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Complementary pictures of the N-boson problem

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We consider a system of N identical bosons which interact in one dimension via attractive pair potentials and obey nonrelativistic quantum mechanics. This system is studied from two complementary viewpoints. The equivalent two-body method approximates the system as a collection of $(N-1)$ independent two-particle systems with coupling constants enhanced by the factor $N/2$, and this yields energy lower bounds. The collective-field method approaches the problem from the standpoint of the limit as $N\rightarrow\infty$ and leads to energy upper bounds. We find that Gaussian trial functions for the N-particle Hamiltonian and Gaussian trial densities in the collective-field theory lead to precisely the same energy upper bounds. The upper bounds provided by field theory allow for systematic improvement via a variational principle. Details are worked out for two exactly soluble problems, namely, the harmonic oscillator and the attractive δ potential.

I. INTRODUCTION

The purpose of this paper is to relate two complementary views of the ground state of a system of N identical bosons. On the one hand, the permutation symmetry of the problem causes the system to behave somewhat like many copies of a two-particle system; on the other hand, it can be viewed as an infinite system which is reached by a limiting process in which the particle number is increased without bound while the product of the coupling constant and the particle number is kept fixed. These complementary viewpoints provide energy lower bounds and energy upper bounds valid for all finite N.

We consider therefore a system of N identical bosons which interact via pair potentials and obey nonrelativistic quantum mechanics. Although the methods we shall use can be extended to three or more spatial dimensions, in this paper we study the ground-state energy of the problem in one dimension. The Hamiltonian for the N-particle system (with the kinetic energy of the center of mass removed) is given by

$$H=\frac{1}{2m}\sum_{i=1}^{N}p_i^2-\frac{1}{2Nm}\left[\sum_{i=1}^{N}p_i\right]^2+\sum_{\substack{i,j=1\\ i<j}}^{N}\gamma f(x_{ij}/a)\ , \tag{1.1}$$

where m is the mass of a particle, $x_{ij}=x_i-x_j$ is a pair distance, $f(x)$ is the potential shape, a is a length parameter, and γ is the coupling constant.

For attractive pair potentials, such an N-boson system collapses in the sense that the energy per particle $|\varepsilon|/N$ increases without bound as N increases. However, the energy ε is in general related to N and the coupling constant γ by an equation of the form

$$E=F_N(v)\ , \tag{1.2}$$

where the dimensionless energy and coupling parameters E and v are given by

$$E=\frac{m\varepsilon a^2}{\hbar^2(N-1)},\quad v=\frac{m\gamma a^2 N}{2\hbar^2}\ . \tag{1.3}$$

We call the graph (v,E) an energy trajectory. In the case of pure power-law potentials with shapes $f(x)$ given by

$$f(x)=|x|^q,\quad q>0 \tag{1.4}$$

scaling arguments show that the corresponding energy trajectories are given by

$$E=F_N^{(q)}(v)=F_N^{(q)}(1)v^{2/(q+2)}\ . \tag{1.5}$$

It is fortunate that we do have two exactly soluble problems at our disposal, namely the harmonic oscillator[1] for which

$$f(x)=x^2 \quad\text{and}\quad E=F_N(v)=F_2(v)=F_\infty(v)=v^{1/2}\ , \tag{1.6}$$

and also the attractive δ potential[2] for which

$$f(x)=-\delta(x) \quad\text{and}\quad E=F_N(v)=-\frac{1}{6}\left[1+\frac{1}{N}\right]v^2\ . \tag{1.7}$$

These two very different examples demonstrate the importance of the functional equation (1.2): In this form, the trajectory functions F_N vary relatively slowly with N. For the harmonic oscillator, the trajectory functions are actually all the same because in that case $F_N(v)=v^{1/2}$ for all $N\geq 2$.

The principal objective of this article is to study the relationship between the general trajectory F_N and the extreme trajectories F_2 and F_∞ which is the limit of F_N as $N\rightarrow\infty$. This endeavor reveals some interesting relationships between two earlier approaches to the N-boson problem, namely the equivalent two-body method[3,4] and the collective-field method.[5,6] In Secs. II and III these methods are described in the same framework so that the relationship between them becomes clear.

