Propagation and stability of kinks in driven and damped nonlinear Klein-Gordon chains

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We consider the propagation of kinks in an elastic chain in a bistable or multistable potential under the action of a driving force. Each element of the chain is subject to a damping force proportional to its velocity. We show that both the propagation velocity of the kinks as a function of the driving field, and the kink width as a function of propagation velocity, are determined by characteristic functions which depend only on the form of the potential. These functions can be found by considering a single particle moving in the upside-down potential of the chain. The general properties of these functions are studied and illustrated by several examples. The stability of these driven kinks is discussed. Interestingly we find in addition to the expected discrete localized eigenmodes a two-dimensional continuum of oscillatory modes with a localized envelope.

I. INTRODUCTION

We are concerned with the structure, propagation, and stability of transition regions (domain walls) in one-dimensional (1D) multistable continuous media[1] described by a field equation of the driven, damped, nonlinear Klein-Gordon type,

$$I\,\partial^2\theta/\partial t^2+\gamma\,\partial\theta/\partial t-\kappa\,\partial^2\theta/\partial x^2=-\partial V/\partial\theta\ , \tag{1.1a}$$

with $I\geq 0$, $\gamma>0$, $\kappa>0$. Here,

$$V(\theta)=V_0 g(\theta)-F\theta\ , \tag{1.1b}$$

where $V_0 g(\theta)$ is a potential scaled by the amplitude, V_0, which has at least two minima, and F is a constant driving force (see Fig. 1). The minima will be shifted by the force F and will eventually disappear at critical values of F by merging with an adjacent maximum. The function

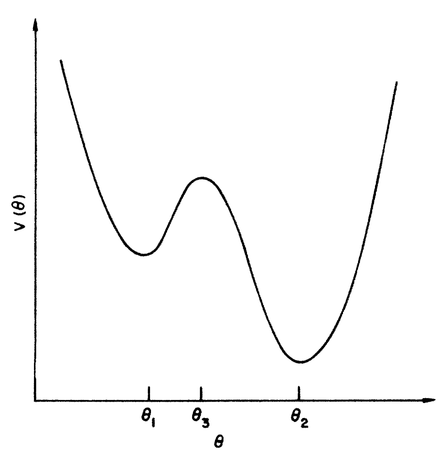

FIG. 1. Bistable potential with driving force potential $-F\theta$. The local minima are at θ_1,θ_2; the intermediate maximum is at θ_3.

$g(\theta)$ is assumed to be analytic (unless stated otherwise) and of such a form that the following conditions are satisfied. (1) No new minima appear by the application of F. This requires that $g(\theta)$ consist of convex sections around the minima, connected by concave sections around the maxima. (2) The higher minimum at $\theta=\theta_1$ disappears at a critical value $F_{\max}$ of the force by merging with the intermediate maximum at $\theta=\theta_3$,

$$g(\theta_1)=g(\theta_3);\ \ g''(\theta_1)=0\ \text{ for } F=F_{\max}\ . \tag{1.2}$$

Models of this type have been used to describe a variety of driven and damped nonlinear systems. Examples are the θ^{2n} potentials used in the theory of phase transitions, and periodic potentials (see Ref. 1 for further references). A particular realization consists of an elastic string with elastic tension κ, mass per unit length I, and damping constant γ, under the influence of a spatially uniform anharmonic force $-\partial V/\partial\theta$.

We are interested in transition regions connecting two minima $\theta_1(F)$ and $\theta_2(F)$ of $V(\theta)$ of traveling-wave (TW) form, i.e., in solitary TW solutions ("kinks") $\theta(x-ut)$ of Eq. (1.1), which depend on x and t only in the combination $\xi=x-ut$, where u is the propagation velocity along x. The TW's are solutions of the ordinary differential equation

$$I(u_s^2-u^2)\frac{d^2\theta}{d\xi^2}+u\gamma\frac{d\theta}{d\xi}=\frac{\partial V}{\partial\theta}\ , \tag{1.3}$$

where u_s, defined by $u_s^2=\kappa/I$, is the sound velocity along the string in the absence of the potential. This problem is equivalent to the motion of a single particle of mass m and friction η in a potential $U(\theta)$,

$$m\,d^2\theta/d\tau^2+\eta\,d\theta/d\tau=-\partial U/\partial\theta\ , \tag{1.4}$$

