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Quantum nondemolition measurements in optical cavities

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We analyze schemes for performing quantum nondemolition (QND) measurements in optical cavities. We consider three schemes: (1) measurement of a quadrature phase amplitude using a parametric process, (2) measurement of a quadrature phase amplitude using the optical Kerr effect in a nonlinear fiber, and (3) measurement of the photon number also using the Kerr effect in a fiber. We show that in the second scheme an enhancement of the QND effect may be obtained by making the cavity finesse for the signal larger than that for the probe.

I. INTRODUCTION

The act of measurement may add quantum noise to a signal. There is a class of measurements known as quantum nondemolition (QND) measurements for which the back action noise arising from the measurement may be evaded.[1] A number of quantum nondemolition schemes have been suggested in an optical context. Approximate QND schemes in parametric amplifiers and frequency converters have been analyzed by Milburn, Lane, and Walls.[2] An exact QND scheme using a combination of a parametric amplifier and a frequency converter has been analyzed by Hillery and Scully[3] and by Yurke.[4] These schemes involved a measurement of the quadrature phase amplitude of the field. An experimental demonstration of a QND measurement of the quadrature phase of an optical field has been given by Levenson *et al.*[5] using four wave mixing in optical fibers. Different QND schemes to measure photon number using four-wave mixing have been suggested by Milburn and Walls[6] and Imoto, Haus, and Yamomoto.[7]

In this paper we shall analyze QND measurements in optical cavities considering (i) a two-mode interaction using a $\chi^{(2)}$ nonlinearity, (ii) a four-mode interaction using a $\chi^{(3)}$ nonlinearity, and (iii) a photon number scheme using a $\chi^{(3)}$ nonlinearity.

II. TWO-MODE QND MEASUREMENT IN PARAMETRIC SYSTEMS

Consider two modes of the electromagnetic field coupled together by a parametric interaction. We assume the coupling is described by the Hamiltonian[3,4]

$$\hat{H}=\hbar\frac{\chi}{2}\hat{X}_1\hat{Y}_1 , \tag{2.1}$$

where $\hat{X}_1$ and $\hat{Y}_1$ are quadrature phase amplitudes defined by

$$\hat{X}_1=a+a^{\dagger} , \tag{2.2a}$$

$$\hat{X}_2=-i(a-a^{\dagger}) , \tag{2.2b}$$

$$\hat{Y}_1=b+b^{\dagger} , \tag{2.2c}$$

$$\hat{Y}_2=-i(b-b^{\dagger}) , \tag{2.2d}$$

where a,b are the annihilation operators for the two modes. This Hamiltonian could describe a crystal with a $\chi^{(2)}$ nonlinearity driven at two frequencies $\omega_s=\omega_a+\omega_b$ and $\omega_d=\omega_a-\omega_b$ with equal coupling strength for the frequency conversion and amplification process. Alternatively it could describe sequential frequency conversion and parametric amplification as described by Yurke.[4] The variable X_1 is a QND observable since it commutes with the Hamiltonian and hence satisfies the back action evading criterion.[1] Mode a is taken to be the signal, and we wish to measure the signal quadrature phase X_1. This is achieved by measuring the quadrature phase Y_2 of mode b. The input and output quadratures of the device are related by

$$\hat{X}_1^{\mathrm{OUT}}=\hat{X}_1^{\mathrm{IN}} , \tag{2.3a}$$

$$\hat{Y}_2^{\mathrm{OUT}}=\hat{Y}_2^{\mathrm{IN}}+G\hat{X}_1^{\mathrm{IN}} , \tag{2.3b}$$

where G is the gain of the device.

If the nonlinear crystal is placed inside an optical cavity the result of a narrow band analysis[4] shows that the gain is enhanced by the cavity finesse. The gain in Eq. (2.3b) becomes G', where $G'=4G/\gamma$ with γ related to the transmission coefficient of the end mirror of the cavity.

We now give a wideband analysis of this measurement scheme in an optical cavity. Our treatment will be based on that of Gardiner and Collett[8] (see also Ref. 9). In an interaction picture defined by

$$\hat{U}(t)=e^{-i\omega_a a^\dagger a t - i\omega_b b^\dagger b t} ,$$

the Hamiltonian becomes

$$H(t)=\hbar\frac{\chi}{2}\hat{X}_1\hat{Y}_1 + i\hbar\kappa_a[a\hat{\Gamma}_a^\dagger(t)e^{-i\omega_a t}-\text{H.c.}] + i\hbar\kappa_b[b\hat{\Gamma}_b^\dagger(t)e^{-i\omega_b t}-\text{H.c.}] , \quad (2.4)$$

where $\hat{\Gamma}_a(t)$ and $\hat{\Gamma}_b(t)$ are bath operators for the cavity losses. For a single-ended cavity the bath is simply the radiation field outside the cavity. This external field can be separated into an input and output contribution. The output field is simply the Heisenberg form of the input field as determined by the time evolution of the coupled internal and external modes. We assume that the input field can be further separated into two components one with the carrier frequency ω_a and another with carrier frequency ω_b. These frequencies are assumed to be sufficiently widely separated that the field at carrier frequency ω_a drives only the cavity mode at frequency ω_a, while the external component at carrier frequency ω_b drives only the cavity mode at frequency ω_b. Let these external input fields be represented by $a^{\text{IN}}(t)$ and $b^{\text{IN}}(t)$ for which the positive frequency components are given by

$$a^{\text{IN}}(t)=\int_B \frac{d\omega}{2\pi}\left[\frac{\hbar\omega}{2}\right]^{1/2} a^{\text{IN}}(\omega)e^{-i\omega t} , \quad (2.5a)$$

$$b^{\text{IN}}(t)=\int_B \frac{d\omega}{2\pi}\left[\frac{\hbar\omega}{2}\right]^{1/2} b^{\text{IN}}(\omega)e^{-i\omega t} . \quad (2.5b)$$

The bandwidth of integration B is defined by

$$B:\omega_{a,b}-\frac{\Omega_c}{2}\le\omega\le\omega_{a,b}+\frac{\Omega_c}{2} ,$$

where Ω_c is the cavity-free spectral range.

Then the bath operators may be written

$$\hat{\Gamma}_a(t)=\left[\frac{2}{\hbar\omega_a}\right]^{1/2} a^{\text{IN}}(t) , \quad (2.6a)$$

$$\hat{\Gamma}_b(t)=\left[\frac{2}{\hbar\omega_b}\right]^{1/2} b^{\text{IN}}(t) . \quad (2.6b)$$

In a Lorentzian approximation to the cavity response[10,11]

$$\kappa_a=\sqrt{\gamma_a} , \quad (2.7a)$$

$$\kappa_b=\sqrt{\gamma_b} , \quad (2.7b)$$

where $\gamma_a/2$ and $\gamma_b/2$ are the cavity line widths at frequencies ω_a and ω_b, respectively, and Ω_c is the free spectral range of the cavity.