The equivalent two-body method immediately provides the lower trajectory bound $F_2(v) \le F_N(v)$. Also, since ε is at the bottom of the spectrum of H, by using a trial function ψ, we can find an upper estimate to ε for a given N. However, ψ must be translation invariant and also symmetric under the permutation of the particle indices. It is difficult in general to design trial functions which have these two symmetries and which also lead to tractable computations. One exception to this is the Gaussian trial function which, because of a unique factoring property, yields an energy upper bound valid for all $N \ge 2$. Consequently one obtains an upper bound F_G to the trajectory function F_∞.

Meanwhile, with the aid of a trial density ϕ, the collective-field method also provides an upper estimate F_ϕ for F_∞. It is interesting that if the density ϕ is Gaussian, then the upper estimate for F_∞ which we obtain is precisely the same F_G which we find by using a Gaussian trial function ψ in the Rayleigh quotient $(\psi, H\psi)/(\psi,\psi)$ and minimizing with respect to scale. However, now we can do better than F_G because, in general, the difficulty of solving a variational problem for the density ϕ is easier than that of the corresponding problem for a trial function ψ.

Our main results can therefore be summarized by the following trajectory inequalities:

$$F_2(v) \le F_N(v) \le F_\infty(v) \le F_\phi(v) \le F_G(v) , \tag{1.8}$$

in which we have supposed that the density ϕ is at least as "good" as the Gaussian. In the special case of the harmonic oscillator all these trajectory functions coalesce into the common curve $F(v) = v^{1/2}$. All the various N-body energy estimates can be recovered from the corresponding F functions by the general equation

$$\varepsilon = \left[\frac{\hbar^2}{ma^2}\right](N-1)F\left[\frac{m\gamma a^2 N}{2\hbar^2}\right] . \tag{1.9}$$

In Sec. IV we apply our results to the δ potential, which provides a useful and interesting test, since in this case we know that, for all $N \ge 2$, $F_N(v)$ is given *exactly* by (1.7).

II. THE EQUIVALENT TWO-BODY METHOD

The equivalent two-body method has a history going back to the dawn of contemporary nuclear physics.[3] Soon after the neutron was discovered in 1932, Wigner and later Feenberg and others tried various ways of relating the ground-state energy of a few-nucleon system to that of a specially constructed two-body system with a new mass and coupling constant. Sometimes the relationship was actually that of a variational upper bound, but it was usually regarded at that time simply as an *ad hoc* approximation. The first rigorous results[4] of this type were given by Post in 1956, who constructed a two-particle Hamiltonian whose lowest energy was a lower bound to the energy of the N-particle system. The initial result was found to be good for tightly bound boson systems, but it was less effective for fermion systems. This area of lower-bound theory has been extended to deal with fermions,[7] atomlike systems,[8] nonlocal interactions,[9] and also the energies of excited N-particle states.[10] We shall present here only a very brief account of the special result which we need. A further sharpening of the bound and an independent review may be found in an article by Hill.[11]

One of the interesting points about the lower bounds is the fact that their quality depends on the system of relative coordinates used.[7,10] We suppose that new coordinates are defined $\xi = Bx$, where $\xi = [\xi_i]$ and $x = [x_i]$ are column vectors of the new and old coordinates, ξ_1 is the center-of-mass coordinate, and $\xi_2 = (1/\sqrt{2})(x_1 - x_2)$ is a pair distance. Our methods *require* these two coordinates and consequently the matrix B which must, of course be invertible, has without any further loss of generality the form

$$B = \begin{bmatrix} \frac{1}{\sqrt{N}} & \frac{1}{\sqrt{N}} & \frac{1}{\sqrt{N}} & \frac{1}{\sqrt{N}} & \cdots & \frac{1}{\sqrt{N}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots\;\vdots\;\vdots & \vdots \end{bmatrix} ,$$

where the rows 2 to N are orthogonal to the first row. The column vectors Π and p of the new and old momenta are therefore related by $\Pi = [B^{-1}]^T p$. The Hamiltonian (1.1) can be rewritten in the form

$$H = \sum_{\substack{i,j=1 \\ i<j}}^{N} \left[\frac{1}{2Nm}(p_i - p_j)^2 + \gamma f(x_{ij}/a)\right] . \tag{2.1}$$

