Possible new frequency shifts for the $2S_{1/2} \rightarrow 2P_{1/2}$ transition in a hydrogen atom within a specific boundary

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Changes in the frequency of the $2S_{1/2} \rightarrow 2P_{1/2}$ transition in hydrogen when the atom is placed between two parallel, perfectly conducting plates are investigated. It is shown that the transition frequency depends on the distance between the plates and that a change in the frequency of $\Delta\nu = -0.894$ MHz results for plates separated by 1 μm. This is sufficiently large to be measurable.

I. INTRODUCTION

The Lamb shift of the $2S_{1/2}$ state in the hydrogen atom is well known and arises from the interaction of the bound electron with the fluctuating vacuum electric field.[1,2] Quantization of the radiation field in free space leads to the important fact that the vacuum field does not really vanish, but rather fluctuates. This nonvanishing field leads to a zero-point energy $\sum_\sigma \hbar\omega_\sigma/2$, which is often considered unobservable and hence is disregarded. In 1948, however, Casimir[3] pointed out that this fluctuation could be observed by imposing a specific boundary condition in a given geometry. Namely, the difference between the zero-point energy inside the volume L^2b ($L \gg b$) with perfect conducting boundaries and the free value for this same volume is finite and observable, i.e.,

$$\sum_{l,j,n} \hbar\omega_{ljn}/2 - [bL^2/(2\pi c)^3] \int_0^\infty (\hbar\omega/2) d^3\omega = -(\pi^2 L^2/720)\hbar c b^{-3} .$$

This fact was confirmed[4] by the measurement of the force exerted on two parallel perfectly conducting plates. Accordingly, one may speculate that the introduction of boundary conditions of specified geometry would alter vacuum fluctuations, which would in turn modify atomic properties. Specifically, when the space around an atom is restricted by perfect conductors, we expect new shifts of the atomic energy levels in addition to the Lamb shift since it is due to vacuum fluctuations of the electric field in free space. Thus it is very interesting to investigate thoroughly the energy spectra of a hydrogen atom when it is placed between two parallel perfectly conducting plates. Below we calculate this effect in the context of quantum electrodynamics.

II. THEORY

The frequency shifts due to vacuum fluctuations can be evaluated by analyzing radiative corrections and effects of vacuum polarization on electron scattering by an external electromagnetic field. Here, we use natural units, i.e., $\hbar = c = 1$, throughout the calculation.

A. Radiative corrections

Let us evaluate the Feynman diagram, Fig. 1(a), in a region of space with specific boundaries that consist of two parallel perfectly conducting square plates of size L separated a distance b from each other. Since the momentum of quantum-electromagnetic normal modes in a rectangular cavity with conducting walls $L \times L \times b$ is expressed as

$$|\mathbf{k}| = [(\pi l/L)^2 + (\pi j/L)^2 + (\pi n/b)^2]^{1/2} ,$$

the matrix element for the radiative correction, Fig. 1(a), is given by[1,2]

$$\Lambda_{(a)}(p_F, p_I) = (i\alpha/4\pi^3) \int_{-\infty}^{\infty} dk_0 [(2\pi)^3/bL^2] \sum_{l,j,n} \gamma^\mu (\not{p}_F - \not{k} - m)^{-1} \not{a} (\not{p}_I - \not{k} - m)^{-1} \gamma_\mu (k^2 - \lambda_{\min}^2 + i\varepsilon)^{-1} , \quad (2.1)$$

where γ^μ are the Dirac matrices, $\not{p} = \gamma^\lambda p_\lambda$, and

$$\not{k} = \gamma_0 k_0 - \gamma_1(\pi l/L) - \gamma_2(\pi j/L) - \gamma_3(\pi n/b) \equiv \gamma_0 k_0 - \gamma_1 k_1 - \gamma_2 k_2 - \gamma_3 k_3 , \quad (2.2)$$

with

$$k_1 \equiv (\pi l/L), \quad k_2 \equiv (\pi j/L), \quad k_3 \equiv (\pi n/b) . \quad (2.3)$$

As mentioned in Sec. I, the matrix element (2.1) gives a result which is the sum of the Lamb shift in free space plus the correction caused by the boundary.

The photon propagator $(k^2 - \lambda_{\min}^2)^{-1}$ is conventionally regulated as $(k^2 - \lambda_{\min}^2)^{-1} - (k^2 - \lambda^2)^{-1}$ with a large regulator mass λ. In this case, the matrix element (2.1) can be rewritten as

$$\Lambda_{(a)}(p_F,p_I)=(i\alpha/4\pi^3)\int_{-\infty}^{\infty}dk_0[(2\pi)^3/bL^2]\sum_{l,j,n}\gamma^\mu(\not p_F-\not k-m)^{-1}\not a(\not p_I-\not k-m)^{-1}\gamma_\mu\int_{\lambda^2_{\min}}^{\lambda^2}d\xi(k^2-\xi)^{-2}$$
$$=(4\pi i\alpha)(2\pi)^{-4}\int_{-\infty}^{\infty}dk_0[(2\pi)^3/bL^2]\sum_{l,j,n}\int_{\lambda^2_{\min}}^{\lambda^2}d\xi\,\gamma^\mu(\not p_F-\not k+m)\not a(\not p_I-\not k+m)$$
$$\times\gamma_\mu(k^2-2kp_F)^{-1}(k^2-2kp_I)^{-1}(k^2-\xi)^{-2}\ , \tag{2.4}$$

where we have used the result

$$(\not p-\not k+m)(\not p-\not k-m)=p^2-2kp+k^2-m^2=k^2-2kp\ , \tag{2.5}$$

with $p^2=m^2$.

The identity

$$(\tilde A\tilde B)^{-1}=\int_0^1 dy[\tilde A y+\tilde B(1-y)]^{-2} \tag{2.6}$$

gives

$$(k^2-2kp_F)^{-1}(k^2-2kp_I)^{-1}=\int_0^1 dy\,(k^2-2kp_y)^{-2}\ , \tag{2.7}$$

where

$$p_y=yp_I+(1-y)p_F\ . \tag{2.8}$$

Then, the matrix element is found in the form

$$\Lambda_{(a)}(p_F,p_I)=(4\pi i\alpha)\int_0^1 dy\int_{\lambda^2_{\min}}^{\lambda^2}d\xi(2\pi)^{-4}\int_{-\infty}^{\infty}dk_0[(2\pi)^3/bL^2]\sum_{l,j,n}\gamma^\mu(\not p_F-\not k+m)$$
$$\times\not a(\not p_I-\not k+m)\gamma_\mu(k^2-2kp_y)^{-2}(k^2-\xi)^{-2}$$
$$=(24\pi i\alpha)\int_0^1 dx\,x(1-x)\int_0^1 dy\int_{\lambda^2_{\min}}^{\lambda^2}d\xi(2\pi)^{-4}\int_{-\infty}^{\infty}dk_0[(2\pi)^3/bL^2]$$
$$\times\sum_{l,j,n}\gamma^\mu(\not p_F-\not k+m)\not a(\not p_I-\not k+m)\gamma_\mu$$
$$\times[k^2-2xkp_y-(1-x)\xi]^{-4}\ . \tag{2.9}$$

The numerator in (2.9) can be decomposed as

$$\gamma^\mu(\not p_F-\not k+m)\not a(\not p_I-\not k+m)\gamma_\mu=\gamma^\mu(\not p_F\not a\not p_I-\not p_F\not a\not k-\not k\not a\not p_I+\not k\not a\not k+m\not p_F\not a+m\not a\not p_I-m\not k\not a-m\not a\not k+m^2\not a)\gamma_\mu\ . \tag{2.10}$$