The Hamiltonian in Eq. (2.4) has been written on the assumption that the amplitude of the external field has units $(\text{s})^{-1/2}$ while the amplitude of the internal field is dimensionless. With this choice of units the commutation relations for the external fields are

$$[a^{\text{IN}}(\omega),a^{\text{IN}^\dagger}(\omega)]=[b^{\text{IN}}(\omega),b^{\text{IN}^\dagger}(\omega)]=2\pi\delta(\omega-\omega') . \quad (2.8)$$

The commutation relations for the bath operators are then given by

$$[\hat{\Gamma}_a(t),\hat{\Gamma}_a(t')]=\int_B \frac{d\omega}{2\pi}\left[\frac{\omega}{\omega_a}\right]e^{-i\omega(t-t')}$$

and a similar expression for $\hat{\Gamma}_b(t)$. Changing the variable to $\omega'=\omega-\omega_a$ and replacing $\pm\Omega_c$ by $\pm\infty$ in the limits of integration (which assumes that the cavity resonances are quite narrow), the commutation relation in Eq. (2.9) reduces to

$$[\hat{\Gamma}_a(t),\hat{\Gamma}_a^\dagger(t')]=e^{-i\omega_a(t-t')}\delta(t-t') . \quad (2.9)$$

The approximation leading to Eq. (2.9) is equivalent to the assumption

$$\hat{\Gamma}_a(t)\simeq\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}a^{\text{IN}}(\omega)e^{-i\omega t}+\text{H.c.}$$

The quadrature phase operators for the cavity modes obey the following quantum stochastic differential equations

$$\frac{d\hat{X}_1(t)}{dt}=-\frac{\gamma_a}{2}\hat{X}_1(t)-\kappa_a\hat{X}_1^{\text{IN}}(t) , \quad (2.10a)$$

$$\frac{d\hat{Y}_2(t)}{dt}=-\chi\hat{X}_1(t)-\frac{\gamma_b}{2}\hat{Y}_2(t)-\kappa_b\hat{Y}_2^{\text{IN}}(t) , \quad (2.10b)$$

where

$$\hat{X}_1^{\text{IN}}(t)\equiv\left[\frac{2}{\hbar\omega_a}\right]^{1/2}[a^{\text{IN}}(t)e^{i\omega_a t}+\text{H.c.}] , \quad (2.11a)$$

$$\hat{Y}_2^{\text{IN}}(t)\equiv -i\left[\frac{2}{\hbar\omega_b}\right]^{1/2}[b^{\text{IN}}(t)e^{i\omega_b t}-\text{H.c.}] . \quad (2.11b)$$

The output fields are related to the input fields and the cavity fields by[8,9]

$$\hat{X}_1^{\text{OUT}}(t)-\hat{X}_1^{\text{IN}}(t)=\kappa_a\hat{X}_1(t) . \quad (2.12a)$$

$$\hat{Y}_2^{\text{OUT}}(t)-\hat{Y}_2^{\text{IN}}(t)=\kappa_b\hat{Y}_2(t) . \quad (2.12b)$$

The solution of Eq. (2.10a) is

$$\hat{X}_1(t)=e^{-(\gamma_a/2)(t-t_0)}\hat{X}_1 - \kappa_a\int_{t_0}^{t}e^{-(\gamma_a/2)(t-s)}\hat{X}_1^{\text{IN}}(s)ds ,$$

which, for $t_0\to-\infty$, may be replaced by the asymptotic result

$$\hat{X}_1(t)=-\kappa_a\int_{-\infty}^{t}e^{-(\gamma_a/2)(t-s)}\hat{X}_1^{\text{IN}}(s)ds . \quad (2.13)$$

Our analysis thus ignores all initial transients.

Substituting Eqs. (2.11a) and (2.5a) into (2.13) and making the approximation implicit in Eq. (2.9) we find

$$\hat{X}_1(t)=-\kappa_a\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left[\left(\frac{\gamma_a}{2}-i\omega\right)^{-1}\times a^{\mathrm{IN}}(\omega+\omega_a)e^{-i\omega t}+\mathrm{H.c.}\right] . \tag{2.14}$$

If we define positive and negative frequency components of $\hat{X}_1(t)$ by[12]

$$\hat{X}_1(t)=\int_0^{\infty}\frac{d\omega}{2\pi}[\hat{x}_1(\omega)e^{-i\omega t}+\hat{x}_1^{\dagger}(\omega)e^{i\omega t}] ,$$

where

$$\hat{x}_1(\omega)\equiv a(\omega_a+\omega)+a^{\dagger}(\omega_a-\omega) , \tag{2.15}$$

then Eq. (2.14) requires that

$$\left(\frac{\gamma}{2}-i\omega\right)a(\omega+\omega_a)=-\kappa_a a^{\mathrm{IN}}(\omega+\omega_a) . \tag{2.16}$$

Proceeding in a similar way for Eq. (2.11b) we find that

$$\begin{pmatrix}\frac{\gamma_a}{2}-i\omega & 0\\ \chi & \frac{\chi_b}{2}-i\omega\end{pmatrix}\begin{pmatrix}\hat{x}_1(\omega)\\ \hat{y}_2(\omega)\end{pmatrix}=-\begin{pmatrix}\kappa_a\hat{x}_1^{\mathrm{IN}}(\omega)\\ \kappa_b\hat{y}_2^{\mathrm{IN}}(\omega)\end{pmatrix} , \tag{2.17}$$

where $\hat{y}_2$ are the positive frequency components of $\hat{Y}(t)$ and are defined by

$$\hat{y}_2(\omega)=-i[b(\omega_a+\omega)-b^{\dagger}(\omega_a-\omega)] , \tag{2.18}$$

and where $\hat{x}_1^{\mathrm{IN}}(\omega)$ and $\hat{y}_2^{\mathrm{IN}}(\omega)$ are the positive frequency components of $\hat{X}_1^{\mathrm{IN}}(t)$ and $\hat{Y}_2^{\mathrm{IN}}(t)$.