where we associate the time τ with the variable $\xi=x-ut$ and where m, η, and U are given by

$$m = I(u_s^2 - u^2)\ ,$$
$$\eta = u\gamma\ , \tag{1.5}$$
$$U = -V\ .$$

The condition that the solution $\theta(\xi)$ connects two minima θ_1 and θ_2 of $V(\theta)$ restricts the mass in Eq. (1.4) to non-negative values, i.e., the propagation velocity u to subsonic values $|u| \le u_s$. Kinks connecting two uniform states, at least one of which is a local maximum of $V(\theta)$, on the other hand, may have supersonic propagation velocities. Depending on the sign of u, the friction constant η can be of either sign. (We could actually also restrict the friction to positive values by the identification $\tau \hat{=} \pm\xi$ for $u > 0$ and $u < 0$, respectively.)

We normalize the force F in such a way that for $F = 0$ the two minima of $V(\theta)$ under consideration are of equal height. For fixed values of the mass m and the damping constant η and values of the force F below a critical value $F_c(m,\eta)$, the motion of the particle starting at the maximum $\theta = \theta_2(F)$ of $U(\theta)$ will not reach the maximum at $\theta = \theta_1(F)$, but will end up in the intermediate minimum at $\theta = \theta_3(F)$, corresponding to an unstable phase. For values $F > F_c(m,\eta)$, on the other hand, the motion will overshoot the maximum at $\theta = \theta_2(F)$ and its final course will depend on the details of the potential $V(\theta)$. For the case of a sinusoidal potential, the problem of determining $F_c(m,\eta)$ has been discussed in Refs. 2–5.

Thus, the kink solutions connecting two minima of $V(\theta)$, in which we are primarily interested, correspond to the motion at $F = F_c(m,\eta)$, starting for $\tau = -\infty$ with zero velocity at the maximum $\theta = \theta_2$ and terminating for $\tau = +\infty$ with zero velocity at the maximum $\theta = \theta_1$ of $U(\theta)$. By using Eqs. (1.3) and (1.4), the dependence of F_c on m and η yields the force $F(u) = F_c(I(u_s^2 - u^2), u\gamma)$ at which the kink with propagation velocity u exists. Note that in the usual terminology, the solitary wave moving at $F > 0$ with positive velocity is called an antikink.

The TW solutions display two general types of degeneracy: (1) Since Eq. (1.3) is invariant under translations $\xi \to \xi + a$, any TW solution $\theta(\xi)$ is a member of a continuous family $\theta(\xi + a)$, $-\infty < a < +\infty$ (Goldstone degeneracy). (2) Since Eq. (1.3) is invariant under the transformation $\xi \to -\xi$, $u \to -u$, there is for any TW solution $\theta(\xi; u, F)$ a solution $\theta(-\xi; -u, F)$. In particular, for any solitary TW with $\theta(-\infty) = \theta_2$, $\theta(+\infty) = \theta_1$ traveling in a certain direction, there exists one with $\theta(-\infty) = \theta_1$, $\theta(+\infty) = \theta_2$ traveling in the opposite direction. In the mechanical analogy, with every motion $\theta(\tau,\eta)$ there is associated a motion $\theta(-\tau,-\eta)$. In particular, for any motion with positive damping starting at $\tau = -\infty$ at the higher maximum $\theta = \theta_2$ and terminating at $\tau = +\infty$ at the lower maximum $\theta = \theta_1$ there exists a motion with negative damping starting at θ_1 and terminating at θ_2. Additional degeneracies occur if the potential function $g(\theta)$ is left invariant under some transformation of the field variable θ; if $g(\theta)$ is symmetric about the maximum $\theta = 0$, $g(\theta) = g(-\theta)$ such that $\theta_1 = -\theta_2$, then there is for any TW solution $\theta(\xi, u, F)$ a solution $-\theta(\xi; u, -F)$. If $g(\theta)$ is periodic with period Λ, $g(\theta) = g(\theta + \Lambda)$, then any TW solution $\theta(\xi)$ is a member of a family $\theta(\xi) + n\Lambda$, $n = 0, \pm 1, \pm 2, \ldots$.

