Squeezing and frequency jump of a harmonic oscillator

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The theoretical basis of a mechanism for squeezing through a frequency jump of a harmonic oscillator is examined. It is found that a related squeezing transformation exists, but its interpretation in terms of a frequency change is not possible. A reinterpretation of the results in terms of a scaling transformation in coordinate or momentum variables is given. Moreover, a sudden jump is not essential, and squeezing can evolve continuously. The equations of motion for such a process are obtained and the interaction responsible for squeezing is obtained.

I. INTRODUCTION

Squeezed states of harmonic oscillators and photons have received considerable attention in recent years.[1] A new approach for calculating the normally ordered form of squeeze operators was presented in an earlier paper.[2] [This approach is based on the IWOP (integration within an ordered product of operators) technique.[3,4]] A clear interpretation of the squeezing as a scaling transformation in coordinate or momentum variables is a very interesting result of this approach. In this paper we investigate the related question of the existence of a squeezing transition caused by a frequency jump in a harmonic oscillator. Using the IWOP technique we investigate a squeezing transformation which, at first sight, appears to be related to the frequency jump $\omega\rightarrow\omega'$ of a harmonic oscillator. However, a deeper examination reveals that ω' cannot be interpreted as the new frequency. Then we go a step further to show that a sudden frequency jump is not essential, and the squeezing transition can continuously evolve if $\omega'=\omega'(t)$, provided $\omega'(t)$ is related to the squeezing interaction in a certain manner. The interaction Hamiltonian and the equations of motion for the squeezing process are derived. It is pointed out that the squeezing transition can be properly interpreted as a rescaling in coordinate or momentum variables *if* ω *remains unchanged.*

In Sec. II the squeezing transformation is introduced and its normally ordered form is derived using the IWOP technique. It is also shown that the same results can be derived in coordinate, momentum, or the canonical coherent state[5] representations. The time evolution of the system is considered in Sec. III and the results are discussed in Sec. IV. It is pointed out that a previous suggestion[6] of a squeezing mechanism through the frequency change of a harmonic oscillator is based on an incorrect interpretation of the squeezing transformation. Possible sources of such a misinterpretation are also discussed in Sec. IV.

II. SQUEEZING TRANSFORMATION

Consider a harmonic oscillator with a *unit mass* and frequency ω ($\hbar=1$)

$$H_\omega=\tfrac{1}{2}P^2+\tfrac{1}{2}\omega^2Q^2=\omega(a_\omega^\dagger a_\omega+\tfrac{1}{2}),\quad [a_\omega,a_\omega^\dagger]=1\ , \tag{1}$$

$$H_\omega\,|\,n\,\rangle_\omega=\omega(n+\tfrac{1}{2})\,|\,n\,\rangle_\omega,\quad a_\omega\,|\,0\,\rangle_\omega=0\ , \tag{2}$$

$$Q=\frac{1}{\sqrt{2\omega}}(a_\omega+a_\omega^+),\quad P=i\left[\frac{\omega}{2}\right]^{1/2}(a_\omega^+-a_\omega)\ ,\qquad a_\omega=\frac{1}{\sqrt{2}}\left[\sqrt{\omega}Q+\frac{i}{\sqrt{\omega}}P\right]. \tag{3}$$

The subscript ω is to emphasize the ω dependence of the operators and the state vectors. However, from here on, we set $a_\omega\equiv a$. The basis vector in coordinate representation is

$$|\,q\,\rangle_\omega=\left[\frac{\omega}{\pi}\right]^{1/4}\exp\left[-\frac{\omega}{2}q^2+\sqrt{2\omega}qa^\dagger-\tfrac{1}{2}a^{\dagger 2}\right]|\,0\,\rangle_\omega\ , \tag{4}$$

with

$$\int_{-\infty}^{+\infty}dq\,|\,q\,\rangle_{\omega\,\omega}\langle\,q\,|=1,\quad Q\,|\,q\,\rangle_\omega=q\,|\,q\,\rangle_\omega\ .$$

