Domany and Kinzel's model[12] with $P(1\,|\,0,0)=0$, $P(1|1,1)=1$, and $P(1|0,1)=\frac{1}{2}$, where

$$P(v_{i,t+1}\,|\,v_{i-1,t},v_{i+1,t})$$

denotes the conditional probability which governs the time evolution. This case is just on the phase transition point and the model is known to show some critical behaviors.

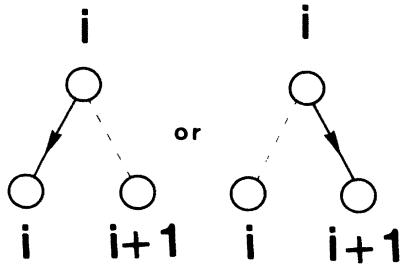

FIG. 1. In case A, the particle at the ith site stays at the same site or jumps to the $(i+1)$th site with probability $\frac{1}{2}$.

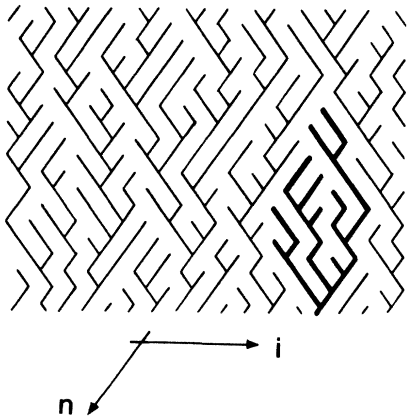

FIG. 2. An example of particle trajectories of case A. It is drawn in oblique coordinates in order to show that this model is equivalent to the model of rivers.

In case B, each particle jumps to any site with equal probability. So, contrary to case A, there is no spatial effect.

The existence of the injection of unit mass particles is critical for asymptotic behavior of the system. Without injection, the number of particles decreases with time and the density of particles approaches zero. Moreover, in $d<2$ dimensions, the system will finally reach the trivial state where only one particle possesses the total mass. The injection keeps the system nontrivial by filling up vacant sites. As is shown in the following, our system asymptotically approaches a statistically stationary state.

We analyze a few statistical properties of the systems at sufficiently large n. Especially, we focus our attention on the distribution of particle mass,

$$p(s,n)ds=\frac{1}{N}\sum_i \int_s^{s+ds} ds'\delta(s_i(n)-s') . \tag{5}$$

The results of computer simulation are shown in Fig. 3. Here we plot the cumulative distribution,

$$P(\geq s)\equiv \int_s^{\infty} p(s')ds' , \tag{6}$$

instead of $p(s)$. In both case A and case B, we obtain the following power-law asymptotic distribution independent of initial conditions;

$$P(\geq s)\propto s^{-\alpha} , \tag{7}$$

where the exponent is given by

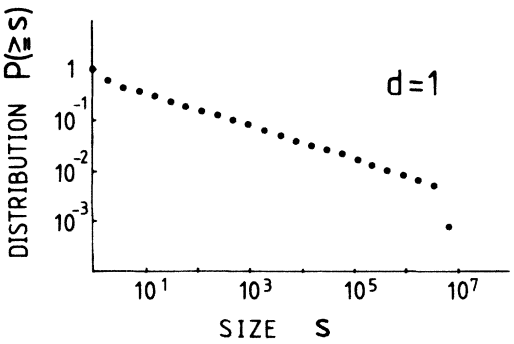

FIG. 3. The cumulative mass distribution of case A after sufficiently large time steps. It clearly obeys a power law. The cutoff size (about $s=10^7$ in this figure) tends to be shifted to infinity as the time step becomes larger and larger.

$$\alpha = \begin{cases} 0.331 \pm 0.006 & \text{(case A)} \quad (8a) \\ 0.499 \pm 0.009 & \text{(case B)} \; . \quad (8b) \end{cases}$$

In Secs. II B and II C, we are going to discuss these asymptotic forms of the mass distribution.

