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Uniform semiclassical approximation to supersymmetric quantum mechanics

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Recently, A. Comtet, A. D. Bandrauk, and D. K. Campbell [Phys. Lett. **150**, 159 (1985)] introduced a modified semiclassical quantization rule motivated by the structure of supersymmetric quantum mechanics. In this paper, following their modified semiclassical framework, we provide a method to obtain wave functions in a uniform approximation. We demonstrate our method by constructing wave functions for the Morse, Rosen-Morse, and anharmonic oscillator potentials. Accurate fits to the wave functions are found for all three potentials.

I. INTRODUCTION

Until very recently, the use of symmetry concepts was restricted to transformations only between bosons or only between fermions. On the other hand, in systems with both kinds of particles, there are a number of symmetry operations that can transform bosons into fermions and vice versa. A natural description of the underlying symmetries of such mixed systems is provided by the concept of supersymmetry.[1] Because of the unified description it provides, there have been many searches for physical manifestations of supersymmetry.[2–5] These investigations show that when supersymmetry is realized in nature, it is generally broken to some extent. To simulate supersymmetry breaking in field theories, Witten introduced supersymmetric quantum mechanics.[6] This formalism investigates Hamiltonians which are squares of conserved, anticommuting charges. Supersymmetric quantum mechanics has been extensively studied in the past five years.[7–21]

Insight into the quantization of supersymmetric Hamiltonians is provided by the use of semiclassical methods.[22–24] Applications to date, however, have tended to concentrate on the energy spectra whereas wave functions provide a more stringent test of quantization procedures. Wave functions are also needed for the calculation of transition rates. The purpose of this article is to provide a method for obtaining accurate semiclassical wave functions of supersymmetric Hamiltonians. This will be achieved in a uniform approximation.

A supersymmetric quantum mechanical system is described by the Hamiltonian[6]

$$H=-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\phi^2(x)-\frac{\hbar}{\sqrt{2m}}[\theta^*,\theta]\phi'(x)\ , \tag{1a}$$

where $x(t)$ is a bosonic variable, $\theta(t)$ is its fermionic partner, and $\phi(x)$ is related to the ground-state wave function as illustrated below.

Introducing the matrix representation $\theta=\sigma_+$ and $\theta^*=\sigma_-$, Eq. (1a) can be rewritten as a two-component system with the Hamiltonians[6]

$$\begin{aligned}H_\pm&=-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\phi^2(x)\pm\frac{\hbar}{\sqrt{2m}}\phi'(x)\\&\equiv-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V_\pm(x)\ .\end{aligned} \tag{1b}$$

Introducing the operators

$$B^\dagger=\frac{\hbar}{\sqrt{2m}}\frac{\partial}{\partial x}+\phi(x) \tag{2a}$$

and

$$B=-\frac{\hbar}{\sqrt{2m}}\frac{\partial}{\partial x}+\phi(x)\ , \tag{2b}$$

H_+ and its supersymmetric partner H_- can be written as

$$H_+=B^\dagger B,\quad H_-=BB^\dagger\ . \tag{3}$$

The two-component system can be explicitly defined by introducing

$$Q=\begin{bmatrix}0&0\\B&0\end{bmatrix},\quad Q^\dagger=\begin{bmatrix}0&B^\dagger\\0&0\end{bmatrix} \tag{4}$$

with which the supersymmetric Hamiltonian takes the form

$$H=\{Q,Q^\dagger\}=\begin{bmatrix}H_+&0\\0&H_-\end{bmatrix}\ . \tag{5}$$

Q and $Q^\dagger$ are conserved charges

$$[H,Q]=[H,Q^\dagger]=0\ . \tag{6}$$

The eigenstates of H can be written as the column vector

$$\psi=\begin{bmatrix}\psi^{(+)}\\\psi^{(-)}\end{bmatrix} \tag{7}$$

which satisfies

$$H\psi=E\psi\ . \tag{8}$$

From Eqs. (2) and (8), one obtains

$$B^{\dagger}\psi^{(-)}=\sqrt{E}\,\psi^{(+)}\ , \tag{9a}$$

$$B\psi^{(+)}=\sqrt{E}\,\psi^{(-)}\ . \tag{9b}$$

The special form of Eq. (1b) follows from general factorization techniques[25] for an arbitrary one-dimensional Hamiltonian $\tilde{H}$. In fact, using the exact ground-state energy E_0, and the exact ground-state wave function ψ_0 of $\tilde{H}$, the superpotential $\phi(x)$ can be expressed as[26]

$$\phi(x)=-\frac{\hbar}{\sqrt{2m}}[d\psi_0(x)/dx][\psi_0(x)]^{-1} \tag{10}$$

so that

$$\tilde{H}-E_0=H_-\ . \tag{11}$$

We note that Eq. (10) can be rewritten as

$$\psi_0^{(-)}(x)\sim\exp\left[-\frac{\sqrt{2m}}{\hbar}\int\phi(x)dx\right] \tag{12}$$

and thus H_- has a normalizable ground state.

