Renormalization-group theory for the eddy viscosity in subgrid modeling

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Renormalization-group theory is applied to incompressible three-dimensional Navier-Stokes turbulence so as to eliminate unresolvable small scales. The renormalized Navier-Stokes equation now includes a triple nonlinearity with the eddy viscosity exhibiting a mild cusp behavior, in qualitative agreement with the test-field model results of Kraichnan. For the cusp behavior to arise, not only is the triple nonlinearity necessary but the effects of pressure must be incorporated in the triple term. The renormalized eddy viscosity will not exhibit a cusp behavior if it is assumed that a spectral gap exists between the large and small scales.

I. INTRODUCTION

For fully developed turbulence, excitations of spatial scales arise that exceed the resolution possible in current and foreseeable computer simulations. This classic problem has resulted in many (usually *ad hoc*) attempts[1–3] at modeling these small (so called "subgrid") scales and their effect on the resolvable large (so called "supergrid") scales. Recently, however, the technique of renormalization-group theory (RNG) has attracted considerable attention[4–16] as a systematic attempt at subgrid scale modeling. There are, at present, basically two distinct RNG approaches being utilized, each with somewhat different objectives in mind.

(i) In the ϵ-expansion method,[4–10] one invariably introduces a zero-mean Gaussian random forcing term into the Navier-Stokes equation. This white-noise forcing term is determined by its correlation function, which is assumed to obey a subgrid wave-number power-law spectrum. A small parameter ϵ is introduced appropriately into the exponent of this power law. One now calculates the effect on the Navier-Stokes equation of removing a small subgrid wave-number shell. One of the resulting effects is to introduce higher-order nonlinearities into the renormalized equation. By recourse to the small ϵ expansion, it can be shown that these higher-order nonlinearities are, in proper RNG jargon,[11] "irrelevant." Unfortunately, to recover the Kolmorogov energy spectrum one is forced to choose $\epsilon=4$. In this extension from $\epsilon \ll 1$ to $\epsilon=4$, it is tacitly assumed that the higher-order nonlinearities can still be neglected. Under some postulated equivalence, and without any experimentally adjustable parameters, Yakhot and Orszag,[6–8] find numerical values for important constants of turbulent flows (e.g., the Kolmogorov constant for the inertial range spectrum, the Batchelor constant, the turbulent Prandtl number, etc.), as well as the scaling of the eddy viscosity in the inertial range.

(ii) In this RNG approach,[12–16] and this is the approach utilized here, one proceeds by successive elimination of subgrid wave-number shells, which leads to an integro-difference recursion relation for the eddy viscosity in the supergrid range. RNG is then applied to this recursion relation, which on iteration tends to a fixed point—the required (wavenumber dependent) eddy viscosity. Now unlike the ϵ-expansion procedure (i), both free decay (with given Kolmogorov energy spectrum) and forced turbulence (with spectral forcing chosen to reproduce the Kolmogorov energy spectrum) can be handled and there is no introduction of a small ϵ parameter. Of course, this leads to the problem of closure—but it must also be remembered that the ϵ-expansion procedure must face this closure problem as well when ϵ is set to its required finite value ($\epsilon=4$).

In this paper we shall apply the difference recursion RNG technique (originally applied to the linear problem of passive scalar diffusion[12]) to Navier-Stokes turbulence, taking proper account of symmetries. The RNG procedure by which the subgrid shells are removed iteratively is outlined in Sec. II (with details presented in Appendix A). In Sec. III the renormalized viscosity is calculated numerically and compared both to that found by the iterative averaging RNG procedure of McComb[13–15] (which by some technique, not properly understood by us, does not introduce triple-order nonlinearities) and to the closure models of Kraichnan[17] and Chollet and Lesieur.[18] It is shown that not only is the triple nonlinearity necessary for the RNG eddy viscosity to exhibit a cusp behavior near the subgrid or supergrid wavenumber cutoff, but also the presence of the pressure in the Navier-Stokes equation is required. We summarize our results in Sec. IV. In Appendix B it is shown that the presence of a spectral gap between the subgrid and supergrid scales will eliminate the triple nonlinearity in the renormalized Navier-Stokes equation. Thus, without this triple nonlinearity, the resulting spectral gap RNG eddy viscosity will not exhibit a cusp behavior. Finally, in Appendix C, a linearized model calculation (following a similar model discussed by Rose[12]) is introduced to exam-

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ine the direct effect of the triple nonlinearity in the renormalized Navier-Stokes equation.

II. NAVIER-STOKES TURBULENCE AND RENORMALIZATION-GROUP PROCEDURE

We consider incompressible turbulence, and the Navier-Stokes equation in wave-number space (utilizing the summation convention over repeated subscripts),

$$(\partial/\partial t+\nu_0 k^2)u_\alpha(\mathbf{k},t) = \int d^3j\, M_{\alpha\beta\gamma}(\mathbf{k})u_\beta(\mathbf{j},t)u_\gamma(\mathbf{k}-\mathbf{j},t)\ . \quad (1)$$

The incompressibility condition

$$k_\beta u_\beta(\mathbf{k},t)=0 \quad (2)$$

has been employed to eliminate the pressure gradient

$$\nabla^2 p = -\partial^2(u_\alpha u_\beta)/\partial x_\alpha \partial x_\beta\ ,$$

resulting in the quadratic nonlinear coupling coefficient $M_{\alpha\beta\gamma}$ given by

$$M_{\alpha\beta\gamma}(\mathbf{k})=[k_\beta D_{\alpha\gamma}(\mathbf{k})+k_\gamma D_{\alpha\beta}(\mathbf{k})]/2i\ , \quad (3)$$

where

$$D_{\alpha\gamma}(\mathbf{k})=\delta_{\alpha\gamma}-k_\alpha k_\gamma/k^2\ . \quad (4)$$

ν_0 is the molecular viscosity. Note the symmetry relation $M_{\alpha\beta\gamma}=M_{\alpha\gamma\beta}$, with $k_\alpha M_{\alpha\beta\gamma}(\mathbf{k})=0$.