Making use of the formulas given in Appendix A [(A28) and (A29), respectively],

$$(2\pi)^{-4}\int_{-\infty}^{\infty}dk_0[(2\pi)^3/bL^2]\sum_{l,j,n}(1;k_\sigma)(k^2-2pk-\Delta)^{-4}=i(1/96\pi^2)(1;p_\sigma)(\Delta+p^2)^{-2}$$
$$+i(1/128\pi b)(1;p_\sigma)(\Delta+p^2)^{-5/2}\ ,$$

$$(2\pi)^{-4}\int_{\infty}^{\infty}dk_0[(2\pi)^3/bL^2]\sum_{l,j,n}k_\sigma k_\rho(k^2-2pk-\Delta)^{-4}=i(1/96\pi^2)[p_\sigma p_\rho-\eta_{\sigma\rho}(\Delta+p^2)/2](\Delta+p^2)^{-2}$$
$$+i(1/128\pi b)[p_\sigma p_\rho-\eta_{\sigma\rho}(\Delta+p^2)/3](\Delta+p^2)^{-5/2}\ ,$$

we obtain

$$\Lambda_{(a)}(p_F,p_I)=-(24\pi\alpha)\int_0^1 dx\,x(1-x)\int_0^1 dy\int_{\lambda^2_{\min}}^{\lambda^2}d\xi\{(96\pi^2)^{-1}[x^2p_y^2+(1-x)\xi]^{-2}(W+w/2)$$
$$+(128\pi b)^{-1}[x^2p_y^2+(1-x)\xi]^{-5/2}(W+w/3)\}\ , \tag{2.11}$$

where

$$W=\gamma^\mu[\not p_F\not a\not p_I-x\not p_F\not a\not p_y-x\not p_y\not a\not p_I+x^2\not p_y\not a\not p_y-mx(\not p_y\not a+\not a\not p_y)+m(\not p_F\not a+\not a\not p_I)+m^2\not a]\gamma_\mu\ , \tag{2.12}$$

$$w=2\gamma^\mu\not a\gamma_\mu[x^2p_y^2+(1-x)\xi]\ . \tag{2.13}$$

The first term on the right-hand side in Eq. (2.11) gives the conventional radiative corrections in free space. The important term is the second term, which gives the modifications to the free-space result. The second term depends on the distance b and is precisely the quantity to be calculated in our present investigation. The quantities W and w can be written as

$$W=-2\not p_I \not a \not p_F+2x\not p_y \not a \not p_F+2x\not p_I \not a \not p_y-2x^2\not p_y \not a \not p_y-4mx(\not p_y \not a+\not a \not p_y)+2m(\not p_F \not a+\not a \not p_F+\not p_I \not a+\not a \not p_I)-2m^2\not a \ , \tag{2.14}$$

$$w=-4\not a[x^2p_y^2+(1-x)\xi] \tag{2.15}$$

by use of identities[1,2]

$$\gamma^\mu \not a \gamma_\mu=-2\not a \ , \tag{2.16a}$$

$$\gamma^\mu \not a \not b \gamma_\mu=2(\not a \not b+\not b \not a) \ , \tag{2.16b}$$

$$\gamma^\mu \not a \not b \not c \gamma_\mu=-2\not c \not b \not a \ . \tag{2.16c}$$

Thus we find the b-dependent term takes the form

$$\Lambda^b_{(a)}(p_F,p_I)=-(3\alpha/16b)\int_0^1 dy\int_0^1 dx\int_{\lambda^2_{\min}}^{\lambda^2} d\xi\{x(1-x)[x^2p_y^2+(1-x)\xi]^{-5/2}G_1 -\tfrac{4}{3}\not a x(1-x)[x^2p_y^2+(1-x)\xi]^{-3/2}\} \ , \tag{2.17}$$

$$G_1=-2\not p_I \not a \not p_F+2m(\not p_F \not a+\not a \not p_F+\not p_I \not a+\not a \not p_I)-2m^2\not a+x[2\not p_y \not a \not p_F+2\not p_I \not a \not p_y-4m(\not p_y \not a+\not a \not p_y)]-2x^2\not p_y \not a \not p_y \ . \tag{2.18}$$

The integrals over ξ and x in (2.17) can easily be carried out utilizing the results given in Appendix B. The result is

$$\Lambda^b_{(a)}(p_F,p_I)=-(3\alpha/16b)\int_0^1 dy\, D_y^b \ , \tag{2.19}$$

where

$$D_y^b=(\tfrac{2}{3}\lambda_{\min}^{-1}p_y^{-2}-\tfrac{1}{3}p_y^{-3})[-2\not p_I \not a \not p_F+2m(\not p_F \not a+\not a \not p_F+\not p_I \not a+\not a \not p_I)-2m^2\not a] +\tfrac{2}{3}p_y^{-3}[\ln(2p_y/\lambda_{\min})-1][2\not p_y \not a \not p_F+2\not p_I \not a \not p_y-4m(\not p_y \not a+\not a \not p_y)]-\tfrac{8}{3}p_y^{-3}\not p_y \not a \not p_y-\tfrac{8}{3}\not a p_y^{-1} \ . \tag{2.20}$$

Taking the matrix between states $u(p_F)$ and $u(p_I)$, and referring to the integrals in Appendix C, we obtain, for $q^2 \ll 4m^2$,

$$\Lambda^b_{(a)}(p_F,p_I)=\Gamma^b_{\rm I}\not a+\Gamma^b_{\rm II}(\not q \not a-\not a \not q)+R_b\not a \ , \tag{2.21}$$

where

$$\Gamma^b_{\rm I}=(\alpha/4mb)\{2+\tan^2\theta[3-\ln(4m^2/\lambda^2_{\min})] -(1-\tan^2\theta)(\csc\theta) \times\ln[(1+\sin\theta)/(1-\sin\theta)]+(2m/\lambda_{\min}) \times[1-(2\theta/\sin2\theta)+2\theta\tan\theta]\} \ , \tag{2.22}$$

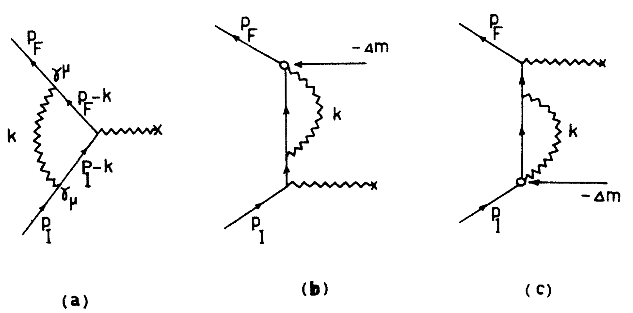

FIG. 1. Feynman diagrams for radiation corrections to electron scattering by an external electromagnetic field.

$$\Gamma^b_{\rm II}=(\alpha/4mb)(4m\cos^2\theta)^{-1}\{\ln(4m^2/\lambda^2_{\min})-4-(\csc\theta) \times\ln[(1+\sin\theta)/(1-\sin\theta)]\} \ , \tag{2.23}$$

$$R_b=(\alpha/4mb)[3-(2m/\lambda_{\min})\ln(4m^2/\lambda^2_{\min})] \ . \tag{2.24}$$

In evaluating Eq. (2.21), we have used $\not p u(\mathbf{p})=mu(\mathbf{p})$, $p_F=p_I+q$, $p_I^2=p_F^2=m^2$, and

$$q^2=4m^2\sin^2\theta \ . \tag{2.25}$$

Using similar techniques, we can evaluate Feynman diagrams [Figs. (1b) and (1c)] as

$$\Lambda^b_{(b)}+\Lambda^b_{(c)}=-R_b\not a \ . \tag{2.26}$$

Thus the sum $\Lambda^b=\Lambda^b_{(a)}+\Lambda^b_{(b)}+\Lambda^b_{(c)}$ can be written as

$$\Lambda^b=\Gamma^b_{\rm I}\not a+\Gamma^b_{\rm II}(\not q \not a-\not a \not q) \ . \tag{2.27}$$

The first term contributes to the shift in the energy levels of the atom, while the second term modifies the electron magnetic moment.