From Eqs. (2.12a) and (2.12b) we also have

$$\begin{pmatrix}\hat{x}_1^{\mathrm{OUT}}(\omega)\\ \hat{y}_2^{\mathrm{OUT}}(\omega)\end{pmatrix}=\begin{pmatrix}\kappa_a & 0\\ 0 & \kappa_b\end{pmatrix}\begin{pmatrix}\hat{x}_1(\omega)\\ \hat{y}_2(\omega)\end{pmatrix}+\begin{pmatrix}\hat{x}_1^{\mathrm{IN}}(\omega)\\ \hat{y}_2^{\mathrm{IN}}(\omega)\end{pmatrix} . \tag{2.19}$$

Eliminating the internal fields we have

$$\hat{x}_1^{\mathrm{OUT}}(\omega)=-\frac{\left(\frac{\gamma_a}{2}+i\omega\right)}{\left(\frac{\gamma_a}{2}-i\omega\right)}\hat{x}_1^{\mathrm{IN}}(\omega) , \tag{2.20a}$$

$$\hat{y}_2^{\mathrm{OUT}}(\omega)=-\frac{\left(\frac{\gamma_b}{2}+i\omega\right)}{\left(\frac{\gamma_b}{2}-i\omega\right)}\hat{y}_2^{\mathrm{IN}}(\omega)+\frac{\chi\sqrt{\gamma_a\gamma_b}}{\left(\frac{\gamma_a}{2}-i\omega\right)\left(\frac{\gamma_b}{2}-i\omega\right)}x_1^{\mathrm{IN}}(\omega) . \tag{2.20b}$$

Equations (2.15a) and (2.15b) indicate that even in the absence of a nonlinear medium the empty cavity introduces a phase shift. It is then convenient to shift the phase of the output fields to compensate for the phase shift introduced by the cavity. Thus we define

$$\hat{x}_1^{\mathrm{obs}}(\omega)=\frac{-\left(\frac{\gamma_a}{2}-i\omega\right)}{\left(\frac{\gamma_a}{2}+i\omega\right)}\hat{x}_1^{\mathrm{OUT}}(\omega) , \tag{2.21a}$$

$$\hat{y}_2^{\mathrm{obs}}(\omega)=\frac{-\left(\frac{\gamma_b}{2}-i\omega\right)}{\left(\frac{\gamma_b}{2}+i\omega\right)}\hat{y}_2^{\mathrm{OUT}}(\omega) . \tag{2.21b}$$

Then

$$\hat{x}_1^{\mathrm{obs}}(\omega)=\hat{x}_1^{\mathrm{IN}}(\omega) , \tag{2.22}$$

$$\hat{y}_2^{\mathrm{obs}}(\omega)=\hat{y}_2^{\mathrm{IN}}(\omega)-\frac{\chi\sqrt{\gamma_a\gamma_b}}{\left(\frac{\gamma_b}{2}+i\omega\right)\left(\frac{\gamma_a}{2}-i\omega\right)}\hat{x}_1^{\mathrm{IN}}(\omega) . \tag{2.23}$$

Equation (2.22) displays the QND nature of the $\hat{X}_1(t)$ quadrature; the corresponding positive frequency component remains unchanged by the interaction. The signal is determined by measuring the output "probe" $\hat{Y}_2^{\mathrm{OUT}}(t)$ or, in the frequency domain, $\hat{y}_2^{\mathrm{OUT}}(\omega)$. The mean signal is

$$\langle\hat{y}_2^{\mathrm{obs}}(\omega)\rangle=\frac{-\chi\sqrt{\gamma_a\gamma_b}}{\left(\frac{\gamma_b}{2}+i\omega\right)\left(\frac{\gamma_a}{2}-i\omega\right)}\langle\hat{x}_1^{\mathrm{IN}}(\omega)\rangle , \tag{2.24}$$

where we have assumed that the probe field is prepared in such a way that $\langle\hat{y}_2^{\mathrm{IN}}(\omega)\rangle=0$. The output quadrature $\hat{Y}_2^{\mathrm{OUT}}(t)$ is measured by heterodyne detection with a local oscillator of carrier frequency ω_b. The noise spectrum of the resulting photoelectron current is determined by[11]

$$V_{y_2}^{\mathrm{obs}}(\omega)=\langle\Delta\hat{y}_2^{\mathrm{obs}}(\omega)^{\dagger}\Delta\hat{y}_2^{\mathrm{obs}}(\omega)\rangle ,$$

where $\Delta\hat{A}\equiv\hat{A}-\langle\hat{A}\rangle$. In this case,

$$V_{y_2}^{\mathrm{obs}}(\omega)=V_{y_2}^{\mathrm{IN}}(\omega)+G^2(\omega)V_{x_1}^{\mathrm{IN}}(\omega) , \tag{2.25}$$

where

$$G^2(\omega)=\frac{\chi^2\gamma_a\gamma_b}{\left(\frac{\gamma_a^2}{4}+\omega^2\right)\left(\frac{\gamma_b^2}{4}+\omega^2\right)} \tag{2.26}$$

is the signal gain squared.

The signal-to-noise ratio is

$$\frac{S}{N} \equiv \frac{|\langle y_2^{\text{obs}}(\omega)\rangle|^2}{V_{y_2}^{\text{obs}}(\omega)}$$

$$= \frac{G^2(\omega)\,|\langle \hat{x}_1^{\text{IN}}(\omega)\rangle|^2}{V_{y_2}^{\text{IN}}(\omega)+G^2(\omega)V_{x_1}^{\text{IN}}(\omega)} \ . \tag{2.27}$$

The gain is a maximum at $\omega=0$, where $G=4\chi\sqrt{\gamma_a\gamma_b}$ in agreement with the narrow band analysis of Yurke, when both modes are equally damped. For sufficiently large gain,

$$\frac{S}{N} \sim \frac{|\langle x_1^{\text{IN}}(\omega)\rangle|^2}{V_{x_1}^{\text{IN}}(\omega)} \ ,$$

thus the observed signal reproduces the statistics of the QND variable. This would be a "good" measurement of the QND variable. If the gain is small a good measurement may still be made by preparing the probe such that

$$V_{y_2}^{\text{IN}}(\omega) \ll 1 \ ,$$

that is, by preparing the probe in a squeezed state.

A measure of the QND coupling is given by the correlation coefficient

$$C \equiv \frac{|\langle x_1^{\text{obs}}(\omega)^\dagger y_2^{\text{obs}}(\omega)\rangle - \langle x_1^{\text{obs}}(\omega)^\dagger\rangle\langle y_2^{\text{obs}}(\omega)\rangle|^2}{V_{y_2}^{\text{obs}}(\omega)V_{x_1}^{\text{obs}}(\omega)}$$

$$= \frac{G^2(\omega)V_{x_1}^{\text{IN}}(\omega)}{V_{y_2}^{\text{IN}}(\omega)+G^2(\omega)V_{x_1}^{\text{IN}}(\omega)} \ .$$

For vacuum input fields

$$C = \frac{G^2(\omega)}{1+G^2(\omega)} \ ,$$

which, for $G \gg 1$, approaches unity indicating perfect correlation. A similarly high correlation is obtained for a squeezed input probe

$$V_{y_2}^{\text{IN}}(\omega) \ll 1 \ .$$

III. FOUR-MODE QND MEASUREMENTS IN OPTICAL FIBERS

A QND measurement has been demonstrated in an optical fiber.[5] The coupling between the QND variable and the probe is due to the optical Kerr effect in an optical fiber. When all relevant four-wave mixing processes in the presence of two strong pump waves are included, the Hamiltonian for this coupling is the four-mode generalization of the two-mode QND schemes considered in Sec. II. The results obtained in the experiment of Levenson *et al.*[5] show a definite correlation between the QND variable and the subsequent optical heterodyne detection of the probe, but the correlation was less than unity. In order to increase the correlation and thus the signal-to-noise ratio, we propose performing the experiment in an optical cavity to enhance the gain.