A. RNG Procedure

In the RNG method one partitions the unresolvable subgrid scales into shells, characterized by a scale factor f, $0<f<1$. The spectrum is partitioned by the wave-number set $\{k_* \equiv k_N \equiv f^N k_0,\ k_{N-1}\equiv f^{N-1}k_0, \ldots, k_1 \equiv f k_0,\ k_0\}$. k_0 is typically chosen to be on the order of the Kolmogorov dissipation wave number, while k_N is the wave number which separates the actual resolvable scales $(k<k_N)$ from the unresolvable scales $(k_N<k<k_0)$. The RNG iterative procedure consists of first eliminating the highest wave-number subgrid shell $k_1<k<k_0$ from Eq. (1) to leave a modified Navier-Stokes equation for the remaining supergrid wave numbers $k<k_1$. It will be shown that the modifications to the Navier-Stokes equation are (a) a renormalized viscosity coefficient and (b) a triple nonlinearity in the fluid velocity. One then proceeds iteratively, removing at the ith step the subgrid shell $k_i<k<k_{i-1}$ until one reaches the actual resolvable scales at the Nth step. Since we are dealing with free decay, it is assumed that in the subgrid scales the inertial energy spectrum $E(k)$ obeys some given power law

$$E(k)\approx k^{-m} \quad \text{for } k_N<|\mathbf{k}|<k_0\ . \quad (5)$$

Theoretically one can proceed with an arbitrary value for the power-law exponent m, but when we present our numerical results for the renormalized eddy viscosity we shall employ the Kolmogorov exponent $m=\pm\frac{5}{3}$. For isotropic, stationary turbulence in the subgrid scales, the equal time velocity covariance is

$$\langle u_\alpha(\mathbf{k},t)u_\beta(\mathbf{k}',t)\rangle = D_{\alpha\beta}(\mathbf{k})\delta(\mathbf{k}+\mathbf{k}')Q(|\mathbf{k}|) \quad \text{for } k_N<|\mathbf{k}|,|\mathbf{k}'|<k_0\ , \quad (6)$$

where $Q(|\mathbf{k}|)$ is related to the energy spectrum $E(k)$ by

$$Q(k)=E(k)/4\pi k^2\ . \quad (7)$$

B. Removal of the first subgrid shell

We now consider the effect of removing the first subgrid shell $k_1<k<k_0$ from the Navier-Stokes equation (1) in the RNG procedure. It is convenient to introduce the notation

$$U_\alpha(\mathbf{k},t)\equiv\begin{cases} u_\alpha^<(\mathbf{k},t) & \text{for } |\mathbf{k}|<k_1 & (8a)\\ u_\alpha^>(\mathbf{k},t) & \text{for } k_1<|\mathbf{k}| & (8b)\end{cases}$$

and to introduce an ensemble average over the particular subgrid shell modes under consideration,

$$\langle u_\alpha^>(\mathbf{k}t)\rangle\equiv 0\ ,$$
$$\langle u_\alpha^<(\mathbf{k},t)\rangle\equiv u_\alpha^<(\mathbf{k},t)\ . \quad (9)$$

For $\mathbf{k}$ in the first subgrid shell, Eq. (1) becomes

$$(\partial/\partial t+\nu_0k^2)u_\alpha^>(\mathbf{k},t)=M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j[u_\beta^>(\mathbf{j},t)+u_\beta^<(\mathbf{j},t)][u_\gamma^>(\mathbf{k}-\mathbf{j},t)+u_\gamma^<(\mathbf{k}-\mathbf{j},t)] \quad \text{for } k_1<|\mathbf{k}|<k_0\ , \quad (10)$$

while for those $\mathbf{k}$ in the supergrid range,

$$(\partial/\partial t+\nu_0k^2)u_\alpha^<(\mathbf{k},t)=M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j[u_\beta^>(\mathbf{j},t)+u_\beta^<(\mathbf{j},t)][u_\gamma^>(\mathbf{k}-\mathbf{j},t)+u_\gamma^<(\mathbf{k}-\mathbf{j},t)] \quad \text{for } |\mathbf{k}|<k_1\ . \quad (11)$$

[It should be noted that the right-hand sides of Eqs. (10) and (11) are very different, not only because of the $\mathbf{k}$ range but also in the ranges of j integrations.]

Following Rose[12] and McComb,[13–15] we assume that in every realization, $u_\alpha^>$ evolves faster than the supergrid velocity field $u_\alpha^<$, so that $\partial u_\alpha^>/\partial t$ can be neglected in Eq. (10). Thus, from Eq. (10),

$$u_\beta^>(\mathbf{j},t)=(1/\nu_0 j^2)M_{\beta\beta'\gamma'}(\mathbf{j})\int d^3j'[u_{\beta'}^>(\mathbf{j}',t)+u_{\beta'}^<(\mathbf{j}',t)][u_{\gamma'}^>(\mathbf{j}-\mathbf{j}',t)+u_{\gamma'}^<(\mathbf{j}-\mathbf{j}',t)] \text{for } k_1<|\mathbf{j}|\le k_0\ . \quad (12)$$

We now substitute Eq. (12) into (11), taking care of symmetries, and perform the subgrid shell ensemble average of Eq. (9) to obtain, for $|\mathbf{k}|\le k_1$,

$$[\partial/\partial t+\nu_1(k)k^2]u_\alpha^<(\mathbf{k},t)=M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\, u_\beta^<(\mathbf{j},t)u_\gamma^<(\mathbf{k}-\mathbf{j},t)$$
$$+2M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\, d^3j'(\nu_0 j^2)^{-1}M_{\beta\beta'\gamma'}(\mathbf{j})u_{\beta'}^<(\mathbf{j}-\mathbf{j}',t)u_{\gamma'}^<(\mathbf{j}',t)u_\gamma^<(\mathbf{k}-j,t)\ . \tag{13}$$

(For details, see Appendix A.) Thus the effect of removing the first subgrid shell on the Navier-Stokes can be seen to do the following two things.