Consider a state vector

$$|\,q\,\rangle_{\omega'}\equiv\left[\frac{\omega'}{\pi}\right]^{1/4}\exp\left[-\frac{\omega'}{2}q^2+\sqrt{2\omega'}qa^\dagger-\tfrac{1}{2}a^{\dagger 2}\right]|\,0\,\rangle_\omega \tag{5}$$

obtained from (4) by replacing ω with ω', but leaving a and $|\,0\,\rangle_\omega$ unchanged. We now evaluate the following operator, constructed with the help of (4) and (5);

$$S\equiv\int_{-\infty}^{+\infty}dq\,|\,q\,\rangle_{\omega'\,\omega}\langle\,q\,|=\frac{(\omega\omega')^{1/4}}{\sqrt{\pi}}\int_{-\infty}^{+\infty}dq:\exp\left[-\frac{q^2}{2}(\omega+\omega')+\sqrt{2}q(\sqrt{\omega'}a^\dagger+\sqrt{\omega}a)-\tfrac{1}{2}(a^2+a^{\dagger 2})-aa^\dagger\right]:$$
$$=\left[\frac{2}{\omega+\omega'}\right]^{1/2}(\omega\omega')^{1/4}:\exp\left[\frac{\omega'-\omega}{2(\omega'+\omega)}a^{\dagger 2}+\left[\frac{2\sqrt{\omega'\omega}}{\omega'+\omega}-1\right]a^\dagger a+\frac{(\omega-\omega')a^2}{2(\omega+\omega')}\right]: \tag{6}$$

where :: denotes a normal ordering. Here, a standard Gaussian integral was used to perform the integration over q, using the IWOP technique, and the following identity was also employed:

$$|0\rangle_{\omega\,\omega}\langle 0| = :\exp(-a^{\dagger}a): . \tag{7}$$

It is encouraging to note that

$$\left[\frac{\omega'+\omega}{2\sqrt{\omega'\omega}}\right]^2 - \left[\frac{\omega'-\omega}{2\sqrt{\omega'\omega}}\right]^2 = 1 \tag{8}$$

because we can now define real parameters r and μ such that

$$\cosh r = \frac{\omega'+\omega}{2\sqrt{\omega'\omega}}, \quad \sinh r = \frac{\omega-\omega'}{2\sqrt{\omega'\omega}},$$
$$\tanh r = \frac{\omega-\omega'}{\omega+\omega'}, \quad \mu \equiv e^{r} = \left[\frac{\omega}{\omega'}\right]^{1/2}. \tag{9}$$

As a result, (6) reduces to the following form:

$$S = \exp\left[-\frac{a^{\dagger 2}}{2}\tanh r\right]\exp[(a^{\dagger}a+\tfrac{1}{2})\ln \operatorname{sech} r]$$
$$\times \exp\left[\frac{a^2}{2}\tanh r\right]. \tag{10}$$

Here we used the identity

$$\exp(\lambda a^{\dagger}a) = :\exp[(e^{\lambda}-1)a^{\dagger}a]: . \tag{11}$$

Equation (10) is the normally ordered form of the squeeze operator[2,4] with a squeezing parameter μ. This result shows that the transformation $|q\rangle_{\omega} \to |q\rangle_{\omega'}$ maps a squeeze operator in Hilbert space so that

$$S\,|q\rangle_{\omega} = \int_{-\infty}^{+\infty} dq'\,|q'\rangle_{\omega'\,\omega}\langle q'|q\rangle_{\omega} = |q\rangle_{\omega'} . \tag{12}$$

Interchanging ω and ω' in (6), one obtains

$$S^{\dagger} = \int_{-\infty}^{+\infty} dq\,|q\rangle_{\omega\,\omega'}\langle q| = S^{-1}, \quad S^{-1}|q\rangle_{\omega'} = |q\rangle_{\omega}, \tag{13}$$

which shows that S is a unitary operator.

In order to clarify the physical meaning of $|q\rangle_{\omega'}$, consider the following transformation:

$$a' = SaS^{-1} = a\cosh r + a^{\dagger}\sinh r$$
$$= \frac{\omega}{\sqrt{2\omega'}}Q + \frac{i}{\omega}\left[\frac{\omega'}{2}\right]^{1/2}P$$
$$= \frac{1}{\sqrt{2}}\left[\sqrt{\omega}Q' + \frac{i}{\sqrt{\omega}}P'\right], \tag{14}$$

$$H'_{\omega}|n\rangle'_{\omega} = \omega(n+\tfrac{1}{2})|n\rangle'_{\omega}, \quad |n\rangle'_{\omega} = S\,|n\rangle_{\omega}, \tag{15}$$