Propagation of kinks and domain walls in driven and damped multistable systems is an old problem[6] and the following citations can represent only a small portion of this work. *Bistable* systems have been studied in Refs. 6–10. Montroll and West[7] have given the analytical solutions of domains in a driven and damped θ^4 chain. An extended discussion of this system has been provided by Nitzan *et al.*[8] Magyari's paper[9] extends the work of Refs. 7 and 8 to include inertial effects and represents an application of the ideas presented in Sec. II of this paper. Landauer[10] discusses the motion of domain walls in the ballast resistor[6] and gives a simple recipe for the calculation of domain-wall velocities in the presence of small driving forces.

The propagation of kinks in a sinusoidal potential has found interest repeatedly.[11–17] Nakajima *et al.*[11] have studied the initial value problem of the partial differential equation (1.1) numerically. McLaughlin and Scott[12] treat the damping perturbatively. Marcus and Imry[13] integrated the traveling-wave equation for the chain with inertia, on the computer. Besides the velocity-force characteristic, they also study the width of the kink as a function of the field. It is characteristic of the numerical work of Refs. 11 and 13 that the calculation has to be repeated if the parameters in Eq. (1.1) are changed. Adams[14] also uses numerical methods but arrives at a formulation which is closely related to the results presented in this paper. Büttiker and Landauer[15] calculated the velocity of kinks in a *purely viscous* chain, with $I \to 0$, $u_s \to \infty$, such that $m \to I u_s^2 = \kappa$. The velocity-force characteristic obtained as the inverse of $F(u)$ was found to be of the form

$$u = u_0 \phi_g(F/V_0)\ , \tag{1.6}$$

where

$$u_0 = (\kappa V_0)^{1/2}/\gamma \tag{1.7}$$

is a velocity unit and the function ϕ_g is determined by the form of $g(\theta)$ but is independent of the parameters of Eq. (1.1). We will extend Eq. (1.6) valid for $I = 0$ to the general case $I \neq 0$. We will show that, with very little algebraic calculation, the results for the purely viscous chain can be used to derive the velocity-force characteristic of the kinks propagating in a chain with inertia.[16]

In Sec. II we reduce the problem of finding the velocity-force characteristic of kinks of Eq. (1.1) to determine a characteristic function $\phi_g(F/V_0)$. We discuss the meaning of the characteristic function in terms of the single-particle problem, Eq. (1.4), and investigate the general features of this function. An additional characteristic function is introduced to discuss the width of kinks as a function of propagation velocity. In Sec. III we consider analytical examples of periodic potentials and discuss the velocity-force characteristic and the width-velocity characteristic. In Sec. IV we investigate the stability of the kink solutions and examine to what extent characteristic functions can be used to obtain the eigenvalue spectrum. Interestingly, our analysis shows that the eigenvalue spectrum of a kink is more complex than one might naively expect. In addition to the expected discrete spectrum of localized modes, we find oscillatory modes with a localized envelope associated with a two-dimensional continuous spectrum.

II. VELOCITY-FORCE CHARACTERISTIC OF KINKS

A. Scaling behavior

We consider a family of potentials $V(\theta)$ of the same form, scaled by the amplitude V_0. The number of parameters in the single-particle equation of motion (1.4) may be reduced by introducing the dimensionless time $\zeta=(V_0/|\eta|)\tau$. The resulting equation,

$$\alpha^2 d^2\theta/d\zeta^2 \pm d\theta/d\zeta = g' - F/V_0 \quad (u>0, u<0) , \tag{2.1}$$

depends only on the parameter $\alpha^2 = mV_0/\eta^2$, the normalized force F/V_0, and, of course, on the form g of the potential. Hence, the critical field F_c becomes a product of V_0 with a function of α only. Alternatively, we can consider α as a function of F/V_0 and it is convenient to introduce a scaling function ϕ_g such that

$$\frac{1}{\alpha} = \phi_g\left[\frac{F}{V_0}\right] , \tag{2.2}$$

where the index g indicates the dependence on the form $g(\theta)$ of the potential. From Eqs. (1.5) and (1.7) we have the relation

$$\alpha^2 = \frac{mV_0}{\eta^2} = \frac{u_0^2}{u^2} - X^2 = X^2\left[\frac{u_s^2}{u^2} - 1\right] , \tag{2.3}$$

where X is the dimensionless parameter

$$X = \left[\frac{IV_0}{\gamma^2}\right]^{1/2} = \frac{u_0}{u_s} . \tag{2.4}$$

Solving (2.3) for u and using (2.2) yields the velocity-force characteristic[16]

$$u = \pm u_0 \phi_g(F/V_0)[1 + X^2\phi_g^2(F/V_0)]^{-1/2} . \tag{2.5}$$

The negative branch determines the velocity of the kinks ($d\theta/d\xi > 0$) and the positive branch the velocity of the antikinks ($d\theta/d\xi < 0$).