B. Theoretical analysis of case A

Instead of treating the distribution density $p(s)$ directly, we introduce its Laplace transform, that is, the characteristic function

$$Z(\rho,n) \equiv \langle e^{-\rho s} \rangle = \sum_{s=1}^{\infty} e^{-\rho s} p(s,n) \; . \quad (9)$$

The asymptotic behavior of $P(\geq s)$ for $s \gg 1$ corresponds to that of $Z(\rho)$ for $|\rho| \ll 1$. More precisely, if the distribution behaves as Eq. (7), we equivalently have

$$Z(\rho) \simeq 1 + c\,|\rho|^{\alpha} \quad (|\rho| \ll 1) \; , \quad (10)$$

where c is a constant.

Although we hope to solve the problem with this characteristic function only, the aggregation process makes it impossible. Consequently we need to introduce a set of r-body characteristic function $\{Z_r\}$ generally defined as

$$Z_r(\rho,n;i_1,i_2,\ldots,i_r) \equiv \langle e^{-\rho(s_{i_1}+s_{i_2}+\cdots+s_{i_r})} \rangle \; , \quad (11)$$

where $\langle \cdots \rangle$ denotes the average taken over the stochastic variables $\{W_{ij}(m);\ m=0,1,\ldots,n-1\}$. Exceptionally, in case A, owing to the short-range interaction, it is sufficient to consider the following r-body characteristic function $Z_r(\rho,n)$ for r-adjacent sites:

$$Z_r(\rho,n) \equiv \langle e^{-\rho(s_i+s_{i+1}+\cdots s_{i+r-1})} \rangle \; . \quad (12)$$

Substituting the evolution equation (1) into Eq. (12), we obtain the basic equation for $Z_r(\rho,n)$,

$$Z_r(\rho,n+1) \equiv \left\langle \prod_j \left[\sum_{k=0}^{r-1} q(i+k-j) e^{-\rho R_j(n)-\rho} + \left[1 - \sum_{k=0}^{r-1} q(i+k-j) \right] e^{-\rho} \right] \right\rangle$$
$$= e^{-\rho r} [Z_{r+1}(\rho,n) + 2Z_r(\rho,n) + Z_{r-1}(\rho,n)]/4 \; . \quad (13)$$

Furthermore, we can convert this time evolution equation into the following simple simultaneous equations for $r=1,2,3,\ldots$, by assuming that $Z_r(\rho,n)$ converges to $Z_r(\rho)$ in the limit $n \to \infty$:

$$Z_{r+1}(\rho) + (2 - 4e^{r\rho}) Z_r(\rho) + Z_{r-1}(\rho) = 0 \; , \quad (14)$$

with the boundary condition

$$Z_0 \equiv \langle 1 \rangle = 1 \; . \quad (15)$$

The solution $Z_1(\rho)$ is readily obtained in the form of the continued fraction,

$$Z_1(\rho) = \cfrac{1}{4e^{\rho} - 2 - \cfrac{1}{4e^{2\rho} - 2 - \cfrac{1}{4e^{3\rho} - 2 - \ddots}}} \; . \quad (16)$$

In particular for $|\rho| \ll 1$, this becomes

$$Z_r(\rho) = \cfrac{1}{2 + 4\rho - \cfrac{1}{2 + 8\rho - \cfrac{1}{2 + 12\rho - \ddots}}}$$
$$\simeq 1 + c\rho^{\alpha}, \quad \alpha = 0.333\cdots \; , \quad (17)$$

where c is a constant. Here the value of α is obtained by numerical calculation. Clearly, Eq. (17) is in good agreement with the result of direct simulation, Eq. (8).

We can make the following geometrical and intuitive explanation for this result. It should be noticed that the mass of a particle is given by the number of connected lattice sites which constitute a riverlike cluster in $(1+1)$-dimensional space-time. That is, the mass distribution in one-dimensional space is equivalent to the distribution of river size in $(1+1)$-dimensional space-time, while the river size is equal to the area size of the drainage basin surrounded by the left and right ridges (see Fig. 4). Since those ridges are trajectories of Markovian random walks, we have to estimate the size distribution of the area surrounded by two random-walk trajectories. Roughly speaking, the area is proportional to the product of its height (or vertical length) and width, and the width is expected to be proportional to the square root of the height. Hence, if we denote the area and its height by s and h, respectively, we get