If we now compute expectations with respect to translation-invariant boson functions, we find from Eq. (2.1) that $\langle H \rangle = \langle \mathcal{H} \rangle$, where the reduced two-body Hamiltonian $\mathcal{H}$ is given by

$$\mathcal{H} = (N-1)\left[\frac{1}{2m\lambda}\Pi_2^2 + \frac{N}{2}\gamma f(\sqrt{2}\xi_2/a)\right] , \tag{2.2}$$

and the parameter λ is equal to the sum of the squares of the elements of the second row of the matrix $[B^{-1}]^T$. For fermion systems our lower-bound methods[7] require more than one pair-distance coordinate so that B cannot be orthogonal and in such cases $\lambda > 1$. The parameter λ is not quite a "coefficient of orthogonality:" More details on this point may be found in the Appendix to Ref. 12. For the ground states of boson systems, however, the best results (that is to say, the highest *lower*-energy bounds) are achieved with classical Jacobi coordinates for which B is orthogonal and therefore the parameter $\lambda = 1$. We shall assume this value for the remainder of the present article. Consequently the trajectory function $F_N(v)$ which we seek is given by the equation

$$F_N(v) = \inf_{\psi} \frac{(\psi, \mathbb{H}\psi)}{(\psi,\psi)} , \tag{2.3}$$

where the Hamiltonian $\mathbb{H}$ is defined in terms of the dimensionless variable

$$x=(x_1-x_2)/a=\sqrt{2}\xi_2/a$$

and the operator $D=d/dx$ by

$$\mathbb{H}=-D^2+vf(x)\ , \tag{2.4}$$

and ψ is a translation-invariant N-boson function. Since the permutation-symmetry constraint increases monotonically with N it is clear that for each *fixed* v the value of $F_N(v)$ increases monotonically with N. That is to say, $F_N(v)\geq F_M(v)$, $N>M$, and consequently we have

$$F_2(v)\leq F_N(v)\leq F_\infty(v)\ , \tag{2.5}$$

provided the limit $N\to\infty$ exists.

We now look at variational estimates of the energy. If we could find a translation-invariant boson function with the single-product form

$$\psi(\xi_2,\xi_3,\ldots,\xi_N)=u(\xi_2)g(\xi_3,\ldots,\xi_N) \tag{2.6}$$

then, by substituting this form in the right-hand side of Eq. (2.3), we see that an upper bound to $F_N(v)$ is given by the Rayleigh quotient

$$F_u(v)=\frac{(u,\mathbb{H}u)}{(u,u)}\ . \tag{2.7}$$

This last expression is exactly what we would use if we were to estimate variationally the bottom of the spectrum of $\mathbb{H}$, a one-particle (or *reduced* two-particle) Hamiltonian. The catch in all this is that (2.6) is a strong constraint for boson functions, and it has in fact been proved[12,13] that the single-product form is achieved if and only if ψ is a Gaussian function. But in this case N disappears from the calculation, and the result (which can still be minimized with respect to a scale parameter) provides an upper-trajectory bound valid for all N: We call this $F_G(v)$. We can now augment the inequalities (2.5) by writing

$$F_2(v)\leq F_N(v)\leq F_\infty(v)\leq F_G(v)\ , \tag{2.8}$$

where $F_G(v)$ is given by using $u(x)=e^{-\alpha x^2}$ in (2.7) and minimizing the resulting expression with respect to α. The corresponding energy inequalities are recovered from the trajectory inequalities by using (1.9).