(i) Renormalize the molecular viscosity ν_0 to

$$\nu_1(k)=\nu_0+\delta\nu_0(k)\ , \tag{14}$$

where

$$\delta\nu_0(k)=2\int d^3j\, Q(|\mathbf{k}-\mathbf{j}|)L_{kj}/(\nu_0 j^2k^2)\ , \tag{15}$$

with the coefficient L_{kj} defined by

$$L_{kj}=-2M_{\alpha\beta\gamma}(\mathbf{k})M_{\beta\beta'\gamma'}(\mathbf{k})D_{\beta'\gamma}(\mathbf{k}-\mathbf{j})D_{\gamma'\alpha}(\mathbf{k})$$
$$=-kj(1-\mu^2)[\mu(k^2+j^2)-kj(1+2\mu^2)]/(k^2+j^2-2kj\mu)\ , \tag{16}$$

$\mathbf{k}\cdot\mathbf{j}=kj\mu$ with $\mu\equiv\cos\theta$. The integration limits in Eq. (15) are $k_1<|\mathbf{k}-\mathbf{j}|\le k_0$ and $k_1<|\mathbf{j}|\le k_0$.

(ii) Include a triple nonlinearity $u^<u^<u^<$. This is a typical biproduct of RNG. (See, for example, RNG for the two-dimensional Ising spin Hamiltonian. One finds that after the first spin decimation, not only is there the original nearest-neighbor interaction but also a diagonal nearest neighbor and four-spin-coupling interactions. These new interactions are weaker than the original interaction. Moreover, Wilson[19] neglects the four-spin-coupling interactions in his RNG and still obtains a very good approximation to the universal critical exponents—although it must be noted that the Ising model is in equilibrium while we are interested in fluid turbulence.)

C. Removal of the *n*th subgrid shell

Thus after removing the first subgrid shell, the Navier-Stokes equation (1) is modified to Eq. (13),

$$[\partial/\partial t+\nu_1(k)k^2]u_\alpha(\mathbf{k},t)=M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\, u_\beta(\mathbf{j},t)u_\gamma(\mathbf{k}-\mathbf{j},t)$$
$$+2M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\, d^3j'(\nu_0 j^2)^{-1}M_{\beta\beta'\gamma'}(\mathbf{j})u_{\beta'}(\mathbf{j}-\mathbf{j}',t)u_{\gamma'}(\mathbf{j}',t)u_\gamma(\mathbf{k}-\mathbf{j},t)\ , \tag{17}$$

where there is now no need for the superscript $<$ notation on the velocity field since the wave numbers are all restricted to $0<k\le k_1$.

To remove the second subgrid shell, we denote the (current) subgrid modes by

$$u_\alpha=u_\alpha^>(k,t)\quad \text{if } k_2<k\le k_1 \tag{18}$$

and the supergrid modes by

$$u_\alpha=u_\alpha^<(k,t)\quad \text{if } k\le k_2\ , \tag{19}$$

and proceed as in Sec. II B but now realize that the triple nonlinearity uuu in Eq. (17) will also contribute in the renormalization procedure. We find that the renormalized Navier-Stokes equation is now given by

$$[\partial/\partial t+\nu_2(k)k^2]u_\alpha^<(\mathbf{k},t)=M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\, u_\beta^<(\mathbf{j},t)u_\gamma^<(\mathbf{k}-\mathbf{j},t)$$
$$+2M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\, d^3j'(\nu_1 j^2)^{-1}M_{\beta\beta'\gamma'}(\mathbf{j})u_{\beta'}^<(\mathbf{j}-\mathbf{j}',t)u_{\gamma'}^<(\mathbf{j}',t)u_\gamma^<(\mathbf{k}-\mathbf{j},t) \tag{20}$$

for $k\le k_2$, with the eddy viscosity

$$\nu_2(k)=\nu_1(k)+\delta\nu_1(k)\ , \tag{21}$$

and

$$\delta\nu_2(k)=2\sum_{i=0}^{1}\int d^3j\, Q(|\mathbf{k}-\mathbf{j}|)L_{kj}/[\nu_i(j)j^2k^2]\ . \tag{22}$$

In Eq. (22), $k_2<|\mathbf{k}-\mathbf{j}|\le k_1$ and $k_{i+1}<|\mathbf{j}|\le k_i$ for $i=0$ or 1. Notice that the $i=1$ term in Eq. (22) is due to the triple nonlinearity in Eq. (17).

Proceeding iteratively, it can be seen that after removing the $(n+1)$th subgrid shell the Navier-Stokes equation becomes

$$[\partial/\partial t+\nu_{n+1}(k)k^2]u_\alpha^<(\mathbf{k},t)=M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\, u_\beta^<(\mathbf{j},t)u_\gamma^<(\mathbf{k}-\mathbf{j},t)$$
$$+2M_{\alpha\beta\gamma}(\mathbf{k})\sum_{i=0}\int d^3j\, d^3j'\, M_{\beta\beta'\gamma'}(\mathbf{j})/[\nu_i(j)j^2]u_{\beta'}^<(\mathbf{j}-\mathbf{j}',t)u_{\gamma'}^<(\mathbf{j}',t)u_\gamma^<(\mathbf{k}-\mathbf{j},t)$$
$$\text{with } |\mathbf{k}|\le k_{n+1}\ . \tag{23}$$

In the ith term of the summation $\sum_i$ the limitations on the $\mathbf{j}$ and $\mathbf{j}'$ integrations are $|\mathbf{j}'|$, $|\mathbf{j}-\mathbf{j}'|$, $|\mathbf{k}-\mathbf{j}|\le k_{n+1}$, but $k_{n-1}<|\mathbf{j}|\le k_{n-i-1}$, $i=0,1,\ldots,n-1$.