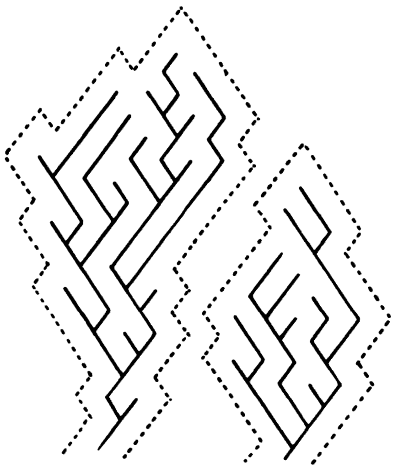

FIG. 4. The particle mass is equal to the area of the drainage basin of the corresponding river pattern surrounded by left and right ridges.

$$s \propto hh^{1/2}=h^{3/2} . \tag{18}$$

Since h can be regarded as the first collision time of two Brownian motions, its distribution $p(h)$ is approximately given by the well-known recurrence time distribution of Brownian motion in one dimension;[13]

$$p(h)\simeq(2\pi)^{-1/2}e^{-1/2h}h^{-3/2}\propto h^{-3/2} . \tag{19}$$

Combining Eqs. (18) and (19) we have

$$P(\geq s)\propto s^{(2/3)[-(3/2)+1]}=s^{-1/3} . \tag{20}$$

This result indicates that the exact value of α in case A is $\frac{1}{3}$.

Another approach to Eq. (13) is possible. Equation (13) can be viewed as a difference equation with discrete space r and time n. Therefore the continuation of Eq. (13) formally leads to the following partial differential equation for Z in the vicinity of $\rho=0$:

$$\frac{\partial Z}{\partial t}=\frac{1}{2}\frac{\partial^2 Z}{\partial x^2}-\rho x Z , \tag{21}$$

where the continuum variables t and x correspond to n and r, respectively. The first term in the right-hand side comes from the effect of aggregation by the nearest-neighbor random walk, while the second term comes from the uniform input. In a steady state these two terms are balanced to yield

$$\frac{1}{2\rho x}\frac{\partial^2}{\partial x^2}Z(\rho,x)=Z(\rho,x) . \tag{22}$$

By introducing a new variable $\xi=\rho^{1/3}x$, we can eliminate the explicit ρ dependence as

$$\frac{1}{2\xi}\frac{\partial^2}{\partial \xi^2}Z(\rho,x)=Z(\rho,x) , \tag{23}$$

which leads to the following scaling solution:

$$Z(\rho,x)=F(\rho^{1/3}x) . \tag{24}$$

Assuming that the solution F is an analytic function, we have the following expression:

$$Z(\rho,x)=c_0+c_1\rho^{1/3}x+c_2(\rho^{1/3}x)^2+\cdots . \tag{25}$$

Now returning to the discrete variable r with the boundary condition Eq. (15), we finally obtain

$$Z_1(\rho)=1+c_1\rho^{1/3}+\cdots . \tag{26}$$

This result also coincides with the previous discussions.

C. Theoretical analysis of case B

Since there is no spatial effect in case B, we can exactly make a mean-field analysis. Time evolution of the mass distribution function $p(s,n)$ is given as follows by considering all possible realizations of the aggregation:

$$p(s+1,n+1)=\sum_{r=1}^{N} a_r \sum_{s_1+s_2+\cdots+s_r=s} \prod_{i=1}^{r} p(s_i,n) , \tag{27}$$

with the boundary condition

$$p(1,n)=a_0 \quad (n\geq 0) , \tag{28}$$

where the coefficient a_r denotes the probability that r particles come together at a site, and it is given by

$$a_r={}_N C_r\left[\frac{1}{N}\right]^r\left[1-\frac{1}{N}\right]^{N-r} , \quad r=0,1,2,\ldots,N . \tag{29}$$

