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Wire perturbations in the Saffman-Taylor problem

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Zocchi, Shaw, Libchaber, and Kadanoff recently discovered that when two wires are symmetrically placed along the center of a Hele-Shaw cell, symmetric but narrow fingers of dimensionless width $\lambda < \frac{1}{2}$ develop. λ decreases as the pushing velocity increases, but at a certain critical finger width the finger suddenly undergoes a transition to the asymmetrical state. We present a simple theory to predict this critical finger width as a function of D, the dimensionless distance between two wires, by assuming that the finger opens up a negative angle at the contact point.

In a recent experiment conducted by Zocchi, Shaw, Libchaber, and Kadanoff,[1] new experimental findings were reported for asymmetric Saffman-Taylor fingers[2-6] in a Hele-Shaw cell. They discovered that a wire in a channel produces a local deformation on the finger surface, and it results in a dramatic change in a finger shape as well as in a finger width. When the wire is placed at the center of the channel, an asymmetrical finger is observed with a substantial reduction in a finger width. Moreover, when they place two wires symmetrically along the center line, they observe a transition from symmetric to asymmetric finger state as a pushing velocity increases. A sharp transition occurs at a critical finger width λ_c, below which asymmetric fingers show up. This new discovery is reminiscent of a recent experiment conducted by Couder and his collaborators.[7,8] They first injected a small bubble along the center of the channel. When the tip of the finger touched the bubble, the bubble became trapped; and after a transient period, a new finger with a narrow width developed. In this case, however, the finger was always symmetric.

The questions to be addressed in this paper are the following, First, why does an asymmetric finger show up with a wire at the center of the channel? Second, why does a transition from symmetric to asymmetric finger state occur when two wires are placed in parallel along the center? How can one predict a critical finger width?

In a previous report,[9] a simple model was proposed to explain Couder's experiment. In this purely phenemelogical picture, the bubble was replaced by a cusp, and the effective opening angle due to this cusp, estimated within a linear theory, was *positive*. Several predictions were made based on this simple picture, and these predictions appear to be in excellent agreement with the experiment.

Therefore, it seems quite natural to try to understand the experiment of Zocchi *et al.* along this direction. As in Couder's experiment, we are not interested in the modified flow field produced by the wire. Rather, we again assume that at the contact point made by the wire, the finger allows a cusp. The effective opening angle at the contact point, however, is *not positive,* because we never observe a symmetric finger with the wire at the center. The proposal in this paper is that the wire is opening up a *negative* angle.[10]

In Fig. 1 are schematic pictures which show how the fingers with positive [Figs. 1(a) and 1(b)] or negative [Fig. 1(c)] opening angle look. Figure 1(a) shows a finger with a positive opening angle and Fig. 1(b) is Couder's finger with a bubble at the tip. Note that the bubble is creating a *smooth* valley, and we interpret this valley as opening up a positive angle by joining both sides of the finger. The cusp appears in this picture at the joining point, and it is above the smooth finger. The strength of the cusp is measured by the discontinuity of the tangential slope, $\Delta\theta$, defined as

$$\Delta\theta = \left[\left(\frac{d\zeta}{dx} \right)_- - \left(\frac{d\zeta}{dx} \right)_+ \right] \Big/ \left[1 + \left(\frac{d\zeta_0}{dx} \right)^2 \right], \quad (1)$$