B. Vacuum polarization

Under an assumption that electron and positron wave functions vanish at the surface of a perfect conductor, the vacuum polarization, Fig. 2, is described by

$$\Pi_{\mu\nu}(q)=\int_{-\infty}^{\infty}dp_0[(2\pi)^3/bL^2]\sum_{l,j,n}\{\mathrm{Tr}[\gamma_\mu(\not p+\not q+m)\gamma_\mu(\not p+m)]\}[(p+q)^2-m^2]^{-1}(p^2-m^2)^{-1}\ , \qquad (2.28)$$

where $p_1=(\pi l/L)$, $p_2=(\pi j/L)$ and $p_3=(\pi n/b)$. By the identity (2.16), we find

$$\Pi_{\mu\nu}(q)=\int_0^1 dz\int_{-\infty}^{\infty}dp_0[(2\pi)^3/bL^2]\sum_{l,j,n}\mathrm{Tr}[\gamma_\mu(\not p+\not q+m)\gamma_\nu(\not p+m)][(p+zq)^2+q^2(z-z^2)-m^2]^{-2}\ . \qquad (2.29)$$

Here we shift the origin of integration and the summation from p to $p-zq$ to get

$$\Pi_{\mu\nu}(q)=\int_0^1 dz\int_{-\infty}^{\infty}dp_0[(2\pi)^3/bL^2]\sum_{l,j,n}\mathrm{Tr}\{\gamma_\mu[\not p+\not q(1-z)+m]\gamma_\nu(\not p-\not qz+m)\}[p^2+q^2(z-z^2)-m^2]^{-2}\ . \qquad (2.30)$$

Making use of identities[1]

$$\mathrm{Tr}(\gamma_\mu\gamma_\nu)=4\eta_{\mu\nu}\ , \qquad (2.31)$$

$$\mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma)=4\eta_{\mu\sigma}\eta_{\nu\rho}-4\eta_{\sigma\nu}\eta_{\rho\mu}+4\eta_{\sigma\rho}\eta_{\mu\nu}\ , \qquad (2.32)$$

we can rewrite (2.30) as

$$\Pi_{\mu\nu}(q)=4\int_0^1 dz\int_{-\infty}^{\infty}dp_0[(2\pi)^3/bL^2]\sum_{l,j,n}\{-(z-z^2)(2q_\mu q_\nu-\eta_{\mu\nu}q^2)-\eta_{\mu\nu}[(p^2/2)-m^2]\}[p^2+q^2(z-z^2)-m^2]^{-2}\ , \qquad (2.33)$$

where $\eta_{00}=-\eta_{11}=-\eta_{22}=-\eta_{33}=1$. Now in order for the current

$$j_\mu=(i\alpha/4\pi^3)\Pi_{\mu\nu}(q)a^\nu(q) \qquad (2.34)$$

to be invariant under gauge transformations

$$a^\nu\to a^\nu+q^\nu\Phi(q)\ , \qquad (2.35)$$

we should have

$$\Pi_{\mu\nu}(q)q^\nu=0\ , \qquad (2.36)$$

which implies

$$\int_{-\infty}^{\infty}dp_0[(2\pi)^3/bL^2]\sum_{l,j,n}[-q^2(z-z^2)-(p^2/2)+m^2][p^2+q^2(z-z^2)-m^2]^{-2}=0\ . \qquad (2.37)$$

Thus we find

$$\Pi_{\mu\nu}(q)=(q_\mu q_\nu-\eta_{\mu\nu}q^2)\Pi_{(1)}(q^2)\ , \qquad (2.38)$$

where

$$\Pi_{(1)}(q^2)=-8\int_0^1 dz\int_{-\infty}^{\infty}dp_0[(2\pi)^3/bL^2]\sum_{l,j,n}(z-z^2)[p^2+q^2(z-z^2)-m^2]^{-2}\ . \qquad (2.39)$$

The effective external field is given by[1]

$$a^\nu_{\mathrm{eff}}(q^2)=\{1-(i\alpha/4\pi^3)[\partial\Pi_{(1)}(q^2)/\partial q^2]_{q^2=0}q^2\}a^\nu(q^2)\ . \qquad (2.40)$$

With the help of formula (A25) in Appendix A we find

$$(\partial\Pi_{(1)}(q^2)/\partial q^2)_{q^2=0}=16\int_0^1 dz\int_{-\infty}^{\infty}dp_0[(2\pi)^3/bL^2]\sum_{l,j,n}(z-z^2)^2(p^2-m^2)^{-3}$$
$$=-i(4\pi^2/15m^2)-i(2\pi^3/15m^3b)\ . \qquad (2.41)$$

Then we have

$$a^\nu_{\mathrm{eff}}(q^2)=\{1-[(\alpha/15\pi m^2)+(\alpha/30m^3b)]q^2\}a^\nu(q^2)\ . \qquad (2.42)$$

The second term is the contribution from the vacuum polarization in free space, while the third term is due to the boundary and is to be added to the radiation correction, (2.22).

The effective potential which the charge feels is then given by

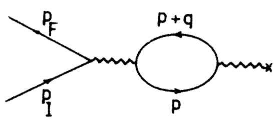

FIG. 2. Vacuum polarization.

$$F(q^2)a_\nu(q)=[1+(\Gamma_{\mathrm{I}}^0+\Delta\Gamma^0)+(\Gamma_{\mathrm{I}}^b+\Delta\Gamma^b)]a_\nu(q)$$
$$\cong a_\nu(q)+[\Gamma_{\mathrm{I}}^0-(\alpha/15\pi m^2)q^2]a_\nu(q)$$
$$+[\Gamma_{\mathrm{I}}^b-(\alpha/30m^3b)q^2]a_\nu(q)\ , \qquad (2.43)$$

where the first two terms are the conventional terms,[2] and for small q^2, Γ_I^0, and Γ_I^b are expressed in the forms

$$\Gamma_{\mathrm{I}}^0\cong(\alpha/2\pi)(2q^2/3m^2)[\ln(m/\lambda_{\min})-\tfrac{3}{8}]\ , \qquad (2.44)$$

$$\Gamma_{\mathrm{I}}^b\cong(\alpha/4mb)(q^2/4m^2)[\tfrac{13}{3}+(8m/3\lambda_{\min})$$
$$-\ln(4m^2/\lambda_{\min}^2)]\ . \qquad (2.45)$$

The result (2.45) goes to infinity for $\lambda_{\min}\to 0$. This difficulty can be eliminated by taking contributions from soft photons into account.

C. Contributions from soft photons and momentum less than k_m

The results obtained in Secs. II A and II B still contain an infrared catastrophe. This catastrophe should not occur in the Lamb shift of a bound electron. The infrared divergences arise because the electron can emit and absorb soft photons without being displaced very far off the mass shell $p^2=m^2$. This difficulty can be overcome by recomputing the vertex corrections under an assumption that emission of photons of energy less than a certain cutoff k_m ($mZ^2\alpha^2\ll k_m\ll m$) is suppressed. Let us divide our calculations into two parts, $mZ\alpha\ll k_m\ll m$ and $mZ^2\alpha^2\ll k_m\ll mZ\alpha$. In the former case we can repeat the previous calculation, taking the modifications of the photon propagator into account. For the latter case, it will be possible to treat the electrons in a nonrelativistic approximation and also to take account of the nuclear potential.