The eddy viscosity recursion relation is given by

$$\nu_{n+1}(k)=\nu_n(k)+\delta\nu_n(k)\ , \tag{24}$$

where

$$\delta\nu_n(k)=2\sum_{i=0}^{n}\int d^3j\, L_{kj}Q(|\mathbf{k}-\mathbf{j}|)[\nu_i(j)j^2k^2]^{-1}\ . \tag{25}$$

The integration limits in Eq. (25) are $k_{n+1}<|\mathbf{k}-\mathbf{j}|\le k_n$ and $k_{i+1}<|\mathbf{j}|\le k_i$ for $i=0,1,\ldots,n$.

RNG transformation for the $(n+1)$th subgrid shell is

$$k\to k_{n+1}\tilde{k}\ , \tag{26}$$

with the renormalized viscosity ν_n^* defined by

$$\nu_n^*(\tilde{k})\equiv Ck_{n+1}^{(m+1)/2}\nu_n(k_{n+1}\tilde{k})\quad\text{for } \tilde{k}\le 1 \tag{27}$$

for some constant C. Thus the renormalized eddy viscosity recursion relation becomes

$$\nu_{n+1}^*(\tilde{k})=f^{(m+1)/2}[\nu_n^*(f\tilde{k})+\delta\nu_n^*(f\tilde{k})]\ , \tag{28}$$

with

$$\delta\nu_n^*(\tilde{k})=2\sum_{i=0}^{n}f^{-i(m+1)/2}\int d^3j\, L_{\tilde{k}\tilde{j}}Q\times(|\tilde{\mathbf{k}}-\tilde{\mathbf{j}}|)/[\nu_{n-i}^*(f^i\tilde{j})\tilde{k}^2\tilde{j}^2] \tag{29}$$

and integration limits ($\tilde{k}\le 1$)

$$1<|\tilde{\mathbf{k}}-\tilde{\mathbf{j}}|\le f^{-1}\ ,$$
$$1<|f^i\tilde{\mathbf{j}}|\le f^{-1},\quad i=0,1,\ldots,n\ . \tag{30}$$

It should be noted that the $i=0$ contribution to $\delta\nu_n^*$ in Eq. (29) arises from the usual Navier-Stokes quadratic nonlinearity, while $i\ge 1$ terms arise from the triple nonlinearity introduced by the RNG transformations.

III. RENORMALIZED EDDY VISCOSITY

The renormalized eddy viscosity is defined as the fixed point ($n\to\infty$) of the recursion relation (28) and (29). This recursion relation has been solved numerically, and we find that a fixed point exists for each $\tilde{k}$, and that this fixed point is independent of the initial value of the molecular viscosity ν_0—as is intuitively expected for the case of strong turbulence.

In Figs. 1 and 2 we plot the $\tilde{k}$ dependence of the renormalized eddy viscosity (for various choices of the parameter f) and compare our results with those of McComb[13–15] and Kraichnan.[17] The parameter f defines the coarseness of the subgrid shell partition. For the finer subgrid partition of $f=0.7$ (Fig. 1), we see that the RNG eddy viscosity exhibits a mild cusp behavior for $\tilde{k}$ close to the subgrid-supergrid cutoff—in qualitative agreement with the test-field model of Kraichnan,[17] the eddy damped quasinormal approximation of Chollet and Lesieur,[18] as well with the recent direct numerical simulation results of Domaradzki et al.[20] but in contrast to the iterative-averaging RNG results of McComb.[13–15]

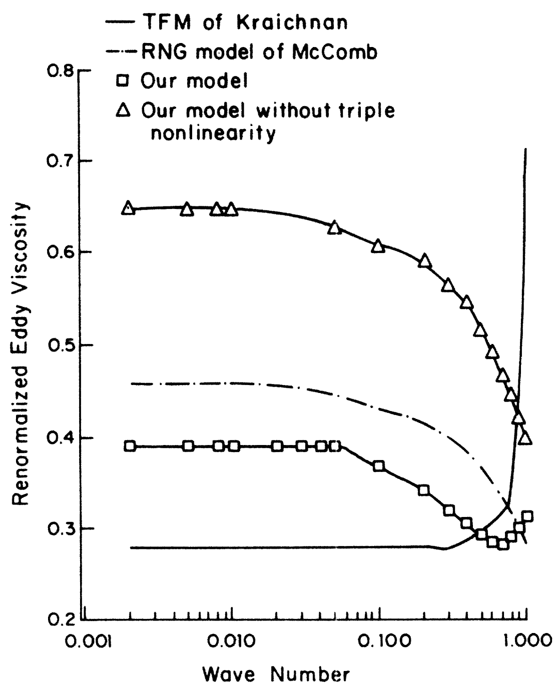

FIG. 1. Scaled renormalized eddy viscosity as a function of the scaled wave number for a relatively fine subgrid partition ($f=0.7$). The unmarked curve, exhibiting the cusp behavior for $\tilde{k}\approx 1$, is the test-field model result of Kraichnan while the curve marked with □ is our result. The curve marked with △ is the $\nu^*(\tilde{k})$ if all triple nonlinear term effects are dropped. Moreover, this curve is also appropriate if the pressure forces in the Navier-Stokes equation are neglected or if in the RNG theory a spectral gap is assumed to exist between the subgrid and supergrid scales. McComb's RNG result (which is claimed to be able to somehow avoid the triple nonlinearity) is shown by —·—·—.