Six modes of the field must be considered in the description of this QND scheme. There are two strong classical pump waves with amplitudes E_x and E_y at frequencies ω_x and ω_y, respectively, and four coupled sideband modes. Because of the nonlinear index of refraction the slowly varying amplitudes of the pump fields evolve according to

$$\frac{dE_x}{dt} = i(k_x+2k_y)E_x \ ,$$

$$\frac{dE_y}{dt} = i(2k_x+k_y)E_y \ ,$$

where the coupling constants are[13]

$$k_j = \left[\frac{12\pi\omega_j}{n^2}\right] f\chi^{(3)}|E_j|^2$$

($j=x,y$) with n the linear refractive index, f a mode overlap factor of the order of one, and $\chi^{(3)}=5\times10^{-15}$ cm^{3}erg is the third-order nonlinear susceptibility.

The sideband modes are shifted above and below each cavity mode at the pump frequency by the same shift $\delta=m\Omega_c$, where m is an integer and Ω_c is the free spectral range of the cavity. The intracavity field due to the sidebands of the frequency ω_x may be written

$$\hat{E}_x^{(s)} = a_{x+}(t)e^{-i(\omega_x+\delta)t} + a_{x-}(t)e^{-i(\omega_x-\delta)t} + \text{H.c.} \ , \tag{3.1}$$

where $a_x\pm$ are annihilation operators for each sideband in a frame rotating at frequencies $\omega_x\pm\delta$. A similar expression may be written for the field due to the sidebands of ω_y,

$$\hat{E}_y^{(s)} = a_{y+}(t)e^{-i(\omega_y+\delta)t} + a_{y-}(t)e^{-i(\omega_h-\delta)t} + \text{H.c.} \tag{3.2}$$

These sideband fields may be written in terms of quadrature phase amplitudes defined with respect to the pump mode frequencies. For example,

$$\hat{E}_x^{(s)}(t) = \hat{X}_1(t)\cos(\omega_x t) + \hat{X}_2(t)\sin(\omega_x t) \ , \tag{3.3}$$

where

$$\hat{X}_1(t)=\hat{X}_A(t)e^{i\delta t}+\text{H.c.} \ , \tag{3.4a}$$

$$\hat{X}_2(t)=\hat{X}_\phi(t)e^{i\delta t}+\text{H.c.} \ , \tag{3.4b}$$

and the amplitude and phase modulation operators are defined, respectively, as

$$\hat{X}_A(t)=a^{\dagger}_{x^+}(t)+a_{x^-}(t) \ , \tag{3.5a}$$

$$\hat{X}_\phi(t)=-i[a_{x^-}(t)-a^{\dagger}_{x^+}(t)] \ . \tag{3.5b}$$

Similar expressions may be written for the field $E_y^{(s)}(t)$.

In practice it is more convenient to work in the "pump interaction" picture. In this frame the annihilation operators are defined by Eqs. (3.1) and (3.2) with the replacements

$$\omega_x \to \omega_x+(k_x+2k_y) \ ,$$

$$\omega_y \to \omega_y+(2k_x+k_y) \ .$$

In this way we take account of the linear phase shifts induced by the pump fields. In what follows we implicitly assume we are working in this frame.

The commutation relations for the operators defined in Eqs. (3.5a) and (3.5b) are

$$[\hat{X}_\phi,\hat{X}_A^\dagger]=-2i \ , \tag{3.6a}$$

$$[\hat{Y}_\phi,\hat{Y}_a^\dagger]=-2i \ , \tag{3.6b}$$

with all other commutators zero. In reference frame defined above the dynamics of the operators $\hat{X}_A$, $\hat{X}_\phi$, $\hat{Y}_A$, and $\hat{Y}_\phi$ is determined by the Hamiltonian

$$\hat{H}=\hbar\left[k_x\hat{X}_A^\dagger\hat{X}_A+k_y\hat{Y}_A^\dagger\hat{Y}_A+2v(k_xk_y)^{1/2}(\hat{X}_A^\dagger\hat{Y}_A+\hat{X}_A\hat{Y}_A^\dagger)+\frac{\hat{\Gamma}_G}{2}(\hat{X}_A\,|E_x|+\hat{Y}_A\,|E_y|+\text{H.c.})\right.$$
$$\left.+\tfrac{1}{2}(\hat{X}_A\hat{\Gamma}^\dagger_{x_\phi}-\hat{X}_\phi\Gamma^\dagger_{x_A}+\hat{Y}_A\Gamma^\dagger_{y_\phi}+\text{H.c.})\right] \ . \tag{3.7}$$

The operator $\hat{\Gamma}_G$ describes the phase noise produced by light scattering in the fiber,[14] and v is a polarization correlation factor [14] $(1\geq v\geq 0)$. The operators $\hat{\Gamma}_{x_A}$, $\hat{\Gamma}_{x_\phi}$, $\hat{\Gamma}_{y_A}$, and Γ_{y_ϕ} describe the coupling of the sideband modes to the external modes of the cavity. As in Sec. I these operators may be written in terms of the input fields at $\omega_x\pm\delta$ and $\omega_y\pm\delta$

$$\hat{\Gamma}_{j_\phi}(t)\equiv i[\hat{\Gamma}_{j^+}(t)e^{-i(\omega_j+\delta)t}-\hat{\Gamma}_{j^-}(t)e^{i(\omega_j-\delta)t}] \ , \tag{3.8a}$$

$$\hat{\Gamma}_{j_A}(t)\equiv[\hat{\Gamma}_{j^+}(t)e^{-i(\omega_j+\delta)t}+\hat{\Gamma}_{j^-}(t)e^{i(\omega_j-\delta)t}] \ , \tag{3.8b}$$

$$\hat{\Gamma}_{j^\pm}(t)=\kappa_j\left[\frac{2}{\hbar(\omega_j\pm\delta)}\right]^{1/2}a_{j\pm}^{\text{IN}}(t) \tag{3.9}$$

with $j=(x,y)$, and where

$$a_{j\pm}^{\text{IN}}(t)$$

are the positive frequency components of the input field at carrier frequencies $\omega_j\pm\delta$. As in Sec. I,

$$\kappa_j=\sqrt{\gamma_j} \ , \tag{3.10}$$

where $\gamma_x/2$ and $\gamma_y/2$ are the widths of the cavity resonances at $\omega_x\pm\delta$ and $\omega_y\pm\delta$, respectively. A key assumption in the derivation of Eq. (3.7) is that the frequency shifts $\omega_x-\omega_y$ and δ are such that all four-wave mixing processes involving only the six modes indicated are phase matched.