For a "chain without inertia," one has $X=0$, and Eq. (2.5) reduces to Eq. (1.6). We could also have gained Eq. (2.5) through *similarity laws* which connect the "purely viscous chain" and the chain with inertia. According to Eq. (2.1), every traveling wave is of the form $\theta=\theta((V_0/u\gamma)\xi, \alpha, F/V_0)$. For a purely viscous chain (subscript vis), the parameter α is given by $\alpha_{\text{vis}}^2 = u_0^2/u_{\text{vis}}^2$, [$X=0$ in Eq. (2.3)]. For a chain with inertia (subscript I), the parameter α is given by Eq. (2.3), $\alpha_I^2 = (u_0^2 - X^2 u_I^2)/u_I^2$. A solution of the chain with inertia is thus similar to a solution of the purely viscous chain if $\alpha_I = \alpha_{\text{vis}}$. Using Eq. (1.6) for u_{vis} and solving for u_I yields again Eq. (2.5). A solution of the purely viscous chain is transferred into a solution of the chain with inertia by replacing κ by $\kappa - Iu^2$ or u_0^2 by $u_0^2 - X^2u^2$.

In a similar way, we can study the parameter dependence of the kink width δ. The characteristic time scale of a solution of Eq. (2.1) depends only on the parameters α^2 and F/V_0 and on the form g of the potential. The solitary motion between two adjacent maxima θ_1, θ_2 of the potential $u(\theta)$ at the critical field F_c can be characterized by a width $\Delta\zeta$ on the ζ axis, defined by

$$\Delta\zeta = |\theta_2 - \theta_1| / |d\theta/d\zeta|_{\max} . \tag{2.6}$$

Since F_c/V_0 is a function of α, $\Delta\zeta$ can be expressed as a scaling function of $\phi_g = 1/\alpha$,

$$\Delta\zeta = \Delta_g(\phi_g) . \tag{2.7}$$

For the kink width δ, we obtain by returning to the original units

$$\delta = (|\eta|/V_0)\Delta\zeta = \delta_0(|u|/u_0)\Delta\zeta , \tag{2.8}$$

where $\delta_0 = (\kappa/V_0)^{1/2}$ is a unit of length. By solving Eq. (2.5) for ϕ_g and using Eq. (2.7), we find the form of the velocity dependence,

$$\delta(u) = \delta_0 \frac{|u|}{u_0} \Delta_g\left[\frac{|u|}{(u_0^2 - X^2u^2)^{1/2}}\right] . \tag{2.9}$$

B. General properties of the scaling functions

In the static case $F=0$, $u=0$, Eq. (1.3) has a first integral

$$\tfrac{1}{2}\kappa(d\theta/d\xi)^2 - V_0[g(\theta) - g_{\min}] = 0 \tag{2.10}$$

which yields a static kink width

$$\delta(0) = \delta_0(\theta_2 - \theta_1)/[2(g_{\max} - g_{\min})]^{1/2} \tag{2.11}$$

and a kink energy (relative to that of the uniform states θ_1, θ_2)

$$E_s = \kappa \int_{-\infty}^{\infty} (d\theta/d\xi)^2 d\xi = (2\kappa V_0)^{1/2} \int_{\theta_1}^{\theta_2} [g(\theta) - g_{\min}]^{1/2} d\theta . \tag{2.12}$$

Thus, the scaling functions ϕ_g and Δ_g satisfy $\phi_g(0)=0$ and $\Delta_g(0)=\infty$ such that $\phi_g\Delta_g = \delta(0)/\delta_0 = \text{const}$.