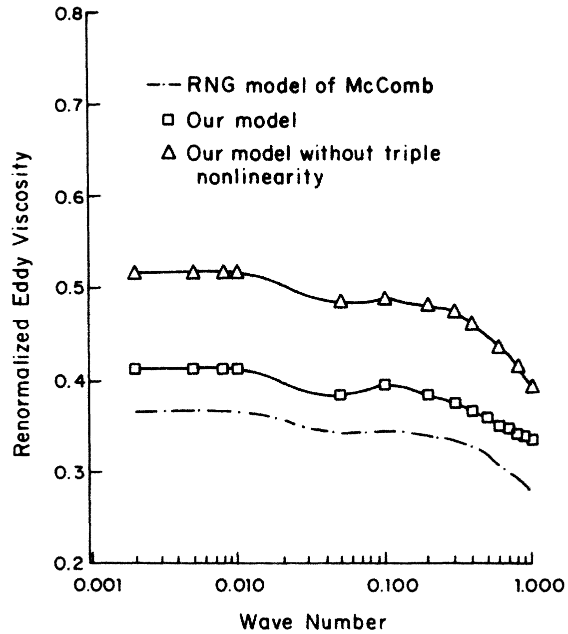

FIG. 2. Scaled renormalized eddy viscosity for a coarser subgrid partition ($f=0.6$) chosen so that now only one memory term (arising from the triple nonlinearity) contributes to ν^*. There is no cusp behavior exhibited—curves are labeled as in Fig. 1.

A. Cusp behavior of eddy viscosity near wave-number cutoff

We will first consider the RNG eddy viscosity. If one totally neglects the contribution of the triple nonlinearity to the eddy viscosity [i.e., retains only the $i=0$ term in $\sum_i$ in Eq. (29)] then the RNG viscosity does not exhibit any cusp behavior. This is shown in Figs. 1 and 2 by the solid curves with symbol Δ. Since the McComb[13–15] iterative-averaging technique is claimed to result in a renormalized Navier-Stokes equation without a triple nonlinearity, the McComb eddy viscosity also does not exhibit any cusp behavior (the dashed curve in Figs. 1 and 2). In fact, McComb's recursion relation is essentially Eqs. (28) and (29) but without the 2 factor in the $\delta\nu_n^*$ equation (which also has only $i=0$ contributing).

However, the appearance of the triple nonlinearity in the eddy viscosity recursion relation is not sufficient for the appearance of the cusp near the supergrid-subgrid cutoff. If the effect of the pressure gradient is dropped from the incompressible Navier-Stokes equation it can be shown that no cusp appears even though triple nonlinearities are generated in the RNG procedure. Indeed, with the neglect of the pressure force the problem will reduce to that of advection of a vector field by a solenoidal velocity field,[21] and this is also closely related to the problem originally treated by Rose.[12] Explicitly, the effect of the pressure can be immediately seen in the nonlinear coupling coefficient L_{kj}, Eq. (16). For the full Navier-Stokes system,

$$L_{kj}=kj(1-\mu^2)[kj-\mu\,|\mathbf{k}-\mathbf{j}|^2]/|\mathbf{k}-\mathbf{j}|^2\ , \tag{31}$$

while for the passive advection problem (no pressure term) the coupling coefficient reduces to

$$L'_{kj}=k^2j^2(1-\mu^2)/|\mathbf{k}-\mathbf{j}|^2\geq 0\ . \tag{32}$$

Since both $|\mathbf{k}-\mathbf{j}|$ and j belong to the subgrid shell, the angle θ is restricted to $\mu\equiv\cos\theta>0$ for k near the cutoff. This ensures that

$$L'_{kj}\geq L_{kj}\ . \tag{33}$$

Moreover, L_{jk} can become negative in certain regions of j space for k near this cutoff. It is this cancellation effect in the two terms of L_{kj} in Eq. (31) that causes the cusp behavior in this wave-number region for Navier-Stokes turbulence. This can be somewhat related to the cancellation effect that Kraichnan[17] finds in his test-field eddy viscosity calculation and which leads to the cusp behavior near the wave-number cutoff (see also Appendix A).

B. The renormalized Navier-Stokes equation

Having considered the effect of the triple nonlinearity on the RNG eddy viscosity, $\nu^*(k)$, we now briefly consider the direct effect of the triple nonlinearity on the final renormalization Navier-Stokes equation (on dropping the ~ notation),

$$[\partial/\partial t+\nu^*(k)k^2]u_\alpha(\mathbf{k},t)=M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\,u_\beta(\mathbf{j},t)u_\gamma(\mathbf{k}-\mathbf{j},t)$$
$$+2M_{\alpha\beta\gamma}(\mathbf{k})\sum_{i=0}\int d^3j d^3j'\,M_{\beta\beta'\gamma'}(\mathbf{j})/[\nu^*(f^ij)j^2]u_{\beta'}(\mathbf{j}-\mathbf{j}',t)u_{\gamma'}(\mathbf{j}',t)u_\gamma(\mathbf{k}-\mathbf{j},t)\ . \tag{34}$$

Unlike the quadratic nonlinearity in Eq. (34), which is energy conserving, the triple nonlinearity is readily shown to be nonenergy conserving. In Appendix C, following Rose,[12] we consider a linearized driven model of Eq. (34) to examine the effect of the triple nonlinearity in Eq. (34). It is shown that the contribution of the pressure force to the triple nonlinearity will lead to a decay of the velocity field that is slower than the velocity decay if the pressure force was absent. Thus one can see that the RNG eddy viscosity for the full Navier-Stokes equation could exhibit a cusp behavior near the subgrid-supergrid cutoff, while such a cusp behavior need not appear if the pressure force is absent. Indeed, this is just what is found in the full numerical solution of Eqs. (27)–(29). This also may account for the somewhat weaker RNG cusp behavior found in Fig. 1, and for the absence of the cusp for the coarser grid partition parameter of $f=0.6$ (Fig. 2). A direct numerical solution of Eq. (34) has not been attempted.