It is now clear that the inequalities in (2.8) all collapse into equalities if and only if the potential has the harmonic-oscillator shape $f(x)=x^2$. The common value obtained in this special case is simply the bottom of the spectrum of $\mathbb{H}$ given in (2.4) with $f(x)=x^2$, that is to say

$$F_2(v)=F_N(v)=F_\infty(v)=F_G(v)=v^{1/2}\ . \tag{2.9}$$

III. THE COLLECTIVE-FIELD METHOD

We first look for a formulation of the collective-field method which will allow us to find the limiting trajectory function $F_\infty(v)=\lim_{N\to\infty}F_N(v)$ for a given potential shape $f(x)$. We obtain Eq. (3.9) in which a variational upper bound $F_\phi(v)$ to $F_\infty(v)$ is provided in terms of the positive field density ϕ defined on $\mathbb{R}$ and normalized to one. We then show that a Gaussian "trial density" ϕ leads to the same upper estimate as we get when a Gaussian boson "trial function" ψ is used to estimate $F_\infty(v)$ via the original Hamiltonian H. This provides an interesting link between the two very different approaches to the many-body problem. From this result we then recover the well-known *exact* solution to the harmonic-oscillator problem for which the potential shape is $f(x)=x^2$.

The collective-field method dates back to the early fifties[5] but recently it has been clarified and presented in a form suitable for our needs by Jevicki and Sakita.[6] There may still be some unresolved problems of a mathematical nature to do with this theory, particularly relating to the prelimit situation when N is finite. However, for the purposes of the present paper, our only interest is in the claims of the theory concerning the limiting energy per particle as N increases without bound, whilst the product γN is held constant: It is this limit that leads to $F_\infty(v)$; see also the note added in proof. We have already resolved the question of the relation of this limiting energy trajectory to the corresponding trajectory for *finite N* because we know from (2.8) that $F_N(v)\leq F_\infty(v)$.

In this section we shall not use relative coordinates, and therefore we shall work with the full Hamiltonian for the N-boson problem *including* the positive center-of-mass term K, that is to say, with the Hamiltonian

$$H+K=\frac{1}{2m}\sum_{i=1}^{N}p_i^2+\sum_{\substack{i,j=1\\ i<j}}^{N}\gamma f(x_{ij}/a)\ . \tag{3.1}$$

By considering translation-invariant boson functions we see that the bottom of the spectrum of H is the same as the bottom of the spectrum of the operator $H+K$. For Bose systems the principal idea is to treat the large-N limit by a special device that builds in from the outset the necessary constraint of Bose symmetry. The most general operator which is symmetric in the $\{x_i\}$ is given formally by the expression

$$\hat{\rho}(x)=\sum_{i=1}^{N}\delta(x-x_i)\ . \tag{3.2}$$

We can use $\hat{\rho}(x)$ to construct boson functions as in the example

$$\sum_{i=1}^{N}\psi(x_i)=\int_{\mathbb{R}}\hat{\rho}(x)\psi(x)dx\ . \tag{3.3}$$

In general, boson functions can be constructed by means of a functional of the form

$$\psi(x_1,x_2,\ldots,x_N)=\Phi[\hat{\rho}]\ . \tag{3.4}$$

Consequently, the requirement that ψ satisfy Schrödinger's equation implies that the functional Φ satisfy a corresponding differential equation. This equation eventually leads to the following approximate expression for the total energy as a functional of a positive field density function ρ defined on $\mathbb{R}$:

$$\varepsilon[\rho]=\frac{\hbar^2}{8m}\int_{\mathbb{R}}\frac{[\rho'(t)]^2}{\rho(t)}dt+\frac{\gamma}{2}\int_{\mathbb{R}^2}\int\rho(s)f[(s-t)/a]\rho(t)ds\,dt\ , \tag{3.5}$$