In Sec. II we first review the primitive semiclassical approximation to the wave functions of the supersymmetric quantum mechanics as presented by Comtet, Bandrauk, and Campbell.[22] Next we summarize the technique of Miller and Good[27] to construct the uniform semiclassical approximation to the wave functions. Finally we introduce the uniform semiclassical approximation to the wave functions of supersymmetric quantum mechanics. In Sec. III we apply our results from Sec. II to the Morse potential, the anharmonic oscillator, and the Rosen-Morse potential. Section IV contains a discussion of our results and conclusions.

II. SEMICLASSICAL APPROXIMATIONS TO SUPERSYMMETRIC QUANTUM MECHANICS

A. Primitive semiclassical approximation to wave functions—supersymmetric Wentzel-Kramers-Brillouin

In this section the eigenfunctions of the Hamiltonian in Eq. (1b) are constructed using the Wentzel-Kramers-Brillouin (WKB) method.[28] In this approximation the solution of the differential equation

$$(H_- -E)\psi^{(-)}=0 \tag{13}$$

is given by

$$\psi^{(-)}(x)=\exp\left[\pm\frac{\sqrt{2m}}{\hbar}\int\left[E-\phi^2+\frac{\hbar}{\sqrt{2m}}\phi'\right]^{1/2}dx\right.$$
$$\left.+\frac{i}{4}\ln\left[2m\frac{E-\phi^2+\frac{\hbar}{\sqrt{2m}}\phi'}{\hbar^2}\right]\right]\ . \tag{14}$$

Except for the harmonic oscillator in any number of dimensions (and problems that can be reduced to it), the function $\phi(x)$ depends on $\hbar$ in a nontrivial way (cf. Sec. III). Nevertheless, as was argued in Ref. 22, one can formally count the powers of $\hbar$ in Eq. (14), expecting the complicated $\hbar$ dependence of ϕ to cancel higher-order corrections, at least to the energy eigenvalues. In this article, we want to see if a similar reasoning can be extended to the calculation of the eigenfunctions. Proceeding in this manner and keeping terms only to first order in $\hbar$ in Eq. (14), one obtains

$$\ln\psi^{(-)}(x)\sim\pm\frac{\sqrt{2m}}{\hbar}\int(E-\phi^2)^{1/2}dx$$
$$\pm\frac{1}{2}\int\frac{\phi'}{(E-\phi^2)^{1/2}}dx+\frac{i}{4}\ln\left[2m\left[\frac{E-\phi^2}{\hbar^2}\right]\right]$$
$$+\frac{i}{4}\frac{\hbar\phi'}{\sqrt{2m}(E-\phi^2)}+O(\hbar^2)\ . \tag{15}$$

Considering the formal order of each term with respect to $\hbar$ in Eq. (15), the primitive semiclassical approximation to ψ_-, correct within a factor of $\exp(\hbar)$, is given as

$$\psi^{(-)}(x)\sim[\phi^2(x)-E]^{-1/4}\{|\phi(x)|+[\phi^2(x)-E^{1/2}]\}^{1/2}$$
$$\times\exp\left[\pm\frac{\sqrt{2m}}{\hbar}\int^x[\phi^2(y)-E]^{1/2}dy\right] \tag{16a}$$

for the classically forbidden region and

$$\psi^{(-)}(x)\sim[E-\phi^2(x)]^{-1/4}\exp\left[\frac{i}{2}\sin^{-1}\left[\frac{\phi(x)}{\sqrt{E}}\right]\right]$$
$$\times\exp\left[\pm\frac{i\sqrt{2m}}{\hbar}\int^x[E-\phi^2(y)]^{1/2}dy\right] \tag{16b}$$

for the classically allowed region.[22] The physically acceptable solution in Eq. (16a) is the exponentially decreasing one, and in Eq. (16b) the combination that behaves like a sine function.[29]