We shall take $\hat{X}_A$ as the QND variable of the signal which we measure by making measurements on $\hat{Y}_\phi$. Thus we regard the sideband field at ω_y as the probe field.

The quantum stochastic differential equations are

$$\frac{d}{dt}\begin{bmatrix}\hat{X}_A\\ \hat{Y}_\phi\\ \hat{Y}_A\\ \hat{X}_\phi\end{bmatrix}=\begin{bmatrix}-\gamma_a/2 & 0 & 0 & 0\\ -4vk & -\gamma_b/2 & -2k_y & 0\\ 0 & 0 & -\gamma_b/2 & 0\\ -2k_x & 0 & -4vk & -\gamma_a/2\end{bmatrix}$$
$$\times\begin{bmatrix}\hat{X}_A\\ \hat{Y}_\phi\\ \hat{Y}_A\\ \hat{X}_\phi\end{bmatrix}-\begin{bmatrix}\kappa_x\hat{X}_A^{\text{IN}}\\ \kappa_y\hat{Y}_\phi^{\text{IN}}+\hat{\Gamma}_G\,|E_y|\\ \kappa_y\hat{Y}_A^{\text{IN}}\\ \kappa_x\hat{X}_\phi^{\text{IN}}+\hat{\Gamma}_G\,|E_x|\end{bmatrix} , \tag{3.11}$$

where we have defined $k\equiv\sqrt{k_xk_y}$. Note that we have assumed that the carrier frequencies of all external modes are resonant with the appropriate cavity mode. Equation (3.11) is of the form

$$\frac{d}{dt}\hat{Z}(t)=-M\hat{Z}(t)+\hat{B} \ . \tag{3.12}$$

To solve these equations we proceed as in Sec. II by writing $\hat{Z}(t)$ in terms of positive and negative frequency components

$$Z(t)=\int_0^\infty\frac{d\omega}{2\pi}[z(\omega)e^{-i\omega t}+z^\dagger(\omega)e^{i\omega t}] \ . \tag{3.13}$$

We then find

$$\hat{z}(\omega)=(M-i\omega)^{-1}b(\omega) \ ,$$

where $\hat{b}(\omega)$ is the positive frequency component of $B(t)$, and

$$(M-i\omega)^{-1}=\frac{1}{D}\begin{bmatrix} \left[\frac{\gamma_a}{2}-i\omega\right]\left[\frac{\gamma_b}{2}-i\omega\right]^2 & 0 & 0 & 0 \\ -4vk\left[\frac{\gamma_a}{2}-i\omega\right]\left[\frac{\gamma_b}{2}-i\omega\right] & \left[\frac{\gamma_a}{2}-i\omega\right]^2\left[\frac{\gamma_b}{2}-i\omega\right] & -2k_y\left[\frac{\gamma_a}{2}-i\omega\right]^2 & 0 \\ 0 & 0 & \left[\frac{\gamma_a}{2}-i\omega\right]^2\left[\frac{\gamma_b}{2}-i\omega\right] & 0 \\ -2k_x\left[\frac{\gamma_b}{2}-i\omega\right]^2 & 0 & -4vk\left[\frac{\gamma_a}{2}-i\omega\right]\left[\frac{\gamma_b}{2}-i\omega\right] & \left[\frac{\gamma_b}{2}-i\omega\right]^2\left[\frac{\gamma_a}{2}-i\omega\right] \end{bmatrix}, \tag{3.14}$$

where

$$D=\left[\frac{\gamma_a}{2}-i\omega\right]^2\left[\frac{\gamma_b}{2}-i\omega\right]^2 .$$

Using the relations between the input and output fields,

$$\hat{z}^{\,\mathrm{OUT}}(\omega)=\begin{bmatrix}\kappa_x & 0 & 0 & 0\\ 0 & \kappa_y & 0 & 0\\ 0 & 0 & \kappa_y & 0\\ 0 & 0 & 0 & \kappa_x\end{bmatrix}\hat{z}(\omega)+\hat{z}^{\,\mathrm{IN}}(\omega) , \tag{3.15}$$

we find

$$x_A^{\mathrm{OUT}}(\omega)=\left[\frac{\frac{\gamma_a}{2}+i\omega}{\frac{\gamma_a}{2}-i\omega}\right]x_A^{\mathrm{IN}}(\omega) , \tag{3.16}$$

$$\hat{y}_\phi^{\,\mathrm{OUT}}(\omega)=-\left[\frac{\frac{\gamma_b}{2}+i\omega}{\frac{\gamma_b}{2}-i\omega}\right]\hat{y}_\phi^{\,\mathrm{IN}}(\omega) -\frac{4vk\sqrt{\gamma_a\gamma_b}}{\left[\frac{\gamma_a}{2}-i\omega\right]\left[\frac{\gamma_b}{2}-i\omega\right]}\hat{x}_A^{\,\mathrm{IN}}(\omega) -\frac{2k_y\gamma_b}{\left[\frac{\gamma_b}{2}-i\omega\right]^2}\hat{y}_A^{\,\mathrm{IN}}(\omega) -\frac{\sqrt{\gamma_b}\,\hat{\Gamma}_G(\omega)\,|E_y|}{\left[\frac{\gamma_b}{2}-i\omega\right]} . \tag{3.17}$$

Again it is necessary to shift the phase of the output fields. The observed variables are then given by

$$\hat{x}_A^{\,\mathrm{obs}}(\omega)=\hat{x}_A^{\,\mathrm{IN}}(\omega) , \tag{3.18}$$

$$\hat{y}_\phi^{\,\mathrm{obs}}(\omega)=\hat{y}_\phi^{\,\mathrm{IN}}(\omega)-\frac{4vk\sqrt{\gamma_a\gamma_b}\,\hat{x}_A^{\,\mathrm{IN}}(\omega)}{\left[\frac{\gamma_a}{2}-i\omega\right]\left[\frac{\gamma_b}{2}+i\omega\right]} -\frac{2k_y\gamma_b}{\left[\frac{\gamma_b^2}{4}+\omega^2\right]}\hat{y}_A^{\,\mathrm{IN}}(\omega)-\frac{\sqrt{\gamma_b}\,\hat{\Gamma}_G(\omega)\,|E_x|}{\left[\frac{\gamma_b}{2}+i\omega\right]} . \tag{3.19}$$