Here $\{a_r\}$ satisfies the following two constraints. One is the normalized probability conservation,

$$\sum_{i=0}^{N} a_i=1 . \tag{30}$$

The other is the particle number conservation held by the uniform and persistent input of unit particles,

$$\sum_{i=1}^{N} i a_i=1 . \tag{31}$$

By Laplace transformation, Eq. (27) becomes the equation for characteristic function

$$Z(\rho,n+1)=e^{-\rho}\sum_{r=0}^{N} a_r Z(\rho,n)^r . \tag{32}$$

If we assume the convergence to $Z(\rho)$ at the limit of $n\rightarrow\infty$, Eq. (32) is reduced to the following algebraic equation:

$$Z(\rho)=e^{-\rho}\sum_{r=0}^{N} a_r Z(\rho)^r . \tag{33}$$

This may be expanded in terms of a new function $y(\rho)=Z(\rho)-1$. In the vicinity of $\rho=0$,

$$\begin{aligned}1+y(\rho)&=\left[1-\rho+\frac{\rho^2}{2}-\cdots\right]\sum_{r=0}^{N} a_r[1+y(\rho)]^r \\ &=\sum_{r=0}^{N} a_r+\sum_{r=1}^{N} r a_r y+\sum_{r=2}^{N}\frac{r(r-1)}{2}a_r y^2+\cdots .\end{aligned} \tag{34}$$

Here, Eqs. (30) and (31) make the coefficients of the y^0 and y^1 terms in the right-hand side vanish, respectively, and thus we obtain

$$y(\rho)\propto\rho^{1/2} , \tag{35}$$

which implies that the exact value of the exponent α in Eq. (8) is

$$\alpha=\tfrac{1}{2} . \tag{36}$$

The preceding derivation indicates that if the conservation of particle number is violated, namely, if the coefficient of the y^1 term in Eq. (34) is not equal to zero, we have $y\propto\rho^1$; consequently, the mass distribution decays exponentially. Therefore, we know that the injection in our model is essential for the power-law mass distribution.

III. GENERALIZATION

In Sec. II, we proposed the aggregation model with injection and proved that the asymptotic distribution of the particle mass obeys the power law in the simple cases A and B. Here, we generalize our model of case A in two ways. One way is to extend the spatial dimension from 1 to a higher integer d, and the other is to enlarge the range of jumps. In each generalization, the model contains one new parameter, and the original model of case B is naturally included as an extreme case. The aim for considering such a generalization is to elucidate the universality of the results in cases A and B. In Secs. III A–III C we report the results obtained by computer simulation. It is shown that in both generalizations the distribution of particle mass always obeys a power law, whose exponent α changes continuously between the two limiting values of $\frac{1}{3}$ and $\frac{1}{2}$, depending on each parameter.

A. The extension to higher spatial dimension

It is straightforward to define our model in a higher-dimensional space. Let a site in d-dimensional Euclidean lattice space by specified by $\mathbf{i}=(i_1,i_2,\ldots,i_d)$. On every site there is one particle with integer mass and it is combined with others by nearest-neighbors random walks. This process can be expressed in the same way as Eqs. (1) and (2),

$$s_{\mathbf{i}}(n+1)=\sum_{\mathbf{j}} W_{\mathbf{ij}}(n)s_{\mathbf{j}}(n)+1 , \tag{37}$$

$$W_{\mathbf{ij}}(n)=\begin{cases}1, & \text{with probability } q(\mathbf{i}-\mathbf{j})\\ 0, & \text{with probability } 1-q(\mathbf{i}-\mathbf{j})\end{cases} \tag{38}$$

where the jumping probability is now given by

$$q(\mathbf{i})=\begin{cases}1/2^d, & i_k=0 \text{ or } 1 \text{ for all } k=1,2,\ldots d\\ 0, & \text{otherwise},\end{cases} \tag{39}$$

which is independent of mass. After all of the particles have jumped according to this probability, unit mass particles are uniformly added on all sites. Then the particles which take a common site are unified to one new particle. Repeating this process, in $(d+1)$-dimensional space-time, we obtain clusters with treelike structures which anisotropically grow in the direction of the time axis. Note that, in contrast to the case $d=1$, the branches of the tree can intermingle each others when $d\geq 2$, just like branches of real trees do.