The formal expansion in Eq. (15) motivates the omission of the additional term $\hbar\phi'$ in the pre-exponential factor $(\phi^2-E)^{-1/4}$. However, retaining the term proportional to ϕ' in the second phase integral in Eq. (15) gives rise to additional pre-exponential factors which do not occur in conventional WKB. It is precisely these terms which lead to the modified quantization condition[22]

$$\sqrt{2m}\int_a^b[E-\phi^2(y)]^{1/2}dy=n\pi\hbar\ , \tag{17}$$

where $\phi^2(a)=\phi^2(b)=E$. Indeed, the second phase integral in Eq. (15), which contains ϕ', is the contribution of the fermionic sector since it originates from the last term in Eq. (1a). Furthermore, this phase integral calculated between the turning points a and b yields

$$\frac{\hbar}{2}\int_a^b\frac{\phi'(y)}{\sqrt{E-\phi^2(y)}}dy=\frac{\pi\hbar}{2}\ , \tag{18}$$

which is exactly equal to the boson zero-point energy, thus canceling its contribution.

B. Uniform semiclassical approximation to wave functions

Since the primitive semiclassical wave functions in Eqs. (16) diverge at the turning points, they are not useful for practical calculations. Therefore, a method which gives approximate wave functions that are uniformly valid at the turning points and away from them is needed. The uniform approximation of Miller and Good[27] is appropriate to this problem. A brief outline of the application of this method to the bound-state problem is given here.

The method involves approximating the solutions of

$$-\hbar^2\frac{d^2}{dx^2}\Psi(x)-2m[E-V(x)]\Psi(x)=0 \tag{19}$$

in terms of the known solutions to a similar equation. In the present case, the bound-state eigenfunctions of a potential well $V(x)$ with two distinct turning points are to be found. To apply the method of Ref. 27, one needs to choose an equation containing an auxiliary potential (mapping function) with the same topology. An appropriate choice is

$$-\hbar^2\frac{d^2}{dS^2}U(S)-(\Gamma-S^2)U(S)=0 \ . \tag{20}$$

This is just the Schrödinger equation for a one-dimensional oscillator and has the solutions

$$U(S)\sim H_n(S/\sqrt{\hbar})\exp(-\tfrac{1}{2}S^2/\hbar) \tag{21}$$

with $(2n+1)\hbar=\Gamma$. In Eqs. (20) and (21), the variable S is a function of x. Its x dependence is determined by requiring that the solution to Eq. (19) be of the form

$$\Psi(x)\sim T(x)U[S(x)] \ . \tag{22}$$

Substituting Eq. (22) into Eq. (19) and choosing

$$T(x)=[S'(x)]^{-1/2} \ , \tag{23}$$

leads to

$$\hbar^2\frac{T''}{T}-(S')^2(\Gamma-S^2)+2m[E-V(x)]=0 \ . \tag{24}$$

Equation (24) implies that an expansion of $S(x)$ in powers of $\hbar$ will contain no first-order terms. Thus, retaining only terms up to first order in $\hbar$ gives $S(x)=S_0(x)$ where $S_0(x)$ is determined by

$$T(x)\equiv[S_0'(x)]^{-1/2}=\left[\frac{\Gamma-S_0^2(x)}{E-V(x)}\right]^{1/4}$$

or equivalently

$$\int^{S_0(x)}(\Gamma-\sigma^2)^{1/2}d\sigma=\sqrt{2m}\int^{x}\sqrt{E-V(y)}dy \ . \tag{25}$$

The approximate wave function is made uniformly valid at the turning points by choosing the zeroes of $\Gamma-S_0^2(x)$ to correspond to those of $E-V(x)$ so that $T(x)$ is finite everywhere. Thus we obtain the usual Bohr-Sommerfeld quantization condition

$$\sqrt{2m}\int_{x_1}^{x_2}\sqrt{E-V(y)}dy=\int_{-\sqrt{\Gamma}}^{\sqrt{\Gamma}}(\Gamma-\sigma^2)^{1/2}d\sigma$$
$$=(n+\tfrac{1}{2})\hbar\pi \ . \tag{26}$$

C. Uniform semiclassical approximation to supersymmetric wave functions

The method of Sec. II B can be modified to provide approximate solutions to Eq. (13) by choosing a different mapping function. The form of Eq. (1b) suggests that a term linear in $\hbar$ should be explicitly present in the mapping function. Since the problem is again that of bound-state eigenfunctions of a potential well with two turning points, a possible choice which preserves this topology is

$$-\hbar^2\frac{d^2}{dS^2}U(S)-(\Gamma+A\hbar-S^2)U(S)=0 \ . \tag{27}$$

The constant A will be determined by requiring that the modified quantization condition, Eq. (17), be recovered. The solutions to Eq. (27) are again of the form given in Eq. (21) except that now $\Gamma=(2n+1-A)\hbar$. Substituting Eq. (22) into Eq. (13) and taking Eq. (23) to hold again yields

$$\hbar^2\frac{T''}{T}-(S')^2(\Gamma+A\hbar-S^2)$$
$$+2m[E-\phi^2(x)]+\sqrt{2m}\,\hbar\phi'=0 \ . \tag{28}$$