$$H'_{\omega} = SH_{\omega}S^{-1} = \omega(a'^{\dagger}a'+\tfrac{1}{2}) = \tfrac{1}{2}(P'^2+\omega^2 Q'^2), \tag{16}$$

$$Q' = SQS^{-1} = \mu Q, \quad P' = SPS^{-1} = P/\mu . \tag{17}$$

The scaling $q \to q' = \mu q$ and $p \to p' = p/\mu$ resulting from (17) implies squeezing, as shown in Ref. 2. According to (16), the squeezed harmonic oscillator can be pictured as having the same frequency ω but rescaled Q and P. Another interpretation of the squeezing process follows from the coordinate basis vector $|q\rangle'_{\omega}$, which can be generated from the ground state $|0\rangle'_{\omega}$ of the squeezed oscillator,

$$|q\rangle'_{\omega} = \left[\frac{\omega}{\pi}\right]^{1/4}\exp\left[-\frac{\omega}{2}q^2 + \sqrt{2\omega}qa'^{\dagger} - \tfrac{1}{2}(a'^{\dagger})^2\right]|0\rangle'_{\omega} . \tag{18}$$

Using (14), (15), (18), and (12), we obtain an interesting result,

$$|q\rangle'_{\omega} = S\,|q\rangle_{\omega} = |q\rangle_{\omega'}, \tag{19}$$

$$Q'|q\rangle'_{\omega} = q\,|q\rangle'_{\omega}, \quad Q\,|q\rangle'_{\omega} = (q/\mu)|q\rangle'_{\omega} = (q/\mu)|q\rangle_{\omega'} . \tag{20}$$

Thus $|q\rangle_{\omega'}$ also represents the coordinate basis vector of the squeezed oscillator. Using (17), one can also calculate the quadrature variances, which exhibit the squeezing property very clearly,

$$\Delta Q = [{}'_{\omega}\langle 0|(Q-\langle Q\rangle)^2|0\rangle'_{\omega}]^{1/2} = \frac{1}{\mu\sqrt{2\omega}}, \tag{21}$$

$$\Delta P = [{}'_{\omega}\langle 0|(P-\langle P\rangle)^2|0\rangle'_{\omega}]^{1/2} = \mu\left[\frac{\omega}{2}\right]^{1/2}, \tag{22}$$

which yield the standard result $\Delta Q\Delta P = \frac{1}{2}$.

Using the IWOP technique, it is straightforward to confirm that the same results are obtained in the momentum or canonical coherent state[5] representations. Consider the operators

$$S_1 = \int_{-\infty}^{\infty} dp\,|p\rangle_{\omega'\,\omega}\langle p| , \tag{23}$$

$$S_2 = \left[\frac{\omega+\omega'}{2\sqrt{\omega\omega'}}\right]^{1/2}\int_{-\infty}^{\infty}\frac{dp}{2\pi}\int_{-\infty}^{\infty} dq\,|p,q\rangle_{\omega'\,\omega}\langle p,q| , \tag{24}$$

where

$$|p\rangle_{\omega} = \left[\frac{1}{\pi\omega}\right]^{1/4}\exp\left[-\frac{p^2}{2\omega} + i\left[\frac{2}{\omega}\right]^{1/2}pa^{\dagger} + \frac{a^{\dagger 2}}{2}\right]|0\rangle_{\omega},$$
$$P\,|p\rangle_{\omega} = p\,|p\rangle_{\omega} \tag{25}$$

$$|p,q\rangle_{\omega} = \exp\left[-\frac{1}{4}\left[\omega q^2 + \frac{1}{\omega}p^2\right] + \frac{1}{\sqrt{2}}\left[\sqrt{\omega}q + \frac{ip}{\sqrt{\omega}}\right]a^{\dagger}\right]|0\rangle_{\omega}, \tag{26}$$

and $|p\rangle_{\omega'}$ ($|p,q\rangle_{\omega'}$) are the same as in (25) [(26)], except $\omega \to \omega'$. We also have

$$\int_{-\infty}^{+\infty} dp\,|p\rangle_{\omega\,\omega}\langle p| = 1,$$
$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} dp\int_{-\infty}^{+\infty} dq\,|p,q\rangle_{\omega\,\omega}\langle p,q| = 1 . \tag{27}$$