1. For the case $mZ\alpha\ll k_m\ll m$

As was done before, for $|\mathbf{k}|=[(\pi l/L)^2+(\pi j/L)^2+(\pi n/b)^2]^{1/2}$, we have

$$\delta\Lambda_\mu=-i\alpha(2\pi)^{-3}[(2\pi)^3/bL^2]\sum_{l,j,n}{}' \int_{-\infty}^{\infty} dk_0(2\pi)^{-1}[\gamma_\nu(\not p_F-\not k+m)\gamma_\mu(\not p_I-\not k+m)\gamma^\nu](k^2-\lambda_{\min}^2+i\varepsilon)^{-1}$$
$$\times[(p_F-k)^2-m^2+i\varepsilon]^{-1}[(p_I-k)^2-m^2+i\varepsilon]^{-1}$$
$$=\alpha(2\pi)^{-3}[(2\pi)^3/bL^2]\sum_{l,j,n}{}' \tfrac{1}{2}[(p_Ip_F)+(kq)](k^2+\lambda_{\min}^2)^{-1/2}[(p_Fk)(p_Ik)]^{-1}\gamma_\mu$$
$$=\alpha(2\pi)^{-3}[(2\pi)^3/bL^2]\sum_{l,j,n}{}' \int_{-1}^{1} dz\ \tfrac{1}{4}[(p_Ip_F)+(kq)](k^2+\lambda_{\min}^2)^{-1/2}(Pk)^{-2}\gamma_\mu\ , \qquad (2.46)$$

where

$$P=\tfrac{1}{2}[(p_F+p_I)+z(p_F-p_I)] \qquad (2.47)$$

and we used the identity (2.16).

Making use of the previous procedure to convert summations into integrals by the Poisson's summation formula on the Fourier transformation and the Euler-Maclaulin formula, we find

$$[(2\pi)^3/bL^2]\sum_{l,j,n}{}'(k^2+\lambda_{\min}^2)^{-1/2}(Pk)^2=4\pi\int_0^{k_m} d\kappa\,\kappa^2(\kappa^2+\lambda_{\min}^2)^{-1/2}(\kappa^2P^2+\lambda_{\min}^2P_0^2)^{-1}$$
$$+(2\pi^2/b)\int_0^{k_m} d\kappa\,\kappa(\kappa^2+\lambda_{\min}^2)^{-1/2}(\kappa^2P^2+\lambda_{\min}^2P_0^2)^{-1}\ , \qquad (2.48)$$

where the first term on the right-hand side is associated with the value in free space, while the second term depends on the boundary. Hereafter, we consider only the b-dependent term. We obtain

$$\delta\Lambda_\mu(b)\cong\gamma_\mu(\alpha/2b)(1-q^2/3m^2)(\lambda_{\min}^{-1}-k_m^{-1})+\gamma_\mu(\alpha/8bm)(q^2/m^2)l_n(k_m/\lambda_{\min})$$
$$=\gamma_\mu(\alpha/2b)(\lambda_{\min}^{-1}-k_m^{-1})-\gamma_\mu(\alpha q^2/16m^3b)[(8m/3\lambda_{\min})-2l_n(2m/\lambda_{\min})]$$
$$+\gamma_\mu(\alpha q^2/16m^3b)[(8m/3k_m)-2\ln(2m/k_m)]\ . \qquad (2.49)$$

Similarly, we can evaluate the contributions from the modification of the electron propagator due to the changes in the photon propagator,

$$\delta\Sigma(b,p_I)=\Sigma(b,p_I,\lambda)-\Sigma(b,p_I,\lambda_{\min})$$

$$=-i\alpha(2\pi)^{-3}[(2\pi)^3/bL^2]\sum_{l,j,n}{}' \int_{-\infty}^{\infty} dk_0(2\pi)^{-1}\gamma_\nu(\not p_I-\not k-m+i\varepsilon)^{-1}\gamma^\nu(k^2-\lambda_{\min}^2+i\varepsilon)^{-1}$$

$$\cong -\alpha(\not p_I-m)(2\pi)^{-3}\tfrac{1}{2}\int_{\kappa<k_m} d^3\kappa[m^2+(p_I\kappa)](p_I\kappa)^{-2}$$

$$=(\not p_I-m)[(\alpha/2b)(k_m^{-1}-\lambda_{\min}^{-1})-(\alpha/2mb)\ln(k_m/\lambda_{\min})] \ . \qquad (2.50)$$

Thus the contribution from the change of vertex caused by modifications of the photon propagator is

$$\gamma_\mu\delta\Gamma^b\equiv\delta\Lambda_\mu(b)+\tfrac{1}{2}\Sigma(b,p_F)(\not p_F-m)^{-1}\gamma_\mu+\tfrac{1}{2}\gamma_\mu(\not p_I-m)^{-1}\delta\Sigma(b,p_I)$$

$$=-\gamma_\mu(\alpha q^2/16m^3b)[(8m/3\lambda_{\min})-2\ln(2m/\lambda_{\min})]+\gamma_\mu(\alpha q^2/16m^3b)[(8m/3k_m)-2\ln(2m/k_m)] \ , \qquad (2.51)$$

where the q-independent term has been omitted because it could be absorbed into the mass renormalization.

Adding the result (2.51) to those derived in Secs. II A and II B we obtain

$$\Gamma_I^b+\Delta\Gamma^b+\delta\Gamma^b=(\alpha/4mb)(q^2/4m^2)[\tfrac{57}{15}+(8m/3k_m)-2\ln(2m/k_m)] \ . \qquad (2.52)$$

2. *For the case* $mZ^2\alpha^2 \ll k_m \ll mZ\alpha$

Let us now calculate contributions from soft photons whose wavelengths are large compared to the size of the atom. As mentioned before, the nonrelativistic calculation is sufficient for $mZ^2\alpha^2 \ll k_m \ll mZ\alpha$. First, we begin with discussions that are independent of photons being soft and then introduce later the restrictions due to soft photons. In second-order perturbation theory, the energy shift due to emission and reabsorption of a photon by an electron in the state β is given in the dipole approximation by

$$\Delta E^{<}=(\alpha/6m^2)(2\pi)^{-3}[(2\pi)^3/bL^2]\sum_{l,j,n}\sum_{i} |\langle\beta|\mathbf{p}|i\rangle|^2k^{-1}(E_\beta-E_i-k)^{-1} \ , \qquad (2.53)$$

which is analogous to the conventional expression.[2] As done in Secs. II A and II B, the summation over l and j,

$$J=\sum_{l,j,n} k^{-1}(E_\beta-E_i-k)^{-1}$$

$$=\sum_{l,j,n}[(\pi l/L)^2+(\pi j/L)^2+(\pi n/b)^2]^{-1/2}\{E_\beta-E_i-[(\pi l/L)^2+(\pi j/L)^2+(\pi n/b)^2]^{1/2}\}^{-1} \ , \qquad (2.54)$$

is replaced by integrals for $L \gg b$ as

$$J=\sum_{n=0}^{\infty}\int_0^\infty dx\int_0^\infty dy[(\pi/L)^2(x^2+y^2)+(\pi n/b)^2]^{-1/2}\{E_\beta-E_i-[(\pi/L)^2(x^2+y^2)+(\pi n/b)^2]^{1/2}\}^{-1}$$

$$=\tfrac{1}{8}(L/\pi)^2\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy[x^2+y^2+(Ln/b)^2]^{-1/2}\{(L/\pi)(E_\beta-E_i)-[x^2+y^2+(Ln/b)^2]^{1/2}\}^{-1} \ . \qquad (2.55)$$

Using the Poisson's summation formula on the Fourier transformation (A7) we find

$$J=C_0+C_b \ , \qquad (2.56)$$

where

$$C_0=(\pi/2)(L/\pi)^2(b/L)\int_0^\infty dr\, r[(L/\pi)(E_\beta-E_i)-r]^{-1} \qquad (2.57)$$

and

$$C_b=(\pi/2)(L/\pi)^2(b/L)\int_0^\infty dr\, r[(L/\pi)(E_\beta-E_i)-r]^{-1}\sum_{s=-\infty}^{\infty} j_0(2\pi bsr/L)$$

$$=(\pi^2/4b)(bL^2/\pi^3)\int_0^\infty dr[(L/\pi)(E_\beta-E_i)-r]^{-1} \ . \qquad (2.58)$$

The last expression in (2.58) can be obtained by use of (A14).