where

$$\int_{\mathbf{R}} \rho(t)dt = N \ . \tag{3.6}$$

As N is increased, the approximation becomes better, and the functional $\varepsilon[\rho]$ approaches an upper estimate to the lowest energy of the system. Since we are interested *only* in the large-N limit we now transform the problem so that this limit can be approached. We define a new density $\phi(t)$ which is normalized to unity on $(-\infty,\infty)$ and we define $F_\phi(v)$ to be, essentially, the *limiting* energy per particle, where from (1.3) $v = m\gamma a^2 N/2\hbar^2$ is kept *constant*. Thus we define

$$\phi(t/a) = a\rho(t)/N$$

and therefore (3.7)

$$\int_{\mathbf{R}} \phi(t')dt' = 1, \quad t' = t/a \ ,$$

$$F_\phi(v) = \lim_{N\to\infty} \left[\frac{ma^2\varepsilon[\rho]}{\hbar^2 N} \right] ,$$

$$v = \frac{m\gamma a^2 N}{2\hbar^2} = \text{const} \ . \tag{3.8}$$

Since $(N-1)/N$ approaches 1 as N increases, we conclude from (3.5), (3.7), and (3.8) that the following functional provides an upper bound to the quantity $F_\infty(v)$ that we seek. Hence

$$F_\infty(v) \le F_\phi(v) = \frac{1}{8}\int_{\mathbf{R}} \frac{[\phi'(t)]^2}{\phi(t)} dt + v \int_{\mathbf{R}^2}\int \phi(s) f(s-t)\phi(t) ds\, dt \ . \tag{3.9}$$

In the step from (3.5) to (3.9) we have first used dimensionless variables $s' = s/a$ and $t' = t/a$ and then dropped the primes on s and t in the final expression. All the results which we shall obtain in this section of the paper are based on (3.9).

The next result is obtained by a simple calculation. We shall give enough of the details so that the calculation can easily be verified. We start with a normalized Gaussian density given by

$$\phi(t) = ce^{-4\alpha t^2}, \quad c = 2\left[\frac{\alpha}{\pi}\right]^{1/2} ,$$

$$\int_{\mathbf{R}} \phi(t)dt = 1 \ . \tag{3.10}$$

This density is now substituted into the right-hand side of (3.9) leading to a function $E(\alpha)$ of the variational parameter α. More interestingly, we can rework the right-hand side of (3.9) so that, by using the change of variable $x = \sqrt{2}t$ and performing *one* of the potential-energy integrals, we obtain the result

$$E(\alpha) = \frac{(u, \mathbb{H}u)}{(u,u)} ,$$

where (3.11)

$$u^2(x) = \phi(t) \ .$$

Consequently, using a Gaussian density in (3.9) or a Gaussian wave function in (2.3) leads to precisely the sample upper estimate for $F_\infty(v)$. When this common upper estimate is minimized with respect to the parameter α, we call the resulting approximate trajectory function $F_G(v)$.

In the special case that $f(x) = x^2$, we therefore find from (3.11), as we did in (2.9), that $F_G(v) = v^{1/2}$. Since we know from (2.9) that in this case, $v^{1/2}$ is also a *lower* bound to $F_\infty(v)$, we again recover the result that $F_\infty(v) = F_G(v) = v^{1/2}$. Of course, from this point of view of collective-field theory *alone*, this still tells us nothing definite about $F_N(v)$, for *finite N*.

The advantage of the collective-field equation (3.9), in general, is that it provides a way of systematically improving on $F_G(v)$. We can simply explore density functions variationally. This is what we do in Sec. IV in the case of the attractive δ potential.

IV. THE δ POTENTIAL

Even at a time when computation has become both cheap and comfortable, it is extremely useful to look at cases for which all the details of a theory can be worked out by exact analytical methods. It turns out that the harmonic oscillator is too good because in this case the trajectory functions all coalesce into one, and also the equivalent two-body method and the collective-field method yield the same results (for the large-N limit). Fortunately, the δ potential separates all the distinct approaches and is not uninteresting since, from the point of view of scaling, it plays the role of an attractive Coulomb potential in one dimension.