where $\zeta(x)$ and $\zeta_0(x)$ are equations for the finger surface

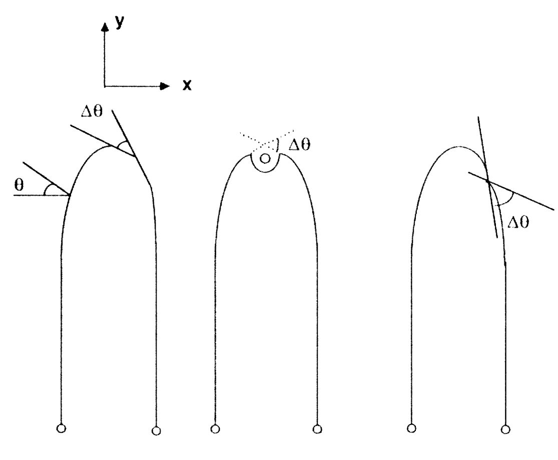

FIG. 1. Schematic pictures of fingers with positive and negative opening angles on the Saffman-Taylor's zero surface tension profile. (a) Cusp appears above the profile and the mismatch angle due to this cusp defined in (1) is positive. (b) Couder's finger with a bubble at the tip. Cusp appears at the joining points of both sides of the finger, and the mismatch angle is positive. (c) Cusp created by a wire in the experiment of Zocchi *et al.* Cusp appears under the finger profile, and the mismatch angle defined in (1) is negative.

with and without surface tension and + and − refer to the limit approaching from left and right, respectively. Conventionally, $\Delta\theta$ is called a mismatch angle, and it is positive in Figs. 1(a) and 1(b). However, when we carefully examine the photograph of a finger with wire, we find that the contact point is slightly pushed backward creating a *sharp* cusp at that point. This cusp is beneath the smooth finger and, by definition (1), the mismatch angle due to this cusp, will, therefore, be *negative*.

In Ref. 9, a simple and elegant way was developed to estimate $\Delta\theta$ at the tip. The strategy in this paper is to explore the analysis further to the case where a cusp appears not at the tip but at an *arbitrary point* on the finger surface. This paper focuses entirely on the symmetric case.

We start our analysis by defining the relevant physical parameters. The system of interest is an effectively two-dimensional channel of width $2W$ and thickness $b \ll W$ along which a fluid of viscosity μ is being pushed by an immiscible second fluid of relatively negligible viscosity. Both fluids are incompressible. We also denote the surface tension, speed of the second fluid, and the width of the asymptotic finger as γ, U, $2\lambda W$, respectively. The angle made by the normal vector against the x axis is denoted as $\theta = \theta_0 + \nu\theta_1$, where θ_0 is the zero-surface-tension solution and θ_1 is the first-order correction. In what follows, we use a variable η, which is the slope of the zero-surface-tension Saffman-Taylor solution. If we assume that the shape correction due the surface tension is small, then the relation between η and the real variable x defined in Fig. 1 is given by

$$\eta = \frac{1-\lambda}{\lambda}\tan\left[\frac{\pi x}{2\lambda}\right] . \tag{2}$$

Suppose now that the cusp appears at η_0 with the mismatch angle $\Delta\theta(\eta_0)$. A simple way of determining η_0 as well as $\Delta\theta(\eta_0)$, within a linear approximation, will be to write an equation of motion for θ (Ref. 6) in the form[9]

$$\nu\frac{d^2\theta}{d\eta^2} + Q1(\eta)\theta(\eta) + \frac{1}{\pi}\mathrm{P}\int_{-\infty}^{+\infty} d\eta' \frac{Q_2(\eta,\eta')\theta(\eta')}{\eta-\eta'} = R(\eta) + \Delta\theta(\eta_0) f(\eta_0)\frac{d}{d\eta}\delta(\eta-\eta_0) , \tag{3}$$

where P denotes the principle value and

$$Q_1(\eta) = \frac{4\beta(1+\eta^2)^{1/2}}{(1+\beta^2\eta^2)^{1/2}} , \tag{4a}$$

$$Q_2(\eta,\eta') = \frac{4\eta\beta^4(1+\eta^2)^{1/4}(1+\eta'^2)^{1/4}}{(1+\beta^2\eta'^2)^{1/2}(1\beta^2\eta'^2)^{3/2}} , \tag{4b}$$

$$R(\eta) = \frac{\eta[3+\beta^2(\eta^2-2)]}{(1+\beta^2\eta^2)^{1/2}(1+\beta^2\eta'^2)^{9/4}} , \tag{4c}$$