Contributions from the soft photons can be obtained by changing the infinite integrals into finite ones, i.e.,

$$C_0=(\pi/2)(bL^2/\pi^3)\int_0^{k_m} dk\, k\,(E_\beta-E_i-k)^{-1}\ , \tag{2.59}$$

$$C_b=(\pi^2/4b)(bL^2/\pi^3)\int_0^{k_m} dk(E_\beta-E_i-k)^{-1}\ , \tag{2.60}$$

where the variable was changed to $r=(L/\pi)k$. Thus Eq. (2.59) gives the expression derived by Bethe for free space.[5] The b-dependent term C_b is just the quantity which leads to the contributions from the soft photons with the boundary conditions considered here. Hereafter, we will only consider the b-dependent term.

Performing the integration in (2.60) and substituting into (2.53), we obtain

$$\Delta E^{<}(b)=(\alpha/3bm^2)\sum_i |\langle\beta|p|i\rangle|^2\ln|(E_i-E_\beta)/(E_\beta-E_i-k_m)|\ , \tag{2.61}$$

where m is the electron mass. Since $k_m \gg E_i-E_n$, where $E_i-E_n\approx(Z\alpha)^2m$, we have

$$\begin{aligned}\ln|(E_i-E_\beta)/(E_\beta-E_i-k_m)| &=\ln[(E_i-E_\beta)/k_m]-\ln[1+(E_i-E_\beta)/k_m]\\ &=\ln[(E_i-E_\beta)/k_m]-\{(E_i-E_\beta)/k_m-\tfrac{1}{2}[(E_i-E_\beta)/k_m]^2+\cdots\}\ .\end{aligned} \tag{2.62}$$

Then (2.61) can be approximated as

$$\Delta E^{<}(b)\approx(\alpha/3bm^2)\left[\sum_i |\langle\beta|\mathbf{p}|i\rangle|^2\ln[(E_i-E_\beta)/k_m]-(1/k_m)\sum_i|\langle\beta|\mathbf{p}|i\rangle|^2(E_i-E_\beta)\right]\ . \tag{2.63}$$

By introducing the Rydberg energy $R_y=m\alpha^2/2$, Eq. (2.63) can be rewritten in the form

$$\begin{aligned}\Delta E^{<}(b)\approx(\alpha/3bm^2)\Bigg[&\sum_i|\langle\beta|\mathbf{p}|i\rangle|^2\ln[(E_i-E_\beta)/R_y]+\xi\sum_i|\langle\beta|\mathbf{p}|i\rangle|^2\ln(2m/k_m)\\ &-(1/k_m)\sum_i|\langle\beta|\mathbf{p}|i\rangle|^2(E_i-E_\beta)+[\ln(R_y/k_m)-\xi\ln(2m/k_m)]m^2\langle\beta|\mathbf{v}^2|\beta\rangle\Bigg]\ .\end{aligned} \tag{2.64}$$

The last term can be absorbed into the mass renormalization since it has the same form as a contribution from a mass counter term,[2] and hence it can be omitted. The procedure suggested by Bethe *et al.*[6] and Harriman[7] yields the first and second terms in the forms

$$\sum_i|\langle\beta|\mathbf{p}|i\rangle|^2\ln[(E_i-E_\beta)/R_y] =(A/R_y)\sum_i|\langle\beta|\mathbf{p}|i\rangle|^2(E_i-E_\beta)\ , \tag{2.65}$$

$$\xi\sum_i|\langle\beta|\mathbf{p}|i\rangle|^2\ln(2m/k_m) =(\xi B/R_y)\ln(2m/k_m)\sum_i|\langle\beta|\mathbf{p}|i\rangle|^2(E_i-E_\beta)\ . \tag{2.66}$$

The detailed derivations of these equations are given in Appendix D. The numerical value of A and B are calculated by computer,

$$A=-0.28929,\quad B=0.50\ .$$

Considering the equivalence[2]

$$\sum_i|\langle\beta|\mathbf{p}|i\rangle|^2(E_i-E_\beta)=\tfrac{1}{2}\langle\beta|[(\mathbf{p},H),\mathbf{p}]|\beta\rangle =\tfrac{1}{2}\langle\beta|\nabla^2V|\beta\rangle \tag{2.67}$$

and taking $\xi=3R_y/4mb$, we find

$$\Delta E^{<}(b)=(\alpha/16m^3b)[(8mA/3R_y)+2\ln(2m/k_m) -(8m/3k_m)]\langle\beta|\nabla^2V|\beta\rangle\ . \tag{2.68}$$

Since the matrix element $\langle\beta|q^2|\beta\rangle$ can be expressed as $\langle\beta|\nabla^2V|\beta\rangle$ for small q, the result (2.68) is added to Eq. (2.52) to yield

$$(\alpha/16m^3b)[\tfrac{57}{15}+(8mA/3R_y)]\langle\beta|\nabla^2V|\beta\rangle\ , \tag{2.69}$$

which leads to an additional shift to the original Lamb shift. Finally, we note that a treatment without dividing into soft and hard photons has been given by Ericksen and Yennie.[8]

D. The additional shift to the transition $2S_{1/2}\rightarrow 2P_{1/2}$ in a hydrogen atom

For the Coulomb potential we have

$$\nabla^2V=4\pi Z\alpha\delta^3(r)\ . \tag{2.70}$$

The additional potential due to the two parallel perfectly conducting square plates at a distance b from each other is given by

$$\Delta U=(\pi Z\alpha^2/4m^3b)[\tfrac{57}{15}+(8mA/3R_y)]\delta^3(r)\ . \tag{2.71}$$

The energy shift can be obtained by evaluating the matrix element of the potential (2.71) with the wave function of hydrogen atoms as

$$\Delta E_{nlm} = \int \psi^*_{nlm}(r)\Delta U \psi_{nlm}(r)d^3r$$
$$= (\pi\alpha/4m^3b)[\tfrac{57}{15} + (8mA/3R_y)]\,|\psi_{nlm}(0)|^2 , \tag{2.72}$$

where

$$|\psi_{nlm}(0)|^2 = \begin{cases} 1/\pi n^3 a_0^3 & \text{for } l=0 \\ 0 & \text{for } l\neq 0 , \end{cases} \tag{2.73}$$

with $a_0 = (\hbar/mc\alpha) = 0.529\times 10^{-8}$ cm. Therefore, the $2S_{1/2}$ level is raised by

$$\Delta E_{200} = (\pi\alpha^2/4m^3b)[\tfrac{57}{15} + (8mA/3R_y)]/8\pi a_0^3 . \tag{2.74}$$

The frequency shift for the transition $2S_{1/2}\rightarrow 2P_{1/2}$ is thus given by dividing ΔE_{200} by the Planck constant h. This shift is entirely caused by the boundaries considered in the present paper.

III. NUMERICAL RESULTS

The shift for the $2S_{1/2}\rightarrow 2P_{1/2}$ transition in a hydrogen atom due to our specifically chosen boundaries is proportional to b^{-1}. It obviously vanishes as b goes to infinity. The numerical value of this shift can be obtained from Eq. (2.74). The result is

$$\Delta\nu = -0.894 b^{-1} \text{ MHz} ,$$

when b is measured in μm. Namely, for $b = 1$ μm, we obtain $\Delta\nu = -0.894$ MHz. This is consistent with value -0.347 MHz calculated by a semiclassical treatment.[9] We have used

$$\alpha = 1/137.035\,987(29) , \tag{3.1}$$

which was recommended by Lauptrup *et al.*[10] Our present results are sufficiently large enough to be observed. Even $\Delta\nu = -0.089$ MHz for $b = 10$ μm can be observed by presently available techniques, as it is larger than existing experimental errors.