Information on the signal quadrature $\hat{X}_A(t)$ is extracted by heterodyne detection of the probe quadrature $\hat{Y}_\phi(t)$. The resulting noise spectrum is determined by

$$V_{y_\phi}^{\mathrm{obs}}(\omega)=V_{y_\phi}^{\mathrm{IN}}(\omega)+\frac{(4vk)^2\gamma_a\gamma_b}{\left[\frac{\gamma_a^2}{4}+\omega^2\right]\left[\frac{\gamma_b^2}{4}+\omega^2\right]}V_{x_A}^{\mathrm{IN}}(\omega) +\frac{4k_y^2\gamma_b^2}{\left[\frac{\gamma_b^2}{4}+\omega^2\right]^2}V_{y_A}^{\mathrm{IN}}(\omega) +\frac{\gamma_b|E_y|^2}{\left[\frac{\gamma_b^2}{4}+\omega^2\right]}\langle\hat{\Gamma}_G^{\dagger}(\omega)\hat{\Gamma}_G(\omega)\rangle . \tag{3.20}$$

From this expression we may compare the different contributions to the variance. The contribution from the QND signal is

$$V^{\mathrm{QND}}(\omega)=\frac{(4vk)^2\gamma_a\gamma_b}{\left[\frac{\gamma_a^2}{4}+\omega^2\right]\left[\frac{\gamma_b^2}{4}+\omega^2\right]}V_{x_A}^{\mathrm{IN}}(\omega) . \tag{3.21}$$

The contribution from the squeezing of the probe is

$$V^{SQ}(\omega)=\frac{4k_y^2\gamma_b^2}{\left[\frac{\gamma_b^2}{4}+\omega^2\right]^2}V_{y_A}^{IN}(\omega)\ , \tag{3.22}$$

and the contribution from phase noise (GAWBS) is

$$V^{G}(\omega)=\frac{\gamma_b\,|E_x|^2\beta_G}{\frac{\gamma_b^2}{4}+\omega^2}\ , \tag{3.23}$$

where $\beta_G=\langle\hat\Gamma_G^\dagger(\omega)\hat\Gamma_G(\omega)\rangle$ is a measure of the noise spectrum of the GAWBS, assumed to be relatively independent of frequency. The relative contribution of the QND signal to the other effects is

$$\frac{V^{QND}}{V^G}=\frac{1}{\beta_G}\left[\frac{4vk}{|E_x|}\right]^2\left[\frac{\gamma_a}{\frac{\gamma_a^2}{4}+\omega^2}\right]V_{x_A}^{IN}(\omega)\ , \tag{3.24}$$

$$\frac{V^{QND}}{V^{SQ}}=\frac{\gamma_a}{\gamma_b}\left[\frac{\frac{\gamma_b^2}{4}+\omega^2}{\frac{\gamma_a^2}{4}+\omega^2}\right]\left[\frac{4vk}{2k_y}\right]^2\frac{V_{x_A}^{IN}(\omega)}{V_{y_A}^{IN}(\omega)}\ . \tag{3.25}$$

For $\omega=0$

$$\frac{V^{QND}}{V^G}\propto\frac{1}{\gamma_a}\ , \tag{3.26}$$

$$\frac{V^{QND}}{V^{SQ}}\propto\frac{\gamma_b}{\gamma_a}\ . \tag{3.27}$$

Thus the QND signal relative to both the GAWBS noise and the squeezing contribution can be enhanced by making $\gamma_a\ll\gamma_b$. Thus it is advantageous to use a cavity with a high finesse for the signal mode. A measure of the QND effect is given by the correlation coefficient

$$C=\frac{|\langle\Delta\hat x_A^{obs}(\omega)^\dagger\Delta\hat y_\phi^{obs}(\omega)\rangle|^2}{V_{y_\phi}^{obs}(\omega)V_{x_A}^{obs}(\omega)}\ ,$$

which, for the probe beam in a coherent state, is given by

$$C=\left[1+\frac{(4vk)^2\gamma_a\gamma_b V_{x_A}^{IN}(\omega)}{\left[\frac{\gamma_a^2}{4}+\omega^2\right]\left[\frac{\gamma_b^2}{4}+\omega^2\right]}+\frac{4k_y^2\gamma_b^2}{\left[\frac{\gamma_b^2}{4}+\omega^2\right]^2}+\frac{\gamma_b\beta_G^2\,|E_x|^2}{\left[\frac{\gamma_b^2}{4}+\omega^2\right]}\right]^{-1}\frac{(4vk)^2\gamma_a\gamma_b V_{x_A}^{IN}(\omega)}{\left[\frac{\gamma_a^2}{4}+\omega^2\right]\left[\frac{\gamma_b^2}{4}+\omega^2\right]}\ . \tag{3.28}$$

When $\gamma_a\ll\gamma_b$ this approaches

$$C\simeq\frac{G^{QND}(\omega)V_{x_A}^{IN}(\omega)}{1+G^{QND}(\omega)V_{x_A}^{IN}(\omega)}\ , \tag{3.29}$$

where the QND gain is

$$G^{QND}(\omega)=\frac{(4vk)^2\gamma_a\gamma_b}{\left[\frac{\gamma_a^2}{4}+\omega^2\right]\left[\frac{\gamma_b^2}{4}+\omega^2\right]}\ . \tag{3.30}$$

Thus for sufficiently large gain $C\to1$, indicating a good QND measurement of $\hat X_A(t)$ is possible.

IV. QND MEASUREMENT OF PHOTON NUMBER

Various schemes to make a QND measurement of the photon number $\hat n\equiv a^\dagger a$ of a field mode have been suggested.[6,7] These schemes make use of the optical Kerr effect whereby the photon number of a signal mode causes a phase shift of a probe beam. We wish to describe such a measurement in the cavity configuration.

We take the Hamiltonian to be

$$\hat H(t)=\hbar\omega_a b^\dagger b+\hbar\omega_a a^\dagger a+\hbar\chi[4a^\dagger ab^\dagger b+(a^\dagger a)^2+(b^\dagger b)^2]+i\hbar\kappa[b\hat\Gamma_b^\dagger(t)-b^\dagger\hat\Gamma_b(t)]\ . \tag{4.1}$$

Note that we are assuming that only the probe cavity mode, represented by the operator b, is coupled out of the cavity to a probe input field. Let this probe input field have a carrier frequency ω_p not necessarily equal to ω_b, the resonant frequency of the nearest cavity mode. In a frame rotating at frequency ω_p the dynamics is described by the Hamiltonian

$$\hat H(t)=\hbar\Delta b^\dagger b+\hbar\chi[4a^\dagger ab^\dagger b+(a^\dagger a)^2+(b^\dagger b)^2]+i\hbar\kappa[b\hat\Gamma_b^\dagger(t)e^{-i\omega_p t}-b^\dagger\hat\Gamma_b(t)e^{i\omega_p t}]\ , \tag{4.2}$$

where $\Delta=\omega_b-\omega_p$. As $\hat n\equiv a^\dagger a$ is a constant of the motion we take it to be the signal QND variable. We point out that the interaction term in Eq. (4.2) contains the additional terms $(a^\dagger a)^2$ and $(b^\dagger b)^2$ which were neglected in previous treatments.[6,7] These terms give rise to self-phase modulation of the probe and signal.