$$f(\eta_0) = \frac{(1+\beta^2\eta_0^2)^{1/2}}{(1+\eta_0^2)^{1/4}} , \tag{4d}$$

and

$$\beta = \frac{\lambda}{1-\lambda} , \tag{4e}$$

$$\nu = \frac{b^2\gamma\pi^2}{12\mu U W^2(1-\lambda)^2} . \tag{4f}$$

Without a delta-function term in the right-hand side, Eq. (3) describes the equation of motion for θ_1. Note that (3) describes the half profile of symmetric finger and thus when the cusp appears at η it is assumed that at $-\eta$ is also a cusp. The true solution of (3) is a smooth finger everywhere except right at η_0. At η_0 we expect a finger to open up an angle, either positive or negative, of magnitude $\Delta\theta(\eta_0)$. Here we are not interested in solving (3). Instead, we ask how the new term modifies the solvability condition.[11-18] Note that the null eigenvectors to (3) remain unaffected by this new term; and since we are dealing with a second-order differential equation, we expect to find two independent null eigenvectors and thus two solvability conditions. Since the experiment only concerns fingers of $\lambda < \frac{1}{2}$, here we also focus on the narrow fingers of $\lambda < \frac{1}{2}$. For the most stable finger profile with the smallest λ, these two solvability conditions can be, in the limit of small ν and small η_0, approximately written as[19]

$$-\frac{\eta_0}{2} \approx \tan\left[\frac{2}{\pi}\left[\frac{1}{B}\right]^{1/2}\frac{\lambda^2}{1-\lambda}\eta_0\right] , \tag{5a}$$

$$\Delta\theta(\eta_0) \approx -\exp\left[\frac{1}{\pi}\left[\frac{1}{B}\right]^{1/2}\frac{\lambda^2}{1-\lambda}\frac{\eta_0^2}{2}\right]\Delta\theta(0) , \tag{5b}$$

where we have defined a new parameter $B = \nu(1-\lambda)^2/\pi^2$ and $\Delta\theta(0)$ is the positive opening angle at the tip, which opens up when we relax the tip but impose a correct boundary condition at the tail. $\Delta\theta(0)$ is given in Ref. 9 [see Eq. (3.3)]. Note that the mismatch angle given by (5b) is *negative*.

For given B, we substitute λ and η_0, which are the solutions of (5a) into (5b). In Fig. 2, $\Delta\theta(\eta_0)/N$ is plotted against η_0 for various values of B, where $N \approx 2.008$ is the multiplicative constant. $\Delta\theta(\eta_0)$ does have different values for different B. But the dependence on B seems to be relatively weak, and we *ignore* this dependence in this work and assume that $\Delta\theta(\eta_0)$ only depends on η_0.

Let us now examine carefully what is happening on the

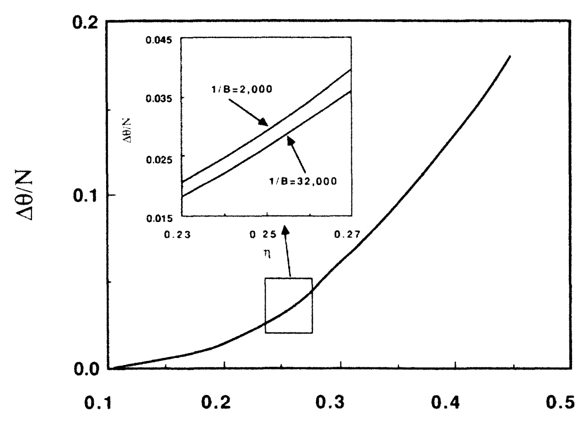

FIG. 2. The mismatch angle, $\Delta\theta/N$, evaluated by (5b) for different values of B. $\Delta\theta/N$ is weakly dependent on B and appears to be a strong function of η, the tangential slope of zero-surface-tension solution.