The form of Eq. (28) leads to an expansion of $S(x)$ in powers of $\hbar$ that now contains a first-order term, in contrast to the result in Sec. II B. Keeping only terms to first order in $\hbar$ gives

$$S(x)\approx S_0(x)+\hbar S_1(x) \ . \tag{29}$$

Substituting Eq. (29) into Eq. (28) and equating powers of $\hbar$ gives

$$2m[E-\phi^2(x)]=(\Gamma-S_0^2)(S_0')^2 \tag{30}$$

and

$$\sqrt{2m}\,\phi'=2(\Gamma-S_0^2)S_0'S_1'+(S_0')^2(A-2S_0S_1) \tag{31}$$

where only terms up to first order in $\hbar$ have been retained. Equation (30) can be immediately solved to give

$$\int^{S_0(x)}(\Gamma-\sigma^2)^{1/2}d\sigma=\sqrt{2m}\int^{x}[E-\phi^2(y)]^{1/2}dy \ . \tag{32}$$

Choosing the zeroes of $\Gamma-S_0^2(x)$ to correspond to those of $E-\phi^2(x)$ produces a wave function that is uniformly valid at the turning points and also gives the quantization condition

$$\sqrt{2m}\int_a^b[E-\phi^2(y)]^{1/2}dy=\tfrac{1}{2}(2n+1-A)\hbar\pi \ . \tag{33}$$

The modified quantization condition of Eq. (17) is obtained by choosing $A=1$. Now Eq. (31) can be solved for $S_1(x)$ with the result

$$S_1(x)=\frac{1}{2[\Gamma-S_0^2(x)]^{1/2}}\left[\sin^{-1}\frac{\phi(x)}{\sqrt{E}}-\sin^{-1}\frac{S_0(x)}{\sqrt{\Gamma}}\right] \tag{34a}$$

for $a<x<b$ and

$$S_1(x)=\frac{-1}{2[S_0^2(x)-\Gamma]^{1/2}}\ln\left[\frac{\phi(x)+[\phi^2(x)-E]^{1/2}}{\sqrt{E}}\,\frac{\sqrt{\Gamma}}{S_0(x)+[S_0^2(x)-\Gamma]^{1/2}}\right] \tag{34b}$$

for $x<a$ or $x>b$.

Thus, to first order in $\hbar$, the approximate wave functions are

$$\psi_n(x)=\frac{N}{(S_0')^{1/2}}H_n(S(x)/\sqrt{\hbar})\exp[-\tfrac{1}{2}S^2(x)/\hbar]\ , \tag{35}$$

where N is a normalization constant and $S(x)$ is given by Eq. (29). Equation (35) is the main result of this paper.

For the ground state, $E=\Gamma=0$ and $a=b=x_0$, so that

$$S_0(x)=\left[2\sqrt{2m}\int_{x_0}^{x}\phi(y)dy\right]^{1/2} \tag{36}$$

and $S_1(x)$ is given by Eq. (34b) everywhere. Thus, up to an overall normalization, the ground-state wave function is

$$\psi_0(x)\sim\exp\left[-\frac{\sqrt{2m}}{\hbar}\int_{x_0}^{x}\phi(y)dy\right]\ , \tag{37}$$

which is the exact wave function as required by the choice of $\phi(x)$ in Eq. (10).

III. APPLICATIONS

In this section we apply the formalism developed in Sec. II to three different one-dimensional systems: The Morse potential, the Rosen-Morse potential, and the anharmonic oscillator. For these systems we calculate not only the bound-state energy spectra, but also the wave functions in the uniform semiclassical approximation to supersymmetric quantum mechanics. In Ref. 22, it was shown that the modified semiclassical quantization condition gives the exact energy eigenvalues for the first two systems. In contrast, this quantization condition does not yield the exact energies for the anharmonic oscillator with the potential energy $V(x)=\alpha x^2+\beta x^6$, but it predicts the correct quantum number dependence for all the energy eigenvalues.[30]