The notation is the same as in Ref. 2, and the evaluation of integrals is also similar. For example, we obtain

$$S_2=\left[\frac{\omega+\omega'}{2\sqrt{\omega\omega'}}\right]^{1/2}\int_{-\infty}^{+\infty}\frac{dp}{2\pi}\int_{-\infty}^{+\infty}dq :\exp\left[-\frac{q^2}{4}(\omega+\omega')-\frac{p^2}{4}\left[\frac{1}{\omega}+\frac{1}{\omega'}\right]+\frac{q}{\sqrt{2}}(\sqrt{\omega'}a^\dagger+\sqrt{\omega}a)+\frac{ip}{\sqrt{2}}\left[\frac{a^\dagger}{\sqrt{\omega'}}-\frac{a}{\sqrt{\omega}}\right]-a^\dagger a\right]:$$

$$=\left[\frac{2\sqrt{\omega'\omega}}{\omega'+\omega}\right]^{1/2}:\exp\left[\frac{\omega'-\omega}{2(\omega+\omega')}a^{\dagger 2}+\left[\frac{2\sqrt{\omega\omega'}}{\omega+\omega'}-1\right]a^\dagger a+\frac{\omega-\omega'}{2(\omega+\omega')}a^2\right]: . \tag{28}$$

Thus $S_2=S$. Similarly one finds that $S_1=S$. Hence the squeezing property can be demonstrated in each of the three representations. As before, we can also show that

$$|p\rangle_{\omega'}=|p\rangle'_{\omega},\quad P|p\rangle_{\omega'}=\mu p\,|p\rangle_{\omega'} . \tag{29}$$

As an application, consider the wave function of the squeezed state, $|n\rangle'_{\omega}$, which can be calculated very simply as follows:

$$ {}_{\omega}\langle q|n\rangle'_{\omega}={}_{\omega}\langle q|\int_{-\infty}^{+\infty}dq\,|q\rangle_{\omega'}\,{}_{\omega}\langle q|n\rangle_{\omega} =\mu\,{}_{\omega}\langle \mu q|n\rangle_{\omega} . \tag{30}$$

III. TIME EVOLUTION

Let $\omega'=\omega'(t)$ with $\omega'(0)=\omega$. We seek the interaction Hamiltonian which can generate the continuous squeezing transformation $|q\rangle_{\omega}\rightarrow|q\rangle_{\omega'(t)}$. For this purpose we first rewrite (10) in the following form:

$$S(t,0)=\exp\left[-\frac{a^{\dagger 2}}{2}\tanh[r(t)]\right]\times\exp\{(a^\dagger a+\tfrac{1}{2})\ln\operatorname{sech}[r(t)]\}\times\exp\left[\frac{a^2}{2}\tanh[r(t)]\right]=\int_{-\infty}^{+\infty}dq\,|q\rangle_{\omega'(t)}\,{}_{\omega}\langle q| , \tag{31}$$

$$S(0,0)=1,\quad \tanh[r(t)]=\frac{\omega'(0)-\omega'(t)}{\omega'(0)+\omega'(t)} . \tag{32}$$

Differentiating (31) with respect to t and using the operator identities

$$e^{va^{\dagger 2}}a=(a-2va^\dagger)e^{va^{\dagger 2}} ,$$

$$e^{va^{\dagger 2}}a^2=(a^2+4v^2a^{\dagger 2}-4va^\dagger a-2v)e^{va^{\dagger 2}} ,$$

we obtain the following equation of motion for S:

$$i\frac{\partial}{\partial t}S(t,0)=V(t)(a^{\dagger 2}-a^2)S(t,0) ,\quad V(t)=\frac{i}{4\omega'(t)}\frac{d\omega'(t)}{dt} ,\quad S(t,t_1)S(t_1,0)=S(t,0),\quad S^\dagger(t,0)=S(0,t) . \tag{33}$$