finger surface when the two wires are symmetrically placed along the center. Since the wires are at fixed positions and the finger width λ decreases as the pushing velocity increases, the contact angle made by the wire on the finger surface will also increase. Now the sudden jump to the asymmetric finger at a certain critical finger width implies that there is a maximum for the contact angle allowed by the system, above which the finger is no longer stable. If we assume that the contact angle is only a function of the slope, then the tip of the finger should shift after the contact angle reaches the maximum, so that the slope at the contact point becomes smaller, thus making the contact angle again less than the allowed value. This is, indeed, what was observed by Zocchi *et al.* After the jump to the asymmetric state, the tip of the finger shifts either to the right or left, and it was reported that the selection of the symmetric state is determined by the wire nearest to the tip. We thus make the following model: The mismatch angle created by the wire on the finger surface is negative. Its absolute value mainly depends on the geometry of the finger shape and is a monotonically increasing function of η_0, the tangential slope of Saffmann-Taylor's zero-surface-tension solution. When the absolute value of the mismatch angle is greater than $\Delta\theta_{\max}$ the finger is unstable and undergoes transition to the asymmetric state. Here $\Delta\theta_{\max}$ is the upper bound for the mismatch angle created by the wire at the finger surface. Note that the mismatch angle is not the same as the contact angle made by the wire. It is too small to be literally interpreted as an observable contact angle. The lack of precise relation between the mismatch angle and the contact angle, however, does not nullify our assumption that there is an upper bound for $\Delta\theta$ as long as the contact angle is proportional to the mismatch angle. Since the absolute value of $\Delta\theta(\eta_0)$ is monotonically increasing as we go down over the finger surface and we assume $\Delta\theta(\eta_0)$ mainly depends on η_0, there will be η_c in η_0 space at which the mismatch angle hits the maximum allowed by the system, above which the symmetric finger is no longer stable and, therefore, should undergo a transition to the asymmetric state in order to make the absolute value of the mismatch angle smaller. Thus, our prediction will be that as long as two wires are placed in η space at $0<\eta<\eta_c$, the finger assumes a symmetric shape but will jump to an asymmetric state when the $\eta_c \leq \eta_0$, where $-\Delta\theta(\eta_c)=\Delta\theta_{\max}$. Determining η_c theoretically requires knowledge of $\Delta\theta_{\max}$ and is beyond the scope of the present approach. In this paper, we set η_c or, equivalently, $\Delta\theta_{\max}$, as a free parameter and fix it by the experimental data.

The relation between η_0 and x is given by (2). For fixed x, as η_0 decreases λ increases. Since the symmetric solutions exist with negative opening angle for $0<\eta_0\leq\eta_c$, the prediction will be that when two wires are placed symmetrically along the center, a dimensionless distance D apart, symmetric solutions exist only for $\lambda_c \leq \lambda \leq \frac{1}{2}$, where $\lambda_c(D)$ and η_c satisfy

$$\eta_c = \frac{1-\lambda_c(D)}{\lambda_c(D)} \tan\left[\frac{\pi D}{4\lambda_c(D)}\right] . \tag{6}$$

Zocchi *et al.* placed two wires, a distance $D=6.64$ mm apart, symmetrically along the center of the channel with $W=5$ cm. They observed a transition to the asymmetric finger at $\lambda_c \approx 0.41$. Substituting the dimensionless $D=6.64/50=0.1328$ to (6), we find

$$\eta_c \approx 0.374 . \tag{7}$$

The number η_c, however, can be adjusted to best fit the experimental data. Once η_c is determined, (6) gives the desired relation between D and $\lambda_c(D)$.

For $D=0.0944$, Zocchi *et al.* observed a transition at $\lambda_c \approx 0.35$, while (6) with $\eta_c=0.374$ predicts $\lambda_c=0.357$. More data are needed to check our prediction (6).

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