A. Morse potential

For the Morse potential we have[22]

$$\phi(x)=(V_0)^{1/2}[1-\exp(-x/a)]-\frac{\hbar}{\sqrt{8m}\,a}\ , \tag{38a}$$

so that the potential V_-, as calculated by Eq. (1b), explicitly reads

$$V_-(x)=V_0[1-\exp(-x/a)]^2+\frac{\hbar^2}{8ma^2}-\frac{\hbar(V_0)^{1/2}}{\sqrt{2m}\,a}\ . \tag{38b}$$

In our calculations we have chosen $V_0=300$ MeV, $a=1$ fm, and $m=931.5$ MeV (nucleon mass). In Table I, we present the energy eigenvalues calculated using both the ordinary WKB (Bohr-Sommerfeld) quantization condition of Eq. (26) and the modified quantization condition of Eq. (17). We must emphasize that either quantization condition gives the exact energy eigenvalues. In Figs. 1(a) and 1(b) we compare the exact wave functions for the first and second excited states with the approximate wave functions obtained by using the standard uniform semiclassical approximation of Ref. 27 [Eq. (22)], and the uniform approximation to the supersymmetric quantum mechanics, developed in Sec. II C [Eq. (35)]. We observe that both approximations are equally good.

B. Anharmonic oscillator

In this case we take

$$\phi(x)=\frac{(V_0)^{1/2}}{3}\left[\frac{x}{a}\right]^3\ , \tag{39}$$

using which we can define two potentials

$$V_\pm(x)=\frac{V_0}{9}\left[\frac{x}{a}\right]^6\pm\frac{\hbar(V_0)^{1/2}}{\sqrt{2m}\,a}\left[\frac{x}{a}\right]^2\ . \tag{40}$$

The potential $V_-(x)$ is a symmetric double well, whereas the potential $V_+(x)$ is a single well (cf. Ref. 15). We plot these two potentials in Fig. 2. We note that, because of supersymmetry, the ground-state energy of $V_-(x)$ is 0, which lies just on the hump in the middle of the potential. Since there are no levels below this energy (i.e., in the region where there are two pockets of the potential), for this particular potential there is no doubling of energy levels due to tunneling. In general, the supersymmetric partner which includes the level with $E=0$ would exhibit this level doubling and this feature can be important in practical calculations.[15]

In Table II, we present the energy eigenvalues. We assume that $V_0=1$ MeV, $a=1$ fm, and $m=931.5$ MeV. The exact energy eigenvalues for the potential $V_-(x)$ are calculated by solving the Schrödinger equation numerically. In this table the column marked SS is obtained by using the modified quantization condition of Eq. (17). We also applied the standard WKB (Bohr-Sommerfeld) quantization condition of Eq. (26) both to the potential $V_-(x)$ (the column marked WKB$_-$) and the potential

TABLE I. Energy levels of the Morse potential, Eq. (38b), with $V_0=300$ MeV, $a=1$ fm, and $m=931.5$ MeV. The first column is the exact energy, the second column (SS) is calculated using the modified quantization condition, Eq. (17), and the third column (WKB) is calculated using the conventional WKB formula, Eq. (26).

n	Exact energy	SS	WKB
0	0.00	0.00	0.00
1	116.57	116.57	116.57
2	191.33	191.33	191.33

$V_+(x)$ (the column marked WKB$_+$). The potentials given in Eq. (40) were previously studied in Ref. 30 to investigate how good the supersymmetry-inspired WKB quantization condition [Eq. (17)] is. We caution the reader that the energies given in Ref. 30 are misprinted and they were corrected by their authors in Ref. 31. Our values displayed in Table II agree with the corrected numbers given in Ref. 31 after inserting the suitable scale factors. An examination of Table II reveals that neither the standard nor the modified quantization condition approximates the energies very well. One can say that Eq. (17) does a slightly better job, although it tends to overpredict the energy eigenvalues. We should also note that, contrary to the naive expectation, the standard WKB applied to the full potentials $V_-(x)$ and $V_+(x)$ does not yield lower and upper bounds to the energies given by the modified quantization condition.

In Figs. 3(a) and 3(b), we compare the wave functions for the first and second excited states of the potential $V_-(x)$. The exact wave functions (solid lines) are calculated by solving the Schrödinger equation numerically. The WKB wave functions for the potential $V_-(x)$ are obtained by using the standard uniform semiclassical approximation of Ref. 27 [Eq. (22)] and the SS wave functions are calculated in the uniform approximation of Sec. II C [Eq. (35)]. We observe that for the first excited level both approximations are very good. For the second excited level, both methods approximate the wave functions well except near the extrema of the wave functions, where WKB seems to do slightly better.