The results of numerical simulation are given in Fig. 5 and Table I, where the cumulative mass distribution

TABLE I. The spatial dimension d and the power exponent of mass distribution α.

d	α
1	0.331±0.006
2	0.465±0.003
3	0.491±0.007
4	0.496±0.010
5	0.500±0.005

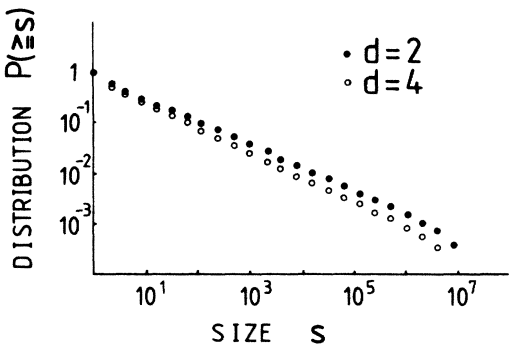

FIG. 5. The cumulative mass distribution for $d=2$ and 4.

$P(\geq s)$ for several values of d are shown. We find the following interesting properties.

(1) Every distribution obeys a power law.

(2) The exponent of the power α increases with d from $\frac{1}{3}$ to $\frac{1}{2}$.

(3) There seems to exist a critical dimension d_c above which α takes the mean-field value, $\frac{1}{2}$. The value of d_c estimated from Table I is 4.

The properties (1) and (2) are definitive, especially property (1) can be proved rigorously. We have also simulated some cases with a slightly modified jumping probability for each d. In any case, the value of α takes an almost identical value as long as the range of jumping is short. Hence we expect that α has such a universality that it is uniquely determined by the spatial dimension d only. As for the third statement, there remains some doubts since we cannot precisely determine the value of α with numerical simulation. Some discussion about the critical dimension will be given in Sec. III C.

B. Extension to long-range jumping

Here we treat the random walk in one-dimensional lattice space, whose jumping probability is given by

$$q(i)\propto i^{-\beta-1} \quad (-1\leq\beta) . \tag{40}$$

We choose such a power-law distribution because it has no characteristic length. If the jumping probability has a finite characteristic length, we expect that it can be reduced to the nearest-neighbor case by considering a suitable spatial renormalization. If this conjecture is correct, we will have a power-law mass distribution with the exponent $\alpha=\frac{1}{3}$ in such cases. We have simulated several cases with Gaussian jumping distributions, and it is confirmed that when the mean deviation of jumping is much smaller than the system size, the distribution becomes a power law with the exponent very close to $\frac{1}{3}$.

The new jumping probability, Eq. (40), includes the previous two cases in Sec. II. Obviously $\beta=-1$ corresponds to the mean-field case, case B. And in the limit $\beta\to\infty$, the new model becomes the nearest-neighbor case since the probability of taking a long jump vanishes. Our aim is to answer the following questions: First, does the

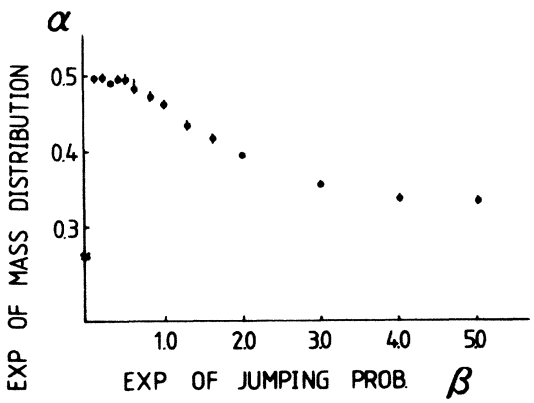

FIG. 6. Relation between the jumping probability exponent β and the exponent of power in mass distribution α.

mass distribution follows a power law for any β? If this is true, then how does the continuous parameter β connect the value of α from $\frac{1}{3}$ to $\frac{1}{2}$?

The results of our numerical simulation clearly give an affirmative answer to the first question. Therefore, we can determine the dependence of the exponent α on β. The relation of α and β are shown in Fig. 6. As expected, α takes the value from $\frac{1}{2}$ to $\frac{1}{3}$ as a continuous monotonous function of β. Furthermore, we can find a critical value for β below which α always takes the mean-field value, $\frac{1}{2}$. From Fig. 6, the critical value denoted by β_c is estimated as $\beta_c=\frac{1}{2}$.