For $F \ll V_0$, one expects that ϕ_g varies linearly with F/V_0, i.e., that there exists a kink mobility

$$\mu = du/dF|_{F=0} = (u_0/V_0)\phi_g'(0) . \tag{2.13}$$

A simple and elegant way to calculate the mobility has been given by Landauer in Ref. 10. By equating the total energy loss due to dissipation in the particle picture, $\eta\int\dot{\theta}^2 dt$, to the total work done by the external force, $-F\int\dot{\theta}dt = F(\theta_2 - \theta_1)$, inserting the static kink solution, and using the first integral (2.10), one obtains

$$\mu = (\theta_2 - \theta_1)\kappa/(\gamma E_s) . \tag{2.14}$$

Comparison with (2.13) yields

$$\phi_{g'}(0) = (\theta_2 - \theta_1)(\kappa V_0)^{1/2}/E_s , \tag{2.15}$$

which is indeed parameter independent by Eq. (2.12).

We go on to consider the behavior of the function ϕ_g at larger fields. Because the potential $U=-V$ changes continuously as we increase F, $\phi_g(F/V_0)$ must be a continuous function. Furthermore, we show now that $\phi_g(F/V_0)$ is monotonically increasing. It is sufficient to consider the single-particle problem, Eqs. (1.4) and (2.1). Consider the critical trajectory belonging to the parameters α and F_c. If the force is increased by δF, a particle started at a local maximum of U with "mass" α^2 will gain enough kinetic energy to reach the next potential maximum with nonzero velocity. The particle will, therefore, continue to follow the potential gradient. In order to have a critical trajectory at $F_c+\delta F$ the particle has to dissipate an additional energy

$$\Delta U=\{\partial[U(\theta_2)-U(\theta_1)]/\partial F\}\delta F=(\theta_2-\theta_1)\delta F>0 \ .$$

This is achieved by decreasing the mass α^2. Thus, according to Eq. (2.3), $\phi_g(F_c+\delta F)>\phi_g(F_c)$. A formal proof of the monotony of ϕ_g can be found in Ref. 2.

Finally, we consider the situation at the field $F_{\max}=V_0 g'_{\max}$ where the higher minimum of $V(\theta)$ at $\theta=\theta_1$ merges with the maximum, i.e., where $V(\theta)$ has a turning point with horizontal slope at θ_1 (existence limit of the state θ_1). We show that, at $F=F_{\max}$, ϕ_g becomes indeterminate, taking on any value between a positive number $1/\alpha_c$ and infinity. Indeed, for constant damping, a particle starting at the maximum $U(\theta)$ at $\theta=\theta_2$ at time $\zeta=-\infty$ will overshoot the horizontal turning point at $\theta=\theta_1$ if its mass α^2 is larger than a critical value α_c^2 but will approach the turning point asymptotically for $\zeta\rightarrow+\infty$ if $\alpha^2<\alpha_c^2$. The above statement then follows from Eq. (2.2). For $\alpha=0$, i.e., $\phi_g=\infty$, Eq. (2.1) becomes $\pm d\theta/d\zeta=g'(\theta)-g'_{\max}$, which shows that the scaling function Δ_g assumes the asymptotic value

$$\Delta_g(\infty)=(\theta_2-\theta_1)/(g'_{\max}-g'_{\min}) \tag{2.16}$$

and the solution $\theta(\zeta)$ connecting the states θ_1 and θ_2 is obtained from

$$\int^{\theta}\frac{d\theta'}{g'(\theta')-g'_{\max}}=\pm\zeta \ . \tag{2.17}$$

Since $g'(\theta)-g'_{\max}\propto\theta-\theta_2$ and $\propto-(\theta-\theta_1)^2$ near θ_2 and θ_1, respectively, $\theta(\zeta)$ starts exponentially from θ_2 but approaches θ_1 as a power law $\theta-\theta_1\propto 1/\zeta$. Actually, this is true for any α in the interval $0\le\alpha\le\alpha_c$, as may be seen from Eq. (2.1) with $g'(\theta)=F_{\max}/V_0+\frac{1}{3}g_1^{(3)}(\theta-\theta_1)^2+\cdots$.