Therefore $S(t,0)$ satisfies all the required properties of a time evolution operator, and we can put (33) in the standard form for the equation of motion in an interaction picture,

$$H^{\mathrm{IP}}(t)S(t,0)=i\partial S(t,0)/\partial t , \tag{34}$$

$$H^{\mathrm{IP}}(t)=e^{iH_0t}[V(t)e^{-i2\omega t}a^{\dagger 2}+V^*(t)e^{i2\omega t}a^2]e^{-iH_0t} , \tag{35}$$

$$H_0=\omega a^\dagger a , \tag{36}$$

where the superscript IP stands for "interaction picture." As a result, there exists a set of state vectors $|q,t\rangle_{\omega}^{\mathrm{IP}}$ which gives the solution of the following equation of motion:

$$H^{\mathrm{IP}}(t)|q,t\rangle_{\omega}^{\mathrm{IP}}=i\frac{\partial}{\partial t}|q,t\rangle_{\omega}^{\mathrm{IP}},\quad |q,0\rangle_{\omega}^{\mathrm{IP}}=|q\rangle_{\omega} \tag{37}$$

where $|q\rangle_{\omega}$ is the basis vector in the Schrödinger picture. Now (34) and (37) imply that

$$S(t,0)=\int_{-\infty}^{+\infty}dq\,|q,t\rangle_{\omega}^{\mathrm{IP}}\,{}_{\omega}^{\mathrm{IP}}\langle q,0| =\int_{-\infty}^{+\infty}dq\,|q,t\rangle_{\omega}^{\mathrm{IP}}\,{}_{\omega}\langle q| . \tag{38}$$

Comparing (31) and (38), one obtains

$$|q,t\rangle_{\omega}^{\mathrm{IP}}=|q\rangle_{\omega'(t)} . \tag{39}$$

Since $\omega'(0)=\omega$, from (19) and (39) it follows that squeezing evolves continuously under the influence of the interaction given by (35) and (33).

IV. DISCUSSION

We have shown that a harmonic oscillator can be squeezed through a continuous time-dependent transformation, Eq. (31), which obeys the equation of motion (33). Sudden frequency jump is not necessary. However, $\omega'(t)$ cannot be identified with the time-dependent oscillator frequency, except that $\omega'(0)=\omega$. The squeezing can be understood in terms of the rescaled variables $q'=\mu q$ and $p'=p/\mu$ if the frequency ω remains unchanged [see Eq. (17) and the discussion following it]. In Ref. 6, $\omega\rightarrow\omega'$ was interpreted as the frequency change. However, the theoretical basis of this interpretation was not investigated and the suggestion of squeezing was based solely on the existence of identity (8). A possible source of such a misinterpretation can be seen through the following transformation:

$$a''=S^{-1}aS=a\cosh r-a^\dagger\sinh r$$
$$=\frac{1}{\sqrt{2}}\left[\sqrt{\omega'}Q+\frac{i}{\sqrt{\omega'}}P\right]$$
$$=a_{\omega'}\ . \tag{40}$$

It is tempting to identify $a''=a_{\omega'}$ with the annihilation operator of the squeezed harmonic oscillator which has a frequency ω'. However, this is incorrect since

$$H''_\omega=S^{-1}H_\omega S=\omega(a^\dagger_{\omega'}a_{\omega'}+\tfrac{1}{2})$$
$$=\mu^2H_{\omega'}$$
$$\equiv\mu^2\left[\frac{P^2}{2}+\frac{\omega'^2}{2}Q^2\right]\ . \tag{41}$$

That is, $H''_\omega\neq H_{\omega'}$ and, therefore, the squeezing transformation (40) does not lead to a new oscillator with frequency ω'.

A similar interpretation emerges if one uses the transformation (14) of this paper. Then (14) and (16) give

$$a'=a_{\omega''},\quad H'_\omega=\frac{1}{\mu^2}H_{\omega''}=\frac{1}{\mu^2}\left[\frac{P^2}{2}+\frac{\omega''^2}{2}Q^2\right], \tag{42}$$

where $\omega''=\omega^2/\omega'=\mu^2\omega$. Again ω'' is not the frequency of the squeezed oscillator since $H'_\omega\neq H_{\omega''}$.

According to (33)–(36), the system that undergoes the squeezing process has the following Hamiltonian:

$$H=\omega a^\dagger a+V(t)e^{-i2\omega t}a^{\dagger 2}+V^*(t)e^{i2\omega t}a^2\ , \tag{43}$$

and it is not simply an oscillator with a time-dependent frequency. Moreover, from (33) we obtain the following expression:

$$\omega'(t)=\omega\exp\left[-i4\int_0^t dt'V(t')\right]\ . \tag{44}$$

Equations (43) and (44) clearly exhibit the origin of squeezing characterized by (8).

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