As in previous sections we write $\hat\Gamma(t)$ in terms of the positive frequency components of the input field

$$\hat\Gamma_b(t)=\left[\frac{2}{\hbar\omega_p}\right]^{1/2}b^{IN}(t)\ . \tag{4.3}$$

We also define the input quadrature phase amplitudes by

$$\hat Y_1^{IN}(t)\equiv\left[\frac{2}{\hbar\omega_p}\right]^{1/2}[b^{IN}(t)e^{i\omega_p t}+\text{H.c.}]\ , \tag{4.4a}$$

$$\hat Y_2^{IN}(t)\equiv-i\left[\frac{2}{\hbar\omega_p}\right]^{1/2}[b^{IN}(t)e^{i\omega_p t}-\text{H.c.}]\ . \tag{4.4b}$$

The quantum stochastic differential equations are

$$\frac{d\hat{Y}_1}{dt}=(\Delta+4\chi\hat{n}_a)\hat{Y}_2+\chi(\hat{Y}_2\hat{n}_b+\hat{n}_b\hat{Y}_2)-\frac{\gamma}{2}\hat{Y}_1-\kappa\hat{Y}_1^{\rm IN}, \tag{4.5a}$$

$$\frac{d\hat{Y}_2}{dt}=-(\Delta+4\chi\hat{n}_a)\hat{Y}_2-\chi(\hat{Y}_1\hat{n}_b+\hat{n}_b\hat{Y}_1)-\frac{\gamma}{2}\hat{Y}_2-\kappa\hat{Y}_2^{\rm IN}, \tag{4.5b}$$

where $n_b\equiv b^\dagger b$. In order to make progress these nonlinear equations must be linearized at least to first order in χ. In the absence of the signal mode the intensity of the cavity mode at frequency ω_b has a bistable steady state with the state equation[11]

$$\gamma I_p=I_c\left[\frac{\gamma^2}{4}+(\Delta+2\chi I_c)^2\right], \tag{4.6}$$

where I_c is the steady state intracavity intensity in units of photon number, and I_p is the input intensity of the probe in units of photon number per second. We now assume that Eqs. (4.5a) and (4.5b) may be approximated by the linear equations that result when n_b is replaced by I_c, the steady-state intracavity intensity. Thus

$$\frac{d\hat{Y}_1(t)}{dt}=(\Delta+2\chi I_c+4\chi\hat{n}_a)\hat{Y}_2(t)-\frac{\gamma}{2}\hat{Y}_1(t)-\kappa\hat{Y}_1^{\rm IN}(t), \tag{4.7a}$$

$$\frac{d\hat{Y}_2(t)}{dt}=-(\Delta+2\chi I_c+4\chi\hat{n}_a)\hat{Y}_1(t)-\frac{\gamma}{2}\hat{Y}_2(t)-\kappa\hat{Y}_2^{\rm IN}(t). \tag{4.7b}$$

Let us further assume that the detuning Δ is chosen to maximize the intracavity intensity at frequency ω_b, that is, we choose $\Delta=-2\chi I_c$. We then expand $Y_i(t)$ in their positive and negative frequency components to obtain

$$\left[\frac{\gamma}{2}-i\omega\right]\hat{y}_1(\omega)=-4\chi\hat{n}_a\hat{y}_2(\omega)-\kappa\hat{y}_1^{\rm IN}(\omega), \tag{4.8a}$$

$$\left[\frac{\gamma}{2}-i\omega\right]\hat{y}_2(\omega)=4\chi\hat{n}_a\hat{y}_1(\omega)-\kappa\hat{y}_2^{\rm IN}(\omega). \tag{4.8b}$$

Solving Eqs. (4.8a) and (4.8b) to linear order in χ and using

$$\kappa\hat{y}_i(\omega)=[\hat{y}_i^{\rm OUT}(\omega)-\hat{y}_1^{\rm IN}(\omega)], \tag{4.9}$$

we find

$$\hat{y}_1^{\rm obs}(\omega)=\hat{y}_1^{\rm IN}(\omega)-G(\omega)\hat{n}_a\hat{y}_2^{\rm IN}(\omega), \tag{4.10a}$$

$$\hat{y}^{\rm obs}(\omega)=\hat{y}_2^{\rm IN}(\omega)+G(\omega)\hat{n}_a\hat{y}^{\rm IN}(\omega), \tag{4.10b}$$

where $\hat{y}_i^{\rm obs}$ is related to $\hat{y}_i^{\rm OUT}$ by a phase shift and

$$G(\omega)=\frac{4\gamma\chi}{\frac{\gamma^2}{4}+\omega^2} \tag{4.11}$$

is the QND gain. The quantity $\hat{y}^{\rm OUT}(t)$ is measured by heterodyne detection with a local oscillator at the carrier frequency ω_p. The frequency components of the resulting signal are determined by $\hat{y}_i^{\rm obs}(\omega)$. The condition for maximum gain $\omega=0$ corresponds to the homodyne beat with the local oscillator.

Let us assume that the detection process is arranged to respond to $\hat{y}_1^{\rm obs}(\omega)$ and further that the input field is prepared in a state such that

$$\langle\hat{y}_1^{\rm IN}(\omega)\rangle=0, \tag{4.12a}$$

$$\langle\hat{y}_2^{\rm IN}(\omega)\rangle=A(\omega), \tag{4.12b}$$

with $A(0)\gg1$. This corresponds to an input state with a large coherent amplitude at the carrier frequency ω_p. Then

$$\langle\hat{y}_1^{\rm obs}(\omega)\rangle=-G(\omega)A(\omega)\langle\hat{n}_a\rangle. \tag{4.13}$$

The noise in the heterodyne is determined by

$$\begin{aligned}V_{y_1}^{\rm obs}(\omega)&=\langle\Delta\hat{y}_1^{\rm obs}(\omega)^\dagger\Delta\hat{y}_1^{\rm obs}(\omega)\rangle\\&=G^2(\omega)\langle\hat{n}_a^2\rangle V_{y_2}^{\rm IN}(\omega)\\&\quad+G^2(\omega)A^2(\omega)V(\hat{n}_a)+V_{y_1}^{\rm IN}(\omega)\\&\quad-G(\omega)\langle\hat{n}_a\rangle[\langle\Delta\hat{y}_2^{\rm IN}(\omega)^\dagger\Delta\hat{y}_1^{\rm IN}(\omega)\rangle+{\rm c.c.}].\end{aligned} \tag{4.14}$$

If the input field has time stationary quadrature phase noise (e.g., a squeezed, coherent, or thermal state) the last term in Eq. (4.14) is zero. Thus

$$V_{y_1}^{\rm obs}(\omega)=G^2(\omega)\langle\hat{n}_a^2\rangle V_{y_2}^{\rm IN}(\omega)+G^2(\omega)A^2(\omega)V(\hat{n}_a)+V_{y_1}^{\rm IN}(\omega). \tag{4.15}$$

If we assume that $G(0)A(0)\gg1$ then the signal-to-noise ratio at maximum gain ($\omega=0$) is

$$\frac{S}{N}=\frac{\langle n_a\rangle^2}{V(n_a)},$$

thus realizing a good QND measurement. Note that in the cavity configuration the effective QND gain is given by $G^{\rm EF}(\omega)=A(\omega)G(\omega)$; thus the cavity response enhances the gain induced by the coherent component on the input probe. In this case there is no real advantage in preparing the probe in a squeezed state as both probe quadrature variances contribute to the noise in Eq. (4.15).