Thus, at $F=F_{\max}$ there exists a whole family of kinks with propagation velocities given by Eq. (2.5). In the chain with inertia ($I\neq0$, $\chi\neq0$), the maximum propagation velocity belonging to $\alpha=0$, i.e., $\phi_g=\infty$, is given by the sound velocity $u_s=u_0/\chi$ (sonic kink) and the corresponding kink widths is given by

$$\delta(u_s)=\delta_0\Delta_g(\infty)/\chi \ . \tag{2.18}$$

u_s increases with decreasing inertia I, and in the purely viscous case ($I=0$, $\chi=0$) one has $u_s=\infty$, i.e., $\delta=\infty$. Since

$$\zeta=(x-ut)V_0/(\gamma\,|\,u\,|\,)\rightarrow-(V_0/\gamma)t \quad \text{for } u\rightarrow\infty \ ,$$

the kink solution (2.17) represents a purely temporal structure describing the advancement of the uniform chain from θ_2 to θ_1.

C. Hamiltonian limit

Finally, we discuss the Hamiltonian limit, i.e., $\gamma\rightarrow0$, $F\rightarrow0$ for the chain with inertia. The resulting propagation velocity, Eq. (2.5), depends on the way in which this limit is carried out. We find $\lim_{F\rightarrow0}\lim_{\gamma\rightarrow0}u=u_s$ and $\lim_{\gamma\rightarrow0}\lim_{F\rightarrow0}u=0$. On the other hand, if we take the limit $\gamma\rightarrow0$ but with $\chi\phi_g(F/V_0)\propto F/\gamma$ held constant, then the velocity u can take on any value[17] between $-u_s$ and u_s.

Therefore, for small damping and $F/\gamma=$const, the kink in the driven and damped chain will differ from a kink in the Hamiltonian chain traveling with the same velocity only by a small perturbation. In fact, an approximation to the velocity-force characteristic[12,18,19] may in this limit be obtained from the balance between power dissipated and work done by the external force,

$$\gamma\int(\partial\theta/\partial t)^2dx=F\int(\partial\theta/\partial t)dx \ , \tag{2.19}$$

by evaluating the integrals for the Hamiltonian kink θ^H. This power-balance equation yields the solvability condition for the lowest-order perturbation equation. Because of the formal Lorentz invariance of the Hamiltonian part of Eq. (1.1a), the kink moving with velocity u is obtained from the static kink θ_s by a Lorentz transformation,

$$\theta^H(x,t;u)=\theta_s(\bar{x}), \quad \bar{x}=(x-ut)/(1-u^2/u_s^2)^{1/2} \ , \tag{2.20}$$

whence

$$\partial\theta^H/\partial t=-[u/(1-u^2/u_s^2)^{1/2}](d\theta_s/d\bar{x}) \ . \tag{2.21}$$

Substituting into Eq. (2.19) and solving for u yields

$$\begin{aligned} u&=\pm u_0\phi'_g(0)(F/V_0)[1+\chi^2\phi'^2_g(0)(F/V_0)^2]^{-1/2}\\ &=\pm\mu F/[1+(\mu F/u_s)^2]^{1/2} \ , \end{aligned} \tag{2.22}$$

where we have used that

$$\gamma\frac{\int(d\theta_s/d\bar{x})^2d\bar{x}}{\int(d\theta_s/d\bar{x})d\bar{x}}=\frac{V_0}{u_0\phi'_g(0)}=\frac{1}{\mu} \tag{2.23}$$

on account of Eqs. (2.12) and (2.14). Similarly, the perturbation approach yields a kink width

$$\delta(u)=\delta_0(1-u^2/u_s^2)^{1/2}=\delta_0\frac{u}{u_0}\frac{(u_0^2-\chi^2u^2)^{1/2}}{u} \ . \tag{2.24}$$

It must be stressed, however, that the validity of this approximation is restricted to small damping (large χ). In fact, comparison of Eqs. (2.22) and (2.24) with the exact results, Eqs. (2.5) and (2.9), shows that the scaling functions $\phi_g(F/V_0)$ and $\Delta_g(z)$ are replaced by $\phi'_g(0)F/V_0$ and z^{-1}, respectively.

On the other hand, for small driving force $F \ll V_0$, the kink in the driven and damped chain may be obtained by a perturbation from the static kink for any value of the damping. This leads to the same power-balance equation (2.19). The results $u=\pm\mu F+O((F/V_0)^3)$ and $\delta=\delta_0+O((F/V_0)^2)$ coincide to lowest order in F/V_0 with Eqs. (2.22) and (2.24).