The N-boson problem with attractive δ pair potentials in one dimension is equivalent to a classical problem to do with electromagnetic waves and a system of mirrors. The complete solution of the problem with pair potential shape $f(x) = -\delta(x)$ has been given by McGuire.[2] As with the harmonic oscillator, the ground-state wave function is a Jastrow function consisting of a product of $\frac{1}{2}N(N-1)$ functions of the pair distances. However, instead of Gaussian factors one has exponential factors of the form $e^{-\alpha|x_{ij}|}$. In our terminology the exact N-body energy is given by the trajectory formula

$$F_N(v) = -\frac{1}{6}\left[1 + \frac{1}{N}\right] v^2, \quad N \ge 2 \ . \tag{4.1}$$

The energies are recovered from this (and of any other trajectory formula) by means of (1.9). Since scaling gives the factor v^2, the various trajectories for this problem are specified, for example, by the single value $F(1)$. The graphs of the trajectories are shown in Fig. 1.

As we found in Sec. III, the application of a Gaussian trial function or, in the collective-field method, of a Gaussian density, leads to the same *upper* estimate to $F_\infty(v)$. If this upper bound is minimized with respect to a scale parameter we call the resulting trajectory function $F_G(v)$ and in the present problem we find for this trajectory function

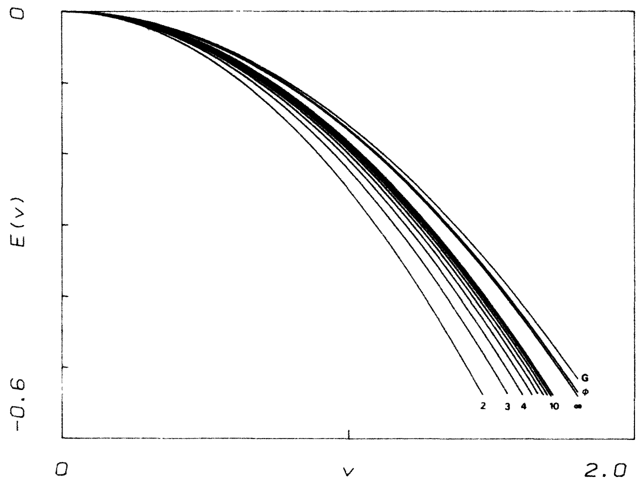

FIG. 1. Energy trajectories $F_N(v)$ for the δ potential. The trajectory $F_G(v)$ is the trajectory found either by using a Gaussian trial wave function or by using a Gaussian density. $F_\phi(v)$ is the trajectory corresponding to our best variational estimate of the field density ϕ.

$$F_G(v)=-\frac{1}{2\pi}v^2>-\frac{1}{6}v^2=F_\infty(v)\ . \tag{4.2}$$

Without help from the collective-field method it would be very difficult, in general, to improve on the Gaussian upper bound (4.2).

For the δ potential $f(x)=-\delta(x)$, the variational upper estimate (3.9) for $F_\infty(1)$ becomes

$$F_\infty(1)\le F_\phi(1)=\int_{\mathbf{R}}\left[\frac{1}{8}\frac{[\phi'(t)]^2}{\phi(t)}-\phi^2(t)\right]dt\ , \tag{4.3}$$

where the positive density function $\phi(x)$ satisfies the normalization condition $\int_{\mathbf{R}}\phi(t)dt=1$. This optimization problem is certainly amenable to numerical methods. However, great care would have to be taken to preserve the bound in (4.3). Our present goal is to look at the theory in the light of an exact analytical computation.

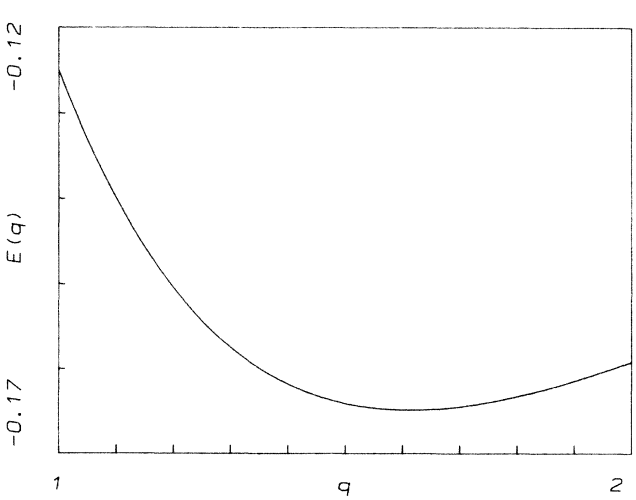

FIG. 2. The field-theoretic expression (4.3) for the energy is first minimized with respect to scale. Shown is the final minimization with respect to the power parameter q.