V. CONCLUSION

We have given a broadband analysis of three QND measurement schemes: (i) an ideal two-mode parametric scheme, (ii) a four-mode scheme based on a third-order nonlinearity in a fiber, and (iii) a photon number scheme based on the optical Kerr effect. The analysis shows that cavities may be used to enhance the QND gain. We have also shown how squeezed states may be used to improve the performance of the first and second schemes. In the four-mode QND scheme we have shown that it is advantageous to use a cavity for which the signal finesse is much larger than that for the probe. This enables the QND gain to dominate GAWBS noise and interference from squeezing of the probe beam.

APPENDIX

In this appendix we briefly describe the derivation of the stochastic differential equations (2.10a), (2.10b), (2.12a), and (2.12b). Further details may be found in Refs. 8 and 9.

Define the integrated field operators by

$$\hat{A}(t) \equiv \int_0^t \hat{\Gamma}_a(s) e^{i\omega_a s} ds \ , \tag{A1a}$$

$$\hat{B}(t) \equiv \int_0^t \hat{\Gamma}_b(s) e^{i\omega_b s} ds \ , \tag{A1b}$$

and the Ito differentials

$$d\hat{A}(t) \equiv \hat{A}(t+dt) - \hat{A}(t) \ , \tag{A2a}$$

$$d\hat{B}(t) \equiv \hat{B}(t+dt) - \hat{B}(t) \ . \tag{A2b}$$

Using Eq. (2.9) (and the approximation implicit in this result) one finds that $\hat{A}(t)$ and $\hat{B}(t)$ obey the following commutation relations:

$$[\hat{A}(t), \hat{A}^{\dagger}(t')] = [\hat{B}(t), \hat{B}^{\dagger}(t')] = \min(t,t') \ . \tag{A3}$$

Thus

$$[d\hat{A}(t), d\hat{A}^{\dagger}(t')] = [d\hat{B}(t), d\hat{B}^{\dagger}(t')] = dt \ . \tag{A4}$$

Following Ref. 9 the output operators $\hat{A}^{\mathrm{OUT}}(t)$ and $\hat{B}^{\mathrm{OUT}}(t)$ are defined as the Heisenberg form of the input operators $\hat{A}(t)$ and $\hat{B}(t)$, for example,

$$\hat{A}^{\mathrm{OUT}}(t) = U^{\dagger}(t)\hat{A}(t)U(t) \ , \tag{A5}$$

where $U(t)$ is the time evolution operator determined by the Hamiltonian in Eq. (2.4) [see Eq. (2.15) of Ref. 9.] Thus

$$d\hat{A}^{\mathrm{OUT}}(t) = U^{\dagger}(t)[U^{\dagger}(dt)\hat{A}(t)U(dt) - \hat{A}(t)]U(t) \ , \tag{A6}$$

where

$$U(dt) = \exp\left[-\frac{i}{\hbar}\hat{H}_0 dt + \kappa_a[a\, d\hat{A}^{\dagger}(t) - \mathrm{H.c.}] + \kappa_b[b\, d\hat{B}^{\dagger}(t) - \mathrm{H.c.}]\right] , \tag{A7}$$

where H_0 is the free Hamiltonian of the coupled internal fields. Expanding the exponential in Eq. (A7) to second order in κ_a and κ_b (and thus to first order in dt) and using Eq. (A4) we find

$$d\hat{A}^{\mathrm{OUT}}(t) = \kappa_a a(t)dt + d\hat{A}^{\mathrm{IN}}(t) \ , \tag{A8}$$

where $a(t) \equiv U^{\dagger}(t)aU(t)$. Equation (A8) is equivalent to

$$\hat{X}_1^{\mathrm{OUT}}(t) = \kappa_a \hat{X}_1(t) + \hat{X}_1^{\mathrm{IN}}(t) \tag{A9}$$

when written in terms of the quadrature phase operators. Equation (2.12b) is derived in a similar way.

To derive Eq. (2.10a) we begin with

$$da(t) = U^{\dagger}(t)[U^{\dagger}(dt)aU(dt) - a]U(t) \tag{A10}$$

and once again expand $U(dt)$ to second order in κ_a and κ_b. The result is

$$da(t) = \frac{i}{\hbar}[\hat{H}_0, a(t)]dt - \frac{\kappa_a^2}{2}a(t)dt - \kappa_a d\hat{A}(t) \ , \tag{A11}$$

which is equivalent to Eq. (2.10a). Equation (2.10b) is found in a similar way.

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[1]C. M. Caves, K. S. Thorne, R. W. P. Drever, V. D. Sandberg, and M. Zimmerman, Rev. Mod. Phys. **52**, 341 (1980).
[2]G. J. Milburn, A. S. Lane, and D. F. Walls, Phys. Rev. A **27**, 2804 (1983).
[3]M. Hillery and M. O. Scully, Phys. Rev. D **25**, 3137 (1982).
[4]B. Yurke, J. Opt. Soc. Am. **B2**, 732 (1985).
[5]M. D. Levenson, R. M. Shelby, M. Reid, and D. F. Walls, Phys. Rev. Lett. **57**, 2473 (1986).
[6]G. J. Milburn and D. F. Walls, Phys. Rev. A **28**, 2065 (1983).
[7]N. Imoto, H. A. Haus, and Y. Yamomoto, Phys. Rev. A **32**, 2287 (1985).
[8]C. W. Gardiner and M. J. Collett, Phys. Rev. A **31**, 3761 (1985).
[9]A. Barchielli, Phys. Rev. A **34**, 1642 (1986).
[10]G. J. Milburn, Phys. Rev. A **36**, 5271 (1987).
[11]G. J. Milburn, M. D. Levenson, R. M. Shelby, and D. F. Walls, J. Opt. Soc. Am. B **4**, 1476 (1987).
[12]C. M. Caves and B. L. Schumaker, Phys. Rev. A **31**, 3068 (1985); **31**, 3093 (1985).
[13]M. D. Levenson, in *Chemical Applications of Nonlinear Raman Spectroscopy*, edited by A. M. Harvey (Academic, New York, 1981).
[14]R. M. Shelby, M. D. Levenson, S. A. Perlmutter, R. G. De Voe, and D. F. Walls, Phys. Rev. Lett. **57**, 691 (1985).