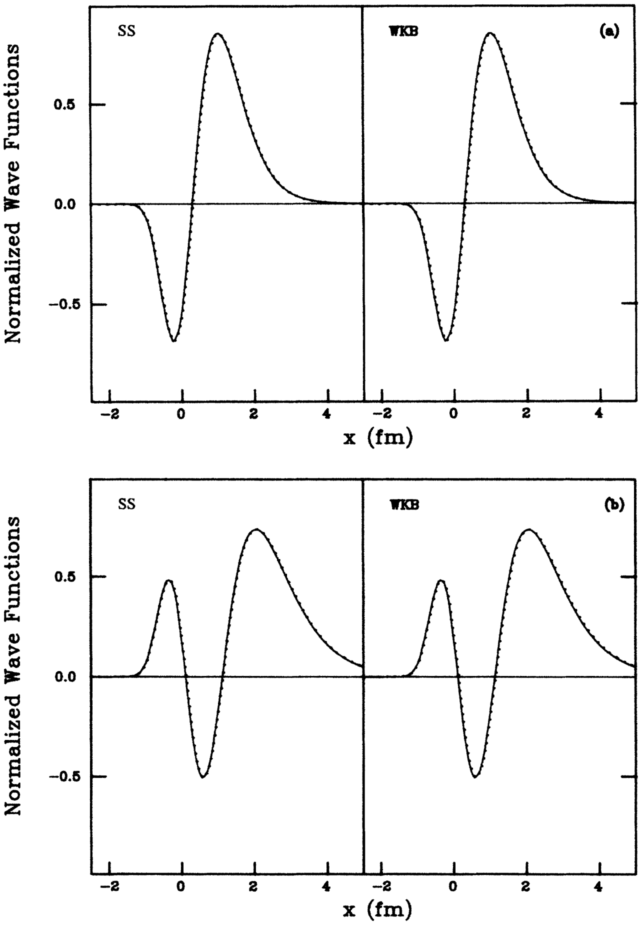

FIG. 1. (a) Normalized wave function for the first excited state of the Morse potential (cf. Table I). In both panels, the exact wave function is the solid line and the approximate wave function is the dotted line. The left panel shows the approximate wave function obtained from the modified uniform approximation and the right panel shows the approximate wave function obtained from the standard uniform semiclassical approximation. (b) Same as (a), but for the second excited state.

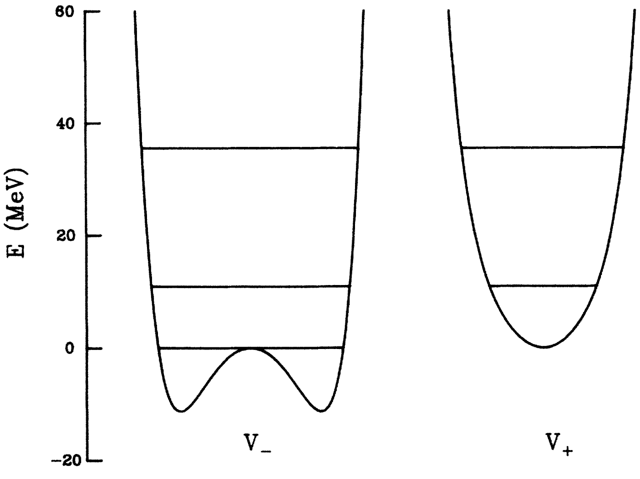

FIG. 2. The potentials $V_\pm(x)$ for the anharmonic oscillator [Eq. (40)] with $V_0=1$ MeV, $a=1$ fm, and $m=931.5$ MeV.

C. Rosen-Morse potential

For the Rosen-Morse potential we have[22]

$$\phi(x)=\frac{\hbar}{\sqrt{8m}\,a}\left[-1+\left[1+\frac{8ma^2V_0}{\hbar^2}\right]^{1/2}\right]\tanh(x/a)\,, \tag{41a}$$

yielding

$$V_-(x)=-V_0\operatorname{sech}^2(x/a)+\frac{\hbar^2}{8ma^2}\left[-1+\left[1+\frac{8ma^2V_0}{\hbar^2}\right]^{1/2}\right]^2\,. \tag{41b}$$