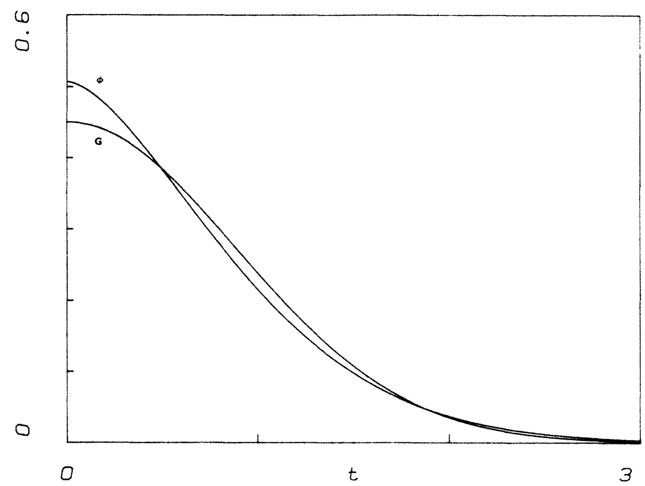

FIG. 3. The best Gaussian density G and the best density ϕ from the class (4.4) for the δ potential.

We tried a variety of one-parameter trial densities but they gave worse results than the Gaussian. Finally, the following two-parameter density, which includes the Gaussian as the special case $q=2$, gave satisfactory results

$$\phi(t)=b^{-1}A(q)e^{-|t/b|^q}\ , \tag{4.4}$$

where b is a scale parameter and $A(q)$ is a normalization factor given by

$$A(q)=\left[\frac{2}{q}\Gamma\left[\frac{1}{q}\right]\right]^{-1}\ . \tag{4.5}$$

If we now substitute (4.4) in (4.3) and minimize the expression with respect to the scale b, we obtain b and the energy as a function of the power parameter q, that is to say, we obtain the expressions

$$b=q2^{(1-1/q)}\Gamma\left[2-\frac{1}{q}\right]\ , \tag{4.6}$$

$$F_\phi(1)=E(q)$$

$$=-\left[q2^{(1+2/q)}\Gamma\left[1+\frac{1}{q}\right]\Gamma\left[2-\frac{1}{q}\right]\right]^{-1}\ . \tag{4.7}$$

Equation (4.7) is consistent with the Gaussian result since $E(2)=-1/2\pi$. It is now safe to use computer technique, and in Fig. 2 we exhibit the graph of $E(q)$ from which we conclude that the minimum is at about $q=1.61$. Hence our upper bound becomes

$$F_\infty(1)\le F_\phi(1)=E(1.61)\le -0.164\,868\ . \tag{4.8}$$

This result is approximately 1% above the exact value $F_\infty(1)=-\frac{1}{6}$. In Fig. 3 we show the graphs of $\phi(t)$ given by (4.4) with $q=1.61$ and also the Gaussian density G with $q=2$.

V. CONCLUSION

The permutation symmetry which is necessarily satisfied by the states of a system of identical bosons is a powerful constraint. By a kind of dynamical analog of a crystallographic principle, the N-boson system is in some respects like $(N-1)$ copies of an equivalent two-body problem whose coupling constant has been strengthened by the factor $N/2$. For a good general approximation, which in the case of the harmonic oscillator yields the exact energy, the equivalent two-body problem should be constructed with the aid of orthogonal relative coordinates. For other purposes, such as treating fermion systems[7] or N-particle excited states,[10] nonorthogonal relative coordinates may be required and in such cases the mass in the equivalent two-body Hamiltonian is changed by a factor $\lambda > 1$.