IV. SUMMARY

By applying RNG procedures to eliminate the subgrid scales, we have found that the renormalized Navier-Stokes equation now involves triple nonlinear interactions. Moreover, it has been shown that this triple nonlinearity (with the inclusion of pressure) makes an essential contribution to the renormalized eddy viscosity. Numerical solution of the RNG recursion relation shows that this eddy viscosity now exhibits a cusplike behavior for wave numbers near the supergrid-subgrid cutoff. This is in qualitative agreement with Kraichnan's test-field model,[17] with Chollet and Lesieur's eddy damped quasinormal approximation,[18] and with the recent direct numerical simulation results of Domaradzki *et al.*[20] In Appendix A it is shown that this triple nonlinearity results from the interaction between subgrid and supergrid velocity fields. This interaction bears some similarity to that needed by Kraichnan[17] to achieve his cusp behavior in the test-field eddy viscosity calculation.

It also appears that the McComb eddy viscosity calculation is inconsistent with the previous eddy viscosity results.[17,18,20] Indeed, in the iterative technique of McComb, it is claimed that no closure problem arises and no triple nonlinearities are generated. Thus in McComb's calculation, the eddy viscosity can not exhibit any cusp behavior near the subgrid-supergrid cutoff. Now the calculations of Kraichnan,[17] Chollet and Lesieur,[18] and Domaradzki *et al.*[20] are also all based directly on the quadratically nonlinear Navier-Stokes equation, but yet all obtain cusp behavior in the viscosity. Now the passive advection of a velocity field will, in the RNG technique, generate triple nonlinearities, but the RNG eddy viscosity does not exhibit a cusp behavior. The argument, basically given by Rose[12] and in the model calculation in Appendix C, is that the triple nonlinearity in the renormalized Navier-Stokes equation will itself contribute to yield extra damping near the wave-number cutoff. This extra damping, together with the RNG viscosity, can be argued to be somewhat equivalent to the cusp eddy viscosity of Chollet and Lesieur.[18] However, in our case, the triple nonlinearity in the renormalized Navier-Stokes equation contributes less damping than the similar term for the passive advection problem. This could account for the appearance of the mild cusp behavior in our calculation (as seen in Fig. 1).

Further, we find that if a spectral gap is assumed to exist between the subgrid and supergrid velocity fields then the RNG eddy viscosity will not exhibit a cusp behavior. It is shown that the spectral gap prevents the appearance of the triple nonlinearity. However, we have also found that the appearance of the triple nonlinearity is not sufficient for the cusp behavior—the pressure force is necessary.

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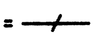
APPENDIX A: DETAILS OF THE RENORMALIZATION-GROUP METHOD FOR THE REMOVAL OF SUBGRID SHELLS

In this appendix we outline the RNG procedure in removing the first subgrid shell $k_1 < k \le k_0$, which results in Eq. (13). It is convenient to proceed using a diagrammatic approach. In the removal of the first subgrid shell, we denote the supergrid propagator by

$(\partial/\partial t + \nu_0 k^2)^{-1} =$ $, \; k \le k_1$ (A1)

and the subgrid propagator by

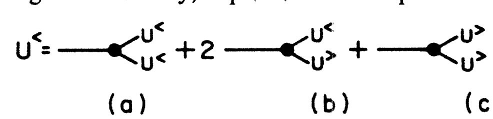
$(\nu_0 k^2)^{-1} =$  $, \; k_1 < k \le k_0 \; .$ (A2)

The nonlinear (vertex) interaction is denoted by

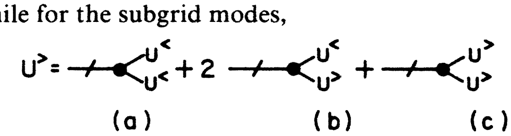
$\lambda_0 \int dk' M_{\alpha\beta\gamma} =$  (A3)

with λ_0 an ordering parameter which is eventually set to unity.

Diagrammatically, Eq. (11) can be represented as


(A4)

while for the subgrid modes,

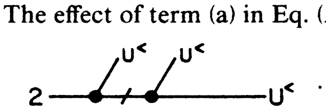

(A5)

Equation (A5) is substituted into Eq. (A4) and then an average is performed over the subgrid scales, keeping terms only to $O(\lambda_0^2)$.

1. Effect on term (b) in Eq. (A4)

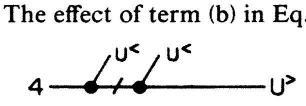
We now consider, term by term, the effect of substituting Eq. (A5) into Eq. (A4).

The effect of term (a) in Eq. (A5) is to produce


(A6)

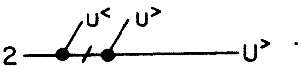
This is the new triple nonlinearity introduced in Eq. (13).

The effect of term (b) in Eq. (A5) produces



which becomes zero on performing the subgrid scale averaging, since $\langle U^{>} \rangle = 0$.

Term (c) in Eq. (A5) yields



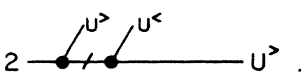
Again, this term is zero on averaging over the homogeneous subgrid scales since the $U^>$ and $U^>$ are connected by the same vertex. This can be seen algebraically since for $\mathbf{p}$ in the subgrid shell this term equals, on subgrid averaging:

$$\int d\mathbf{p}\,d\mathbf{p}'\langle U^>(\mathbf{p}-\mathbf{p}')U^>(\mathbf{p}')U^<(\mathbf{k}-p)\rangle = \int d\mathbf{p}d\mathbf{p}'Q(\mathbf{p}-\mathbf{p}')\delta(\mathbf{p})U^<(\mathbf{k}-\mathbf{p}) =0\ ,$$

since $\mathbf{p}$ is in the subgrid, and so it cannot satisfy $|\mathbf{p}|=0$.