C. Theoretical approaches

Here we introduce some approaches to the preceding generalized models. Since the problem is not so simple, the theory is not complete, unfortunately.

First, we again consider the r-body characteristic function introduced in Sec. II B. In case A, Z_r satisfies the simple evolution equation, Eq. (13), which is, however, quite exceptional. In the generalized models, the evolution of an r-body characteristic function is governed by all the other characteristic functions because any number of particles may jump into one site. That is, the evolution equation inevitably becomes a set of an infinite number of difference equations. We can formally write down the equations, but it is too complicated to be solved directly. Therefore, we introduce a simplification and an approximation.

So far, we let particles jump simultaneously every time step; however, this synchronism can easily be proved to be not essential. We have simulated the systems where only one particle jumps at one time step. Then the evolution becomes necessarily very slow, but the obtained asymptotic mass distributions are identical to those with synchronous jumps. Hence, it is sufficient to consider only one jump at a time step. This modification makes the analysis much simpler. In such a case, an r-body characteristic function at time step n is governed only by $(r-1)$-body, r-body, and $(r+1)$-body characteristic functions at time step $n-1$.

However, there still remains a complicated problem. It is the specification of the r-body characteristic function. In general, the r-body characteristic function is defined by specifying r sites as Eq. (11). Consequently, there are as many r-body characteristic functions as the number of combinations of r distinct sites. As a rough approximation we neglect differences among the r-body characteristic functions and denote them simply by $Z_r(\rho,n)$. Then we have an evolution equation for $Z_r(\rho,n)$,

$$Z_r(\rho,n+1)=e^{-\rho r}\left[\frac{b(r)}{N}Z_{r+1}(\rho,n)+\left[1-2\frac{b(r)}{N}\right]Z_r(\rho,n)+\frac{b(r)}{N}Z_{r-1}(\rho,n)\right], \tag{41}$$

where $b(r)$ is a function intuitively defined as the expectation value of the number of effective perimeter sites. For example, in case A, $b(r)$ equals 1 because the r sites are compact and the jumps can reach, at most, a neighbor site. In case B, all sites can be viewed as directly connecting to the other sites; hence, $b(r)=r$. In the general models we anticipate the following functional form for $b(r)$:

$$b(r)\propto r^{\delta},\quad 0\le\delta\le 1\ . \tag{42}$$

This assumption can be interpreted in geometrical terminology as letting the fractal dimension of the perimeter of the cluster D be given by

$$D=d\delta\ , \tag{43}$$

where d denotes the spatial dimension, and in the long-range jump model, $d=1$.

A steady solution of Eq. (41) with Eq. (15) can be obtained numerically. From the asymptotic behavior we find the following relation between α and δ:

$$\alpha=\frac{1}{3-\delta}\ . \tag{44}$$

As expected, α changes continuously and monotonically from $\frac{1}{3}$ to $\frac{1}{2}$ as δ moves from 0 to 1. Results of Secs. III A and III B indicate that δ is a monotonous function of d or β; however, its explicit functional form has not been elucidated yet.

In the d-dimensional voter model, the following asymptotic behavior is known for the distribution of lifetime of reputation:[14]

$$p(\ge L)\propto\begin{cases}L^{-1/2}, & d=1\\ (\ln L)/L, & d=2\\ L^{-1}, & d=3\end{cases} \tag{45}$$

where L is the lifetime of a repute which corresponds to the height of a river denoted by h in Sec. II B. This quantity has also a deep connection to Eq. (42) in the following sense. Equation (41) can be viewed as a diffusion equation in r space, which describes the Markovian nearest-neighbor random walk of a fictitious particle on the discrete positive space.[14] The probabilities of going

up ($r \to r+1$) or down ($r \to r-1$) are both proportional to $b(r)$. In this space, the area surrounded by the particle's trajectory and the axis $r=0$ is equivalent to the mass s, and the length of the recurrence time is identical to the reputation life time L and the height of river h. Since the distribution of recurrence time of the fictitious random walk should follow Eq. (45), the form of $b(r)$ is determined self-consistently as

$$b(r) \propto \begin{cases} \text{const}, & d=1 \\ r/\ln r, & d=2 \\ r, & d=3 \ . \end{cases} \tag{46}$$

Equation (46) with Eq. (44) implies that $\alpha=\frac{1}{3}$ for $d=1$ and $\alpha=\frac{1}{2}$ for $d>2$. Thus we get $d_c=2$ for the critical dimension, which does not agree with the numerical results $d_c=4$.