Recent experiments have obtained errors less than 0.009 MHz. The best values to date for the Lamb shift of the hydrogen atom are

$$\Delta\nu = 1057.845 \pm 0.009 \text{ MHz}$$

by Lundeen *et al.*,[11]

$$\Delta\nu = 1057.862 \pm 0.020 \text{ MHz}$$

by Andrews *et al.*,[12] and

$$\Delta\nu = 1057.8514 \pm 0.0019 \text{ MHz}$$

by Sokolov *et al.*[13]

IV. CONCLUSION

Through this investigation we have seen that the fluctuating vacuum field is strongly affected by boundaries made from two parallel perfectly conducting plates. When a hydrogen atom is placed between two parallel square plates made from perfect conductors, one can expect to observe a new shift in the frequency of radiation for the transition $2S_{1/2}\rightarrow 2P_{1/2}$, in addition to the usual Lamb shift.

In addition, it might be possible for the hydrogen atom to interact with its own radiation field, which would be reflected by the surfaces. This interaction would also shift radiation frequencies. The effects of the coupling of an excited two-level system with itself through the electric dipole radiation reflected by a nearby mirror have been investigated by numerous physicists.[14–16] This effect has recently been observed as a frequency shift in the emitted radiation.[17] This shift is significant only if the wavelength of the radiation field is of the order of the distance between the electric dipole and the reflecting mirror.

For the transition $2S_{1/2}\rightarrow 2P_{1/2}$ in the hydrogen atom, the wavelength of radiation is about 28.3 cm and it is extremely large compared to the distance $b = 1-10$ μm between two parallel plates. Therefore, reflection of the radiation field by the walls is not feasible.

Finally, we stress the point that the additional frequency shift discussed in this paper is sufficiently large to be observed by presently available techniques, even if it is small compared to the conventional Lamb shift. The shift predicted in the present investigation is a striking result of the quantum-mechanical prediction of a dynamic vacuum.

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APPENDIX A

We evaluate the Feynman integrals

$$\int_{-\infty}^{\infty} dk_0 \left[\frac{(2\pi)^3}{bL^2}\right] \sum_{l,j,n} [k_0^2 - (\pi l/L)^2 - (\pi j/L)^2 - (\pi n/b)^2 + i\varepsilon - \Delta_0]^{-3} . \tag{A1}$$

To do this, we can make use of the integral

$$\int_{-\infty}^{\infty} [k_0^2 + i\varepsilon - (\Delta_0 + \mathbf{k}^2)]^{-1} dk_0 = -i\pi(\Delta_0 + \mathbf{k}^2)^{-1/2} , \tag{A2}$$

which can easily be obtained. Double differentiation of (A2) with respect to Δ_0 gives

$$\int_{-\infty}^{\infty} [k_0^2 + i\varepsilon - (\Delta_0 + \mathbf{k}^2)]^{-3} dk_0 = -i\pi\tfrac{3}{8}(\Delta_0 + \mathbf{k}^2)^{-5/2} . \tag{A3}$$

Then we obtain

$$-i\pi\tfrac{3}{8}\left[\frac{(2\pi)^3}{bL^2}\right]\times\sum_{l,j,n=0}^{\infty}[\Delta_0+(\pi l/L)^2+(\pi j/L)^2+(\pi n/b)^2]^{-5/2}\ , \tag{A4}$$

which is equal to (A1). When $L \gg b$, and where L actually goes to ∞, the summations over l and j in (A4) can be replaced by integrals as

$$(L/\pi)^5\sum_{n=0}^{\infty}\int_0^{\infty}dx\int_0^{\infty}dy[x^2+y^2+(Ln/b)^2+(L/\pi)^2\Delta_0]^{-5/2}$$
$$=\tfrac{1}{4}(L/\pi)^5\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy[x^2+y^2+(Ln/b)^2+(L/\pi)^2\Delta_0]^{-5/2}. \tag{A5}$$

Now consider the summation

$$\sum_{s=-\infty}^{\infty}C_s\equiv\tfrac{1}{8}(L/\pi)^5\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy[x^2+y^2+(Ln/b)^2+(L/\pi)^2\Delta_0]^{-5/2}\ . \tag{A6}$$

Use of the Poisson's summation formula on the Fourier transformation,

$$\sum_{n=-\infty}^{\infty}f(n)=\sum_{s=-\infty}^{\infty}\int_{-\infty}^{\infty}f(t)\exp(2\pi ist)dt\ , \tag{A7}$$

gives

$$\sum_{n=-\infty}^{\infty}[x^2+y^2+(Ln/b)^2+(L/\pi)^2\Delta_0]^{-5/2}=\sum_{s=-\infty}^{\infty}\int_{-\infty}^{\infty}[x^2+y^2+(L/b)^2t^2+(L/\pi)^2\Delta_0]^{-5/2}\exp(2\pi ist)dt$$
$$=\left[\frac{b}{L}\right]\sum_{s=-\infty}^{\infty}\int_{-\infty}^{\infty}[x^2+y^2+z^2+(L/\pi)^2\Delta_0]^{-5/2}\exp(2\pi ibsz/L)dz\ . \tag{A8}$$

Thus

$$\sum_{s=-\infty}^{\infty}C_s=\tfrac{1}{8}(L/\pi)^5(b/L)\sum_{s=-\infty}^{\infty}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dz[x^2+y^2+z^2+(L/\pi)^2\Delta_0]^{-5/2}\exp(2\pi ibsz/L)$$
$$=\tfrac{1}{8}(L/\pi)^5(b/L)\sum_{s=-\infty}^{\infty}\int_0^{\infty}dr\,r^2\int_0^{2\pi}d\psi\int_{-1}^{1}d\xi[r^2+(L/\pi)^2\Delta_0]^{-5/2}\sum_{\mu=0}^{\infty}(2\mu+1)i^{\mu}j_{\mu}(2\pi bsr/L)P_{\mu}(\xi)\ , \tag{A9}$$

where j and P are the spherical Bessel function and the Legendre polynomials. After integration over angles, we obtain

$$\sum_{s=-\infty}^{\infty}C_s=(4\pi/8)(L/\pi)^5(b/L)\times\int_0^{\infty}dr\,r^2[r^2+(L/\pi)^2\Delta_0]^{-5/2}\times\sum_{s=-\infty}^{\infty}j_0(2\pi bsr/L)\ . \tag{A10}$$

Comparing (A5) with (A6), we easily find

$$\tfrac{1}{4}(L/\pi)^5(b/L)\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy[x^2+y^2+(Ln/b)^2+(L/\pi)^2\Delta_0]^{-5/2}=C_0+\sum_{s=-\infty}^{\infty}C_s\ , \tag{A11}$$

where

$$C_0=(4\pi/8)(L/\pi)^5(b/L)\int_0^{\infty}dr\,r^2[r^2+(L/\pi)^2\Delta_0]^{-5/2} \tag{A12}$$

from (A10). The summation in (A10) can be evaluated by use of the Euler-Maclaulin formula

$$\sum_{s=0}^{\infty}f(s)=\int_0^{\infty}f(t)dt-\tfrac{1}{2}[f(x)]_0^{\infty}+\sum_{k=1}^{\infty}(B_{2k}/(2k)!)[f^{(2k-1)}(x)]_0^{\infty}\ , \tag{A13}$$

where B_{2k} is Bernoulli's number. Considering $j_0(\infty)=0$, $j_0^{(2k-1)}(\infty)=j_0^{(2k-1)}(0)=0$, and $j_0(0)=1$, we obtain

$$\sum_{s=0}^{\infty}j_0(2\pi bsr/L)=(L/4br)+\tfrac{1}{2}\ . \tag{A14}$$

Since

$$\sum_{s=-\infty}^{\infty} j_0(2\pi bsr/L)=2\sum_{s=0}^{\infty} j_0(2\pi bsr/L)-1=(L/2br)\ , \tag{A15}$$

$$\sum_{s=-\infty}^{\infty} C_s=(\pi L/4b)(L/\pi)^5(b/L)\times\int_0^{\infty} dr\, r[r^2+(L/\pi)^2\Delta_0]^{-5/2}=\frac{2\pi^2}{3b}\frac{bL^2}{(2\pi)^3}\Delta_0^{-3/2}\ . \tag{A16}$$

Similarly, from (A12) we obtain

$$C_0=\frac{4\pi}{3}\frac{bL^2}{(2\pi)^3}\Delta_0^{-1}\ , \tag{A17}$$

which actually gives the results for free space, i.e., no specific boundaries. Therefore the effects of our boundaries can be obtained from the $\sum_{s=-\infty}^{\infty}C_s$ term in (A11), i.e., (A16).