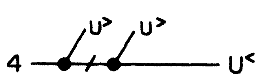
2. Effect on term (c) in Eq. (A4)

Working only to $O(\lambda_0^2)$ the substitution of term (a) in Eq. (A5) into term (c) of Eq. (A4) yields



Under subgrid scale averaging this term vanishes, since $\langle U^> \rangle=0$.

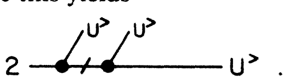
On substituting term (b) of Eq. (A5), we obtain



(A7)

which on subgrid scale averaging yields the renormalization of the response function $\langle U^> U^> \rangle U^<$.

As is usually done in RNG theories in fluid turbulence, we neglect the effect of substituting term (c) in Eq. (A5) since this yields



Neglecting this term is basically a closure approximation. An analogous procedure is performed for removal of the nth subgrid shell. Note that term (A7) will yield a contribution to the renormalized viscosity.

Also note that the term (A6), the new triple nonlinearity, results from the interaction between subgrid and supergrid velocity fields [see Eq. (A4), term (b)]. Moreover, it is shown in this paper that this triple nonlinearity is essential for us to obtain a cusp behavior in the eddy viscosity around the supergrid-subgrid cutoff. It is interesting to note that Kraichnan,[17] in his test-field theory, also requires the interaction between subgrid and supergrid velocities to achieve the cusp behavior in the eddy viscosity.

APPENDIX B: THE EFFECT OF A SPECTRAL GAP ON THE EDDY VISCOSITY

In this appendix we shall consider the application of RNG to Navier-Stokes turbulence with a spectral gap between the large scales and the subgrid scales. This is of interest also because of the recent spectral gap calculations of Biskamp and Welter,[22] Biskamp,[23] and Montgomery and co-workers.[24,25] Nevertheless, it should be noted that computational results indicate that any spectral gap will be rapidly filled in within a few eddy turnover times.

For subgrid wave numbers, the existence of a spectral gap implies that the only wave-number couplings allowed will result in the subgrid scale equation

$$(\partial/\partial t+\nu_0 k^2)u_\alpha^>(\mathbf{k},t) = \int d^3j\, 2M_{\alpha\beta\gamma}(\mathbf{k})u_\beta^\gtrless(\mathbf{j},t)u_\gamma^>(\mathbf{k}-\mathbf{j},t)\ . \tag{B1}$$

For the supergrid wave numbers, the Navier-Stokes equation becomes

$$[\partial/\partial t+\nu_0 k^2]u_\alpha^<(\mathbf{k},t) =M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j[u_\beta^\lessgtr(\mathbf{j},t)u_\gamma^<(\mathbf{k}-\mathbf{j},t) +u_\beta^\lessgtr(\mathbf{j},t)u_\gamma^<(\mathbf{k}-\mathbf{j},t)]\ . \tag{B2}$$

On substituting Eq. (B2) into (B1), and performing the analogous RNG procedure as in Sec. II, we obtain

$$[\partial/\partial t+\nu_1(k)k^2]u_\alpha^<(\mathbf{k},t) =M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\, u_\beta^\lessgtr(\mathbf{j},t)u_\gamma^<(\mathbf{k}-\mathbf{j},t)\ , \tag{B3}$$

where the eddy viscosity

$$\nu_1(k)=\nu_0+\delta\nu_0(k)\ , \tag{B4}$$

with

$$\delta\nu_0(k)=2\int d^3j\, Q(|\mathbf{k}-\mathbf{j}|)L_{kj}/(\nu_0 j^2k^2)\ . \tag{B5}$$

Since the spectral gap exists at each iteration, we can immediately generalize to obtain

$$[\partial/\partial t+\nu_{n+1}(k)k^2]u_\alpha^<(\mathbf{k},t) =M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\, u_\beta^\lessgtr(\mathbf{j},t)u_\gamma^<(\mathbf{k}-\mathbf{j},t)\ , \tag{B6}$$

where the eddy viscosity

$$\nu_{n+1}(k)=\nu_n(k)+\delta\nu_n(k)\ , \tag{B7}$$

with

$$\delta\nu_n(k)=2\int d^3j\, Q(|\mathbf{k}-\mathbf{j}|)L_{kj}/[\nu_n(j)j^2k^2]\ . \tag{B8}$$

Equation (B8) should be compared to Eq. (25). We see that the effect of the spectral gap on the eddy viscosity is exactly the same as the effect of neglecting the influence of the pressure force on the eddy viscosity. In both of these cases there is no triple nonlinearity induced into the renormalized Navier-Stokes equation, and the resulting RNG eddy viscosity does not show a cusp behavior for wave numbers near the supergrid-subgrid cutoff. It should also be noted that the presence of a spectral gap yields a closed set of equations without the need of invoking a closure approximation.

APPENDIX C: THE DIRECT EFFECT OF THE TRIPLE NONLINEARITY IN THE RENORMALIZED NAVIER-STOKES EQUATION

In this appendix we follow Rose in his calculation[12] for the passive scalar advection to estimate the effect of the triple nonlinearity in the renormalized Navier-Stokes equation (34).