We have another fact which also suggests $d_c=2$. By using some statistical properties of random-walk trajectories, we can prove the following relation:

$$\langle s^2(n) \rangle \propto \begin{cases} n^{2+(d/2)}, & d<2 \\ n^3, & d \geq 2 \ , \end{cases} \tag{47}$$

where $S(n)$ denotes the size of a river which contains the origin at $n=0$. This shows that the variance of mass distribution changes its behavior at $d=2$. Again this result is consistent with $\alpha=\frac{1}{3}$ for $d=1$ and $\alpha=\frac{1}{2}$ for $d>2$.

As for the long-range jump case, we first introduce an effective dimension d_{eff} defined as

$$d_{\text{eff}} = \begin{cases} 1, & \beta \geq 2 \\ 2/\beta, & 2>\beta>0 \ . \end{cases} \tag{48}$$

Random walks with the power jumping probability are called Levy flights,[2] which are fully characterized by d_{eff} only. It can be shown that Eq. (47) is also valid if we replace d by d_{eff}. This suggests that the critical dimension of d_{eff} is 2, or equivalently, that means $\beta_c=1$, which also contradicts the numerical estimation $\beta_c=\frac{1}{2}$.

There seem to be two ways to cope with these discords. One is to doubt the numerical results and the other is to believe them. Although fluctuation of the value of α is estimated to be not so large, there might be a systematic shift caused by the finiteness of the system size, which becomes more apparent in the long-range jumping case. From a theoretical point of view, α is expected to be $\frac{1}{3}$ when $\beta>2$ because in that region $d_{\text{eff}}=1$. However, Fig. 6 shows only a very slow convergence to $\frac{1}{3}$ as $\beta \to \infty$. This overestimation of α may be due to the finiteness of the system size. That is, if the system size is comparable to the random walk's mean deviation, then the value of α may be estimated large because of the mean-field effects among the sites within a distance of the deviation. Therefore, the exact value of α in both generalized models might be a little bit smaller.

However, this systematic shift does not explain the discords between theory and simulation at all. For if α is smaller, then d_c becomes larger and β_c becomes smaller. This is a negative correction. Finiteness of the number of time steps can also be shown to work negatively on the correction of exponents. Since we cannot find any mechanism in the simulation to make us underestimate the value of α, we hence conclude from numerical data that d_c or β_c cannot be 2 or 1, respectively.

As mentioned above, we do not have any satisfactory theory for d_c and β_c; therefore, we would like to propose our conjectures, $d_c=4$ and $\beta_c=\frac{1}{2}$, as open questions. It might be stressed that, in view of Eq. (48), our conjectured critical values are consistent with each other. Moreover, it is known[15] that a certain time correlation function in the voter model is strong for $d<4$ and weak for $d>3$, so there are some indications for $d_c=4$. If $d_c=4$, $\beta_c=\frac{1}{2}$ is really the case, our characteristic function analysis must fail (perhaps) because of the nontrivial geometric nature of the clusters.

IV. SUMMARY

Without any injection, an aggregation system will finally reach a trivial stationary state, that is, all particles will stick together forming some giant clusters. On the other hand, if small particles are continuously injected, the system is expected to approach a nontrivial quasistationary state where the rate of aggregation balances with the rate of injection. In such a state the total mass and mean cluster size are divergent; however, the cluster size (or mass) distribution function may be convergent and physically meaningful.

We have introduced and analyzed several random aggregation models with injection, and shown that in all models the mass distributions follow power laws in the final quasistationary state. This result supports the general statement that constant injection of small particles leads an aggregation system to follow a power-law size (or mass) distribution.

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