Thus

$$\int_{-\infty}^{\infty} dk_0\left[\frac{(2\pi)^3}{bL^2}\right]\times\sum_{l,j,n}[k_0^2-(\pi l/L)^2-(\pi j/L)^2-(\pi n/b)^2+i\varepsilon-\Delta_0]^{-3}=-i(\pi^2/2)\Delta_0^{-1}-i(\pi^3/4b)\Delta_0^{-3/2}\ . \tag{A18}$$

Defining that $k_1=(\pi l/L)$, $k_2=(\pi j/L)$, and $k_3=(\pi n/b)$, we have

$$k^2\equiv k_0^2-[(\pi l/L)^2+(\pi j/L)^2+(\pi n/b)^2]\ , \tag{A19}$$

$$pk\equiv p_0k_0-\mathbf{p}\cdot\mathbf{k}=p_0k_0-[p_1(\pi l/L)+p_2(\pi j/L)+p_3(\pi n/b)]\ , \tag{A20}$$

and then (A18) can be written as

$$(2\pi)^{-4}\int_{-\infty}^{\infty} dk_0\left[\frac{(2\pi)^3}{bL^2}\right]\sum_{l,j,n}(k^2+i\varepsilon-\Delta_0)^{-3}=-i(1/32\pi^2)\Delta_0^{-1}-i(1/64\pi b)\Delta_0^{-3/2}\ . \tag{A21}$$

If $k-p$ is substituted for the variable of integration, (A21) becomes

$$(2\pi)^{-4}\int_{-\infty}^{\infty} dk_0\left[\frac{(2\pi)^3}{bL^2}\right]\sum_{l,j,n}(k^2-2kp+i\varepsilon-\Delta)^{-3}=-i(1/32\pi^2)(\Delta+p^2)^{-1}-i(1/64\pi b)(\Delta+p^2)^{-3/2}\ , \tag{A22}$$

where $\Delta=\Delta_0-p^2$.

It is easy to verify the following integral:

$$\int_{-\infty}^{\infty} dk_0k_0\left[\frac{(2\pi)^3}{bL^2}\right]\sum_{l,j,n}[k_0^2-(\pi l/L)^2-(\pi j/L)^2-(\pi n/b)^2+i\varepsilon-\Delta_0]^{-1}=0\ , \tag{A23}$$

and, then, double differentiation with respect to Δ_0 gives

$$\int_{-\infty}^{\infty} dk_0k_0\left[\frac{(2\pi)^3}{bL^2}\right]\sum_{l,j,n}[k_0^2-(\pi l/L)^2-(\pi j/L)^2-(\pi n/b)^2+i\varepsilon-\Delta_0]^{-3}=0\ . \tag{A24}$$

Thus the replacement of k by $k-p$ gives

$$(2\pi)^{-4}\int_{-\infty}^{\infty} dk_0k_0\left[\frac{(2\pi)^3}{bL^2}\right]\sum_{l,j,n}(k^2-2kp+i\varepsilon-\Delta)^{-3}=-i(1/32\pi^2)p_0(\Delta+p^2)^{-1}-i(1/64\pi b)p_0(\Delta+p^2)^{-3/2}\ . \tag{A25}$$

Similarly, we obtain

$$(2\pi)^{-4}\int_{-\infty}^{\infty} dk_0\left[\frac{(2\pi)^3}{bL^2}\right]\sum_{l,j,n}k_j(k^2-2kp+i\varepsilon-\Delta)^{-3}=-i(1/32\pi^2)p_j(\Delta+p^2)^{-1}-i(1/64\pi b)p_j(\Delta+p^2)^{-3/2}\ . \tag{A26}$$

Accordingly, the results (A22), (A25), and (A26) can be combined as

$$(2\pi)^{-4}\int_{-\infty}^{\infty} dk_0\left[\frac{(2\pi)^3}{bL^2}\right]\times\sum_{l,j,n}(1;k_\sigma)(k^2-2kp+i\varepsilon-\Delta)^{-3}=-i(1/32\pi^2)(1;p_\sigma)(\Delta+p^2)^{-1}-i(1/64\pi b)(1;p_\sigma)(\Delta+p^2)^{-3/2}\ . \tag{A27}$$

Differentiation of (A27) with respect to Δ and p_σ gives

$$(2\pi)^{-4}\int_{-\infty}^{\infty} dk_0\left[\frac{(2\pi)^3}{bL^2}\right]\sum_{l,j,n}(1;k_\sigma)(k^2-2pk-\Delta)^{-4}=i(1/96\pi^2)(1;p_\sigma)(\Delta+p^2)^{-2}+i(1/128\pi b)(1;p_\sigma)\times(\Delta+p^2)^{-5/2}\ , \tag{A28}$$

$$(2\pi)^{-4}\int_{-\infty}^{\infty} dk_0\left[\frac{(2\pi)^3}{bL^2}\right]\sum_{l,j,n}k_\sigma k_\rho(k^2-2pk-\Delta)^{-4}=i(1/96\pi^2)[p_\sigma p_\rho-\eta_{\sigma\rho}(\Delta+p^2)/2](\Delta+p^2)^{-2}+i(1/128\pi b)[p_\sigma p_\rho-\eta_{\sigma\rho}(\Delta+p^2)/3](\Delta+p^2)^{-5/2}\ , \tag{A29}$$

where $\eta_{00}=-\eta_{11}=-\eta_{22}=-\eta_{33}=1$ and other elements vanish. Notice that the first terms in (A28) and (A29) are those obtained initially by Feynman[18] for the case of free space, while the second terms are due to our specific boundaries. Therefore we can rewrite Eq. (A27) in the form

$$(2\pi)^{-4}\int_{-\infty}^{\infty}dk_0\left[\frac{(2\pi)^3}{bL^2}\right]\sum_{l,j,n}(1;k_\sigma)(k^2-2kp+i\varepsilon-\Delta)^{-3}-(2\pi)^{-4}\int_{-\infty}^{\infty}d^4k\left[\frac{(2\pi)^3}{bL^2}\right](1;k_\sigma)(k^2-2pk+i\varepsilon-\Delta)^{-3}$$
$$=-i(1/64\pi b)(1;p_\sigma)(\Delta+p^2)^{-3/2}\,, \qquad \text{(A30)}$$

which is the same type of equation given in Sec. I. We can also give the similar expressions for Eqs. (A28) and (A29).