In our calculations we have chosen $V_0=200$ MeV, $a=1$ fm, and $m=931.5$ MeV (nucleon mass). In Table III, the energy eigenvalues calculated using both the ordinary WKB (Bohr-Sommerfeld) quantization condition of Eq. (26) and the modified quantization condition of Eq. (17) are compared with the exact answer. In this case the modified quantization condition (SS) gives the exact answer for the energy eigenvalues,[22] whereas the standard WKB does not. In Figs. 4(a) and 4(b), we compare the exact wave functions for the first and second excited states with the approximate wave functions obtained by using the standard uniform semiclassical approximation of Ref. 27 [Eq. (22)] and the uniform approximation to the supersymmetric quantum mechanics developed in

TABLE II. Energy levels of the anharmonic oscillator, Eq. (40), with $V_0=1$ MeV, $a=1$ fm, and $m=931.5$ MeV. The exact energies are obtained numerically. The second column (SS) is calculated using Eq. (17). The third column (WKB$_-$) is calculated by applying Eq. (26) to $V_-(x)$ and the fourth column (WKB$_+$) is obtained by applying Eq. (26) to $V_+(x)$. n is the quantum number for the potential $V_-(x)$ and k for the potential $V_+(x)$.

n	Exact energy	SS	WKB$_-$	k	WKB$_+$
1	10.923	12.783	11.287	0	10.077
2	35.546	36.157	34.950	1	34.510
3	65.924	66.425	65.391	2	65.151
4	101.827	102.268	101.350	3	101.194
5	142.529	142.924	142.090	4	141.979

Sec. II C [Eq. (35)]. For the first excited level both approximations are equally good. However, for the second excited state the modified uniform approximation agrees with the exact solution very well, whereas the WKB method provides a reasonable, but not as good fit to the correct asymptotic behavior.

IV. CONCLUSIONS

In this article, following Comtet, Bandrauk, and Campbell,[22] we have investigated the modified semiclassical quantization condition motivated by the structure of su-

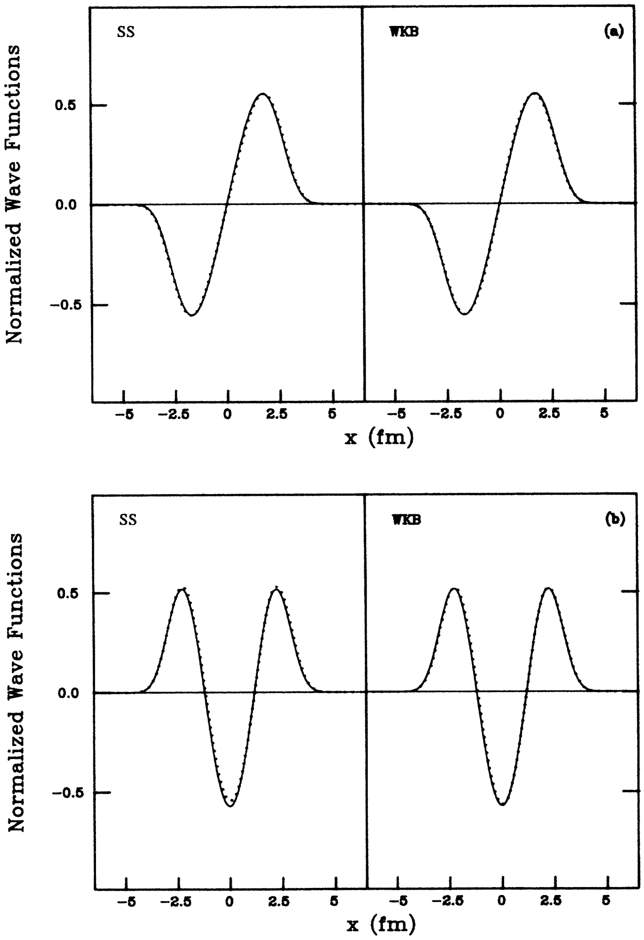

FIG. 3. (a) Normalized wave function for the first excited state of the anharmonic oscillator (cf. Table II). In both panels, the exact wave function is the solid line and the approximate wave function is the dotted line. The left panel shows the approximate wave function obtained from the modified uniform approximation and the right panel shows the approximate wave function obtained from the standard uniform semiclassical approximation. (b) Same as (a), but for the second excited state.