Another quite different approach is to relate the lowest energy E_N of the N-boson system to the corresponding energy E_K of a K-boson system, where $K < N$ and no special factors are introduced into the mass or the potential energy. Bruch and Sawada[14] proved, for example, that for a wide class of Hamiltonians with attractive intermolecular potentials (in $\mathbb{R}^3$) the inequality $E_3 \leq 3E_2$ is valid. For a smaller class of potentials (still in $\mathbb{R}^3$) this inequality has been generalized[15] to

$$\begin{pmatrix} N \\ 2 \end{pmatrix}^{-1} E_N \leq \begin{pmatrix} K \\ 2 \end{pmatrix}^{-1} E_K$$

for $\mu K \leq N$, where the factor $\mu \geq 1$ depends on the class of potentials. For potentials like

$$f(r) = -\alpha r^{-1} + \beta \ln(r) + \gamma r ,$$

which have the property that they are at the same time convex functions of $-r^{-1}$ and concave functions of r, it is found that $\mu = \frac{3}{2}$: One knows, therefore, for all these problems that $E_N \leq \frac{1}{2} N(N-1) E_2$ for all $N \geq 3$.

In contrast to the above results which relate the N-particle problem to a K-particle problem with $K < N$, the collective-field method treats the limiting case as N increases without bound. In this article we have used the collective-field method to obtain an upper estimate $F_\phi(v)$ for the limiting trajectory function $F_\infty(v)$. We have also found that using Gaussian trial functions (in orthogonal relative coordinates) for the N-boson problem leads, via the large-N limit (which can be computed for such functions), to precisely the same trajectory function $F_G(v)$ which is found when a Gaussian density is used as a trial density in the collective-field method.

Our principal result is best summarized by the following statement: For all $N \geq 2$, the energy ε of the N-boson problem is approximated by the inequalities

$$\left[\frac{\hbar^2}{ma^2} \right] (N-1) F_2 \left[\frac{m\gamma a^2 N}{2\hbar^2} \right] \leq \varepsilon \leq \left[\frac{\hbar^2}{ma^2} \right] (N-1) F_\phi \left[\frac{m\gamma a^2 N}{2\hbar^2} \right] \quad (5.1)$$

in which the function $F_2(v)$ is found by solving the two-boson problem and $F_\phi(v)$ is obtained from the variational equation (3.9) for the collective field ϕ.

The collective-field method is very useful, but it is still not completely understood from a purely mathematical viewpoint. It is therefore interesting at this time to have an analysis of its relation to the equivalent two-body method along with a detailed study of two exactly soluble problems. The variational principal (3.9) for computing the upper trajectory estimate $F_\phi(v)$ provides the recipe (5.1) for the immediate improvement of any earlier N-boson results which were dependent on Gaussian wave functions. For example, in the case of the linear potential in one dimension, the ground-state energy is determined[16] with error less than 0.222% for all $N \geq 2$. This last point is perhaps more sharply emphasized by the observation that the improvement is realized *even* for $N = 2$. Thus for the δ potential, our results for the upper bound $F_\phi(v)$ to $F_\infty(v)$ lead via (5.1) to a better upper estimate of the energy for $N = 2$ than can be obtained by applying a Gaussian trial function to the two-body Hamiltonian directly. In this illustration, we get to two by first passing infinity.

Note added in proof. We have now shown[17] that the upper bound (3.9) can be derived rigorously, independently of collective field theory, as an appropriate limit based on the sequence of variational upper bounds provided within conventional quantum mechanics by trial wave functions of the form

$$\Phi(x_1, x_2, \ldots, x_N) = g(x_1) g(x_2) \cdots g(x_N) .$$

This wave function is applied to the Hamiltonian H given here by (1.1), the limit $N \to \infty$, with v constant, is computed, and the density ϕ is then given, essentially, by $\phi(t) = g^2(t)$.

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