APPENDIX B

Under the conditions $\lambda^2\to\infty$ and $\lambda_{\min}^2\to 0$ we will perform the following integrals:

$$\int_0^1 dx\int_{\lambda_{\min}^2}^{\lambda^2}d\xi\,x(1-x)[x^2p_y^2+(1-x)\xi]^{-5/2}$$
$$=-\tfrac{2}{3}\int_0^1 dx\{x[x^2p_y^2+(1-x)\xi]^{-3/2}\}_{\lambda_{\min}^2}^{\infty}$$
$$\approx\tfrac{2}{3}\lambda_{\min}^{-1}p_y^{-2}-\tfrac{1}{3}p_y^{-3}\,, \qquad \text{(B1)}$$

$$\int_0^1 dx\int_{\lambda_{\min}^2}^{\lambda^2}d\xi\,x^2(1-x)[x^2p_y^2+(1-x)\xi]^{-5/2}$$
$$\approx\tfrac{2}{3}p_y^{-3}[\ln(2p_y/\lambda_{\min})-1]\,, \qquad \text{(B2)}$$

$$\int_0^1 dx\int_{\lambda_{\min}^2}^{\lambda^2}d\xi\,x^3(1-x)[x^2p_y^2+(1-x)\xi]^{-5/2}\approx\tfrac{4}{3}p_y^{-3}\,, \qquad \text{(B3)}$$

$$\int_0^1 dx\int_{\lambda_{\min}^2}^{\lambda^2}d\xi\,x(1-x)[x^2p_y^2+(1-x)\xi]^{-3/2}\approx 2p_y^{-1}\,. \qquad \text{(B4)}$$

APPENDIX C

When the variable y is changed into ϕ as

$$y=\tfrac{1}{2}[(\tan\phi/\tan\theta)+1]\,, \qquad \text{(C1)}$$

we have

$$p_y^2=[yp_I+(1-y)p_F]^2$$
$$=m^2-q^2(y-y^2)=[m(\cos\theta/\cos\phi)]^2\,, \qquad \text{(C2)}$$

where $q^2=4m^2\sin^2\theta$ was used. Then we easily evaluate the following integrals:

$$\int_0^1 dy\,p_y^{-2}=(2\theta/m^2\sin 2\theta)\,, \qquad \text{(C3)}$$

$$\int_0^1 dy\,p_y^{-3}=1/m^3\cos^2\theta\,, \qquad \text{(C4)}$$

$$\int_0^1 dy\,p_y^{-3}p_{y\sigma}\ln(p_y^2/\lambda_{\min}^2)=(p_{I\sigma}+p_{F\sigma})\{[\ln(m/\lambda_{\min})+1]/m^3\cos^2\theta-\ln[(1+\sin\theta)/(1-\sin\theta)]/m^3\sin 2\theta\cos\theta\}\,, \qquad \text{(C5)}$$

$$\int_0^1 dy\,p_y^{-1}=\ln[(1+\sin\theta)/(1-\sin\theta)]/2m\sin\theta\,, \qquad \text{(C6)}$$

$$\int_0^1 dy\,p_y^{-3}p_{y\sigma}=(p_{I\sigma}+p_{F\sigma})/2m^3\cos^2\theta\,, \qquad \text{(C7)}$$

$$\int_0^1 dy\,p_y^{-3}p_{y\sigma}p_{y\tau}=(p_{I\sigma}+p_{F\sigma})(p_{I\tau}+p_{F\tau})/4m^3\cos^2\theta$$
$$+(p_{I\sigma}-p_{F\sigma})(p_{I\tau}-p_{F\tau})\{\ln[(1+\sin\theta)/(1-\sin\theta)]-2\sin\theta\}/8m^3\sin^3\theta\,. \qquad \text{(C8)}$$

In the evaluation of (C5), (C6), and (C8) we have used the formula

$$\int d\phi(1/\cos\phi)=\ln[(1+\sin\phi)^{1/2}(1-\sin\phi)^{-1/2}]\,. \qquad \text{(C9)}$$

APPENDIX D

Following the treatment given by Bethe,[6] we derive Eqs. (2.65) and (2.66). The oscillator strength is defined as[7]

$$|\langle nl\,|\,\mathbf{p}\,|\,i\rangle|^2=(mR_y/3\nu)g(i,nl)\,, \qquad \text{(D1)}$$

where

$$\nu=(E_i-E_{nl})/R_y\,. \qquad \text{(D2)}$$

When $(nl)=2s$,

$$g(j,2s)=\begin{cases}1024j(j^2-1)(j^2-4)^{-3}[(j-2)/(j+2)]^{2i} & \text{for } 3\le j\le 49\\ (0.343\,514j^{-3}+0.114j^{-5}) & \text{for } 50\le j\end{cases} \qquad \text{(D3)}$$

for transitions into discrete states, and for the continuous spectrum it is taken in the form

$$dg(i,2s)/d\nu=2\nu^{-2}(4+3\nu^{-1})\exp\{-4(\nu-\tfrac{1}{4})^{-1/2}\text{arccot}[(\nu-\tfrac{1}{4})^{-1/2}/2]\}\{1-\exp[-2\pi(\nu-\tfrac{1}{4})^{-1/2}]\}^{-1}\ , \tag{D4}$$

and the summation over i is replaced by an integration over $d\nu$. Then, by numerical computation, we obtain

$$\sum\!\!\!\!\!\!\int g(i,2s)\equiv\sum_{i=3}^{\infty}g(i,2s)+\int_{1/4}^{\infty}[dg(i,2s)/d\nu]d\nu=(0.026\,43)+(0.973\,57)=1\ , \tag{D5}$$

$$A\equiv\sum\!\!\!\!\!\!\int g(i,2s)[(\ln\nu)/\nu]=-0.289\,29\ , \tag{D6}$$

$$B\equiv\sum\!\!\!\!\!\!\int g(i,2s)/\nu=0.50\ , \tag{D7}$$

where we have used the continuum energy $E_n=R_y/n^2$ expressed with the quasiprincipal quantum number n. The symbol $\sum\!\!\!\!\!\!\int$ denotes a summation and integral for discrete and continuous states. Thus we find

$$\begin{aligned}&\left[\sum\!\!\!\!\!\!\int|\langle 2s|\mathbf{p}|i\rangle|^2\ln[(E_i-E_{2s})/R_y]\right]\Big/\sum\!\!\!\!\!\!\int|\langle 2s|\mathbf{p}|i\rangle|^2(E_i-E_{2s})\\&=(mR_y/3)\sum\!\!\!\!\!\!\int g(i,2s)\nu^{-1}\ln\nu)/(mR_y^2/3)\sum\!\!\!\!\!\!\int g(i,2s)\\&=A/R_y\ .\end{aligned} \tag{D8}$$

This result leads to Eq. (2.65). Similarly, Eq. (2.66) can easily be derived.

[1]S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Academic, New York, 1961), Chap. 15.

[2]J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964), Chap. 8.

[3]H. B. G. Casimir, Proc. Ned. Akad. Wet. **51**, 793 (1948); J. Chem. Phys. **46**, 407 (1949). See also, C. Itzykson and J. B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980), Chap. 3.

[4]M. J. Sparnaay, Physica (Utrecht) **24**, 751 (1958).

[5]H. A. Bethe, Phys. Rev. **72**, 339 (1947).

[6]H. A. Bethe, L. M. Brown, and J. R. Stehn, Phys. Rev. **77**, 370 (1950).

[7]J. M. Harriman, Phys. Rev. **101**, 594 (1956).

[8]G. Ericksen, Doctoral dissertation, University of Minnesota, 1959.

[9]Il.-T. Cheon, *Abstracts of the Proceedings of the Tenth International Conference on Atomic Physics, Tokyo, 1986,* edited by H. Namuri (North-Holland, Amsterdam, 1987), p. 46.

[10]B. E. Lauptrup, A. Peterman, and E. Rafael, Phys. Rep. C **3**, 193 (1972).

[11]S. R. Lundeen and F. M. Pipkin, Metrologia **22**, 1 (1986).

[12]D. A. Andrews and G. Newton, Phys. Rev. Lett. **37**, 1254 (1976).

[13]Yu. L. Sokolov and V. P. Yakovlev, Zh. Eksp. Teor. Fiz. **83**, 15 (1982) [Sov. Phys.—JETP **56**, 7 (1982)].

[14]R. R. Chance, A. Prock, and Silbey, Phys. Rev. A **12**, 1448 (1975).

[15]H. Morawitz, Phys. Rev. **187**, 1792 (1969).

[16]H. Kuhn, J. Chem. Phys. **53**, 101 (1970).

[17]W. R. Holland and D. G. Hall, Phys. Rev. Lett. **52**, 1041 (1984).

[18]R. P. Feynman, Phys. Rev. **76**, 769 (1949); *Quantum Electrodynamics* (Benjamin, New York, 1962).