To be able-to proceed analytically, we linearize the Navier-Stokes equation by separating the velocity field into an advecting part ($\hat{\mathbf{u}}$) and an advected part ($\mathbf{u}$) so that

$$(\partial/\partial t+\nu_0 k^2)u_\alpha(\mathbf{k},t) = \int d^3j\, M_{\alpha\beta\gamma}(\mathbf{k})u_\beta(\mathbf{j},t)\hat{\mathbf{u}}_\gamma(\mathbf{k}-\mathbf{j},t)\ , \quad \text{(C1)}$$

where $\hat{\mathbf{u}}_\gamma$ is a prescribed random variable. We again proceed as in Appendix A to obtain the renormalized equation

$$[\partial/\partial t+\hat{\nu}(k_N)k^2]u_\alpha(\mathbf{k},t)=M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\, u_\beta(\mathbf{j},t)\hat{u}_\gamma(\mathbf{k}-\mathbf{j},t)$$
$$+M_{\alpha\beta\gamma}(\mathbf{k})[\hat{\nu}(k_N)k_N^{4/3}]^{-1}\int d^3j\, d^3j'\, M_{\beta\beta'\gamma'}(\mathbf{j})j^{-2/3}\hat{u}_{\beta'}(\mathbf{j}-\mathbf{j}',t)u_{\gamma'}(\mathbf{j}',t)\hat{u}_\gamma(\mathbf{k}-\mathbf{j},t)\ . \quad \text{(C2)}$$

Following Rose,[12] we have also taken the simplifying limit that the partition grid parameter $f\to 1$. $k_*\equiv k_N$ is the wave number separating the supergrid and subgrid wave numbers.

1. Effect of the pressure term

We now specify the advecting velocity field (in Cartesian coordinates)

$$\hat{u}_x(\mathbf{j})=V[\delta(j_y-k_2')+\delta(j_y+k_2')]\delta(j_x)\delta(j_z)\ ,\quad \hat{u}_y=0=\hat{u}_z \quad \text{(C3)}$$

where wave number $k_2'<k_*$. It is convenient to introduce a time-independent source term[12] S,

$$S(j)-S_0[\delta(j_x-k_1')+\delta(j_x+k_1')]\delta(j_y)\delta(j_z)\ , \quad \text{(C4)}$$

for some amplitude S_0 and wave number $k_1'<k_*$. Moreover, we are interested in wave number k_1' near the supergrid-subgrid cutoff k_* so that $(k_1'^2+k_2'^2)^{1/2}>k_*$. Hence the two supergrid modes $u_\alpha(\pm k_1',0,0)$ are coupled to the other modes only through the triple nonlinearity. Explicitly evaluating Eq. (C2), we find, after some straightforward algebra,

$$u_x(k_1',0,0)=S_{0_x}/\hat{\nu}(k_*)k_1'^2\ , \quad \text{(C5)}$$

$$u_z(k_1',0,0)=S_{0_z}/[\hat{\nu}(k_*)k_1'^2+Ak_1'^2]\ , \quad \text{(C6)}$$

$$u_y(k_1',0,0)=\frac{[S_{0_y}+2Ak_1'k_2'/\hat{\nu}(k_1'^2+k_2'^2)]}{\{\hat{\nu}k_1'^2+Ak_1'^2[1-2k_2'^2/(k_1'^2+k_2'^2)]\}}\ , \quad \text{(C7)}$$

where

$$A=V^2/4\hat{\nu}(k_*)k_*^{4/3}(k_1'^2+k_2'^2)^{1/3}>0\ . \quad \text{(C8)}$$

It should be noted that the terms

$$2Ak_1'k_2'/\hat{\nu}(k_*)(k_1'^2+k_2'^2)$$

and

$$-2Ak_1'^2k_2'^2/(k_1'^2+k_2'^2)$$

in the decay of u_y, Eq. (C5), arise from the pressure effect present in the coupling coefficient $M_{\alpha\beta\gamma}(k)$.

2. Case when the pressure term yields no contribution

To examine the overall effect of the pressure in the decay of the source term, Eq. (C4), we now consider an advecting velocity field with

$$\hat{u}_x(\mathbf{j})=V[\delta(j_x-k_2')+\delta(j_x+k_2')]\delta(j_y)\delta(j_z)\ ,\quad \hat{u}_y=0=\hat{u}_z\ . \quad \text{(C9)}$$

Since the wave-number dependence of the source and advecting velocity are parallel, the pressure term in $M_{\alpha\beta\gamma}$ will have no effect on the decay of u_α. Proceeding as before, we readily find now that

$$u_x(k_1',0,0)=S_{0_x}/\hat{\nu}(k_*)k_1'^2\ , \quad \text{(C10)}$$

$$u_z(k_1',0,0)=S_{0_z}/[\hat{\nu}(k_*)k_1'^2+Ak_1'^2]\ , \quad \text{(C11)}$$

$$u_y(k_1',0,0)=S_{0_y}/[\hat{\nu}(k_*)k_1'^2+Ak_1'^2]\ , \quad \text{(C12)}$$

where A is given by Eq. (C8).

Hence, in the case when the pressure term has no explicit effect in the triple nonlinearity, we find that the decay of the source is enhanced by the presence of the triple nonlinearity in the renormalized Navier-Stokes equation: $\hat{\nu}k_1'^2\to\hat{\nu}k_1'^2+Ak_1'^2$. Thus one could argue that the "presureless" eddy viscosity near the cutoff $k_*\equiv k_N$ need not show significant wave-number cusp dependence since the triple nonlinearity present in the Navier-Stokes equation could itself account for the needed extra dissipation attested to by the Kraichnan test-field model.

However, on comparing Eq. (C7) with (C12), we see that the pressure effect is to reduce the decay of the source term for wave numbers near the cutoff k_*. Thus if one is to reproduce the test-field eddy viscosity results of Kraichnan, one might expect the presence of cusplike behavior in the renormalized viscosity, as we have found in Fig. 1.

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