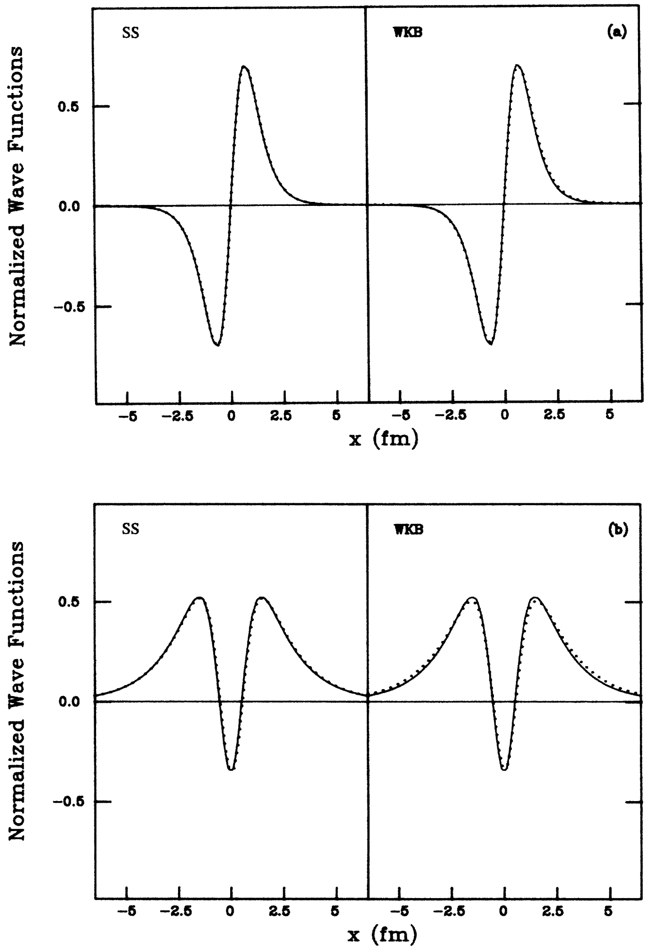

FIG. 4. (a) Normalized wave function for the first excited state of the Rosen-Morse potential (cf. Table I). In both panels, the exact wave function is the solid line and the approximate wave function is the dotted line. The left panel shows the approximate wave function obtained from the modified uniform approximation and the right panel shows the approximate wave function obtained from the standard uniform semiclassical approximation. (b) Same as (a), but for the second excited state.

TABLE III. Energy levels of the Rosen-Morse potential, Eq. (41), with $V_0 = 200$ MeV, $a = 1$ fm, and $m = 931.5$ MeV. The second column (SS) is calculated using Eq. (17) and the third column (WKB) is calculated using Eq. (26).

n	Exact energy	SS	WKB
0	0.00	0.00	4.39
1	89.19	89.19	91.90
2	136.57	136.57	137.60

persymmetric quantum mechanics. In particular, we have constructed accurate semiclassical wave functions in a uniform approximation, which are finite at the turning points. In contrast, the primitive semiclassical wave functions introduced by Comtet and co-workers are incorrect at the turning points. For the ground state, our approximation gives the exact wave function as required by the choice in Eq. (10).

We have compared our wave functions with the exact ones for three different problems. The first one was the Morse potential, for which both the Bohr-Sommerfeld formula and the supersymmetry modified quantization rule give the exact energy eigenvalues. We have found that both the standard uniformized WKB and our supersymmetry modified uniformization provide an excellent approximation to the exact wave functions. The second test case was the anharmonic oscillator, for which neither quantization rule gave exact energies, although the supersymmetry modified rule fared somewhat better. For this case both wave functions were still good approximations except near the extrema of the wave functions, where WKB appeared to be slightly better. The third test case was the Rosen-Morse potential, for which the supersymmetry modified quantization rule gave the exact energy eigenvalues, in contrast to the Bohr-Sommerfeld formula. In this case, our supersymmetry modified rule yielded somewhat better wave functions.

Recently there was much discussion in the literature concerning the exactness of semiclassical bound-state energies for supersymmetric quantum mechanics. It was argued that for shape invariant potentials[32] the supersymmetry modified quantization rule gave exact energy eigenvalues.[31,32] (It was also conjectured that shape invariance is not only sufficient, but also necessary for a potential to be exactly solvable. This conjecture was nevertheless disproved.[33]) For the two shape-invariant potentials we studied, namely, for the Morse and Rosen-Morse potentials, the supersymmetry-modified quantization rule gives the exact energy eigenvalues as shown by Dutt, Khare and Sukhatme;[31] however, our uniform wave functions were not exact, except for the ground state. Obviously, the exactness of the energy eigenvalues in an approximation scheme does not automatically guarantee the exactness of the wave functions, which provide a more sensitive test. Consequently, we cannot conclude from this study that the supersymmetry-inspired semiclassical approximation is better than the standard WKB approximation. It would be interesting to investigate further the connection between shape invariance and the exactness of the wave functions.

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