III. EXAMPLES

We will discuss three versions of Eq. (1.1) given by the periodic extensions of the following examples:

$$\text{(a)}\quad g_+(\theta)=+\frac{1}{2\pi}\theta^2,\quad -\pi\le\theta\le\pi\ ,\qquad V_+(\theta)=V_0 g_+(\theta)-F\theta\ , \tag{3.1}$$

$$\text{(b)}\quad g_-(\theta)=-\frac{1}{2\pi}\theta^2,\quad -\pi\le\theta\le\pi\ ,\qquad V_-(\theta)=V_0 g_-(\theta)-F\theta\ , \tag{3.2}$$

and

$$\text{(c)}\quad g(\theta)=-\cos\theta,\quad V(\theta)=-V_0\cos\theta-F\theta\ . \tag{3.3}$$

Kinks and their propagation in the potential, Eq. (1.1b), with the periodic function, Eq. (3.1), have been studied in Ref. 20. A single-particle problem, Eq. (1.4), with a potential specified by the periodic function, Eq. (3.2), has been considered in Ref. 5. Examples (a) and (b), while simple and analytically tractable, exhibit atypical features due to the cusps of the periodic functions at $\theta=(2n+1)\pi$. For the sinusoidal potential,[11-16] example (c), we will use the numerical results presented in Ref. 15.

In all three cases, the function $g(\theta)$ has the symmetry property $g(\theta)=g(-\theta)$. Therefore, kinks retain their form under $F\to -F$, and the scaling functions Δ and ϕ satisfy

$$\Delta(-\phi)=\Delta(\phi),\quad \phi(-F/V_0)=-\phi(F/V_0)\ . \tag{3.4}$$

A. Example (a)

The periodic function g_+ gives rise to the single-particle potential

$$U(\theta)=V_-(\theta)=-\frac{1}{2\pi}V_0\theta^2+F\theta,\quad -\pi\le\theta\le\pi\ ,$$

which has cusps at the local minima $\theta_n=(2n+1)\pi$. The motion connecting the maxima of U at $\theta_A=\pi F/V_0$ and $\theta_A+2\pi$ is given in terms of the dimensionless time $\zeta=(V_0/|\eta|)\tau$ by

$$\theta(\zeta)=\theta_A+(\pi-\theta_A)e^{\lambda_1\zeta},\quad \zeta\le 0\ ,\qquad \theta(\zeta)=2\pi+\theta_A-(\pi+\theta_A)e^{\lambda_2\zeta},\quad \zeta\ge 0\ , \tag{3.5}$$

describing a kink, and

$$\theta(\zeta)=2\pi+\theta_A-(\pi+\theta_A)e^{-\lambda_2\zeta},\quad \zeta\le 0\ ,\qquad \theta(\zeta)=\theta_A+(\pi-\theta_A)e^{-\lambda_1\zeta},\quad \zeta\ge 0\ , \tag{3.6}$$

describing an antikink. Here,

$$\lambda_{1,2}=[1\pm(1+4\alpha^2/\pi)^{1/2}]/(2\alpha^2)\ ,$$

with $1/\alpha=\phi_+(F_c/V_0)$. The scaling function ϕ_+ is found from the continuity condition for $d\theta/d\zeta$ at $\zeta=0$ as

$$\phi_+(F/V_0)=\frac{2}{\sqrt{\pi}}\frac{F/V_0}{[1-(F/V_0)^2]^{1/2}}\ . \tag{3.7}$$

It has a zero-field slope $\phi'_+(0)=2/\sqrt{\pi}$ and diverges like $(\Delta F/V_0)^{-1/2}$ as $\Delta F\to 0$, where $\Delta F/V_0=1-F/V_0$. The function ϕ_+ is shown in Fig. 2 ($\chi=0$ curve). Note that Eq. (3.6) at $F=V_0$ is the solution of Eq. (2.17) with $g=g_+$. The single particle in the potential V_- with the cusps at the local minima does not exhibit overdamped motion no matter how small the mass α^2. This has the consequence that ϕ_+ exhibits a power-law singularity at $F=V_0$.

The function ϕ_+ determines, with the help of Eq. (2.5), the $u(F)$ characteristic

$$u=\pm\frac{2u_0}{\sqrt{\pi}}\frac{F/V_0}{[1-(1-4\chi^2/\pi)(F/V_0)^2]^{1/2}} \tag{3.8}$$

of the kinks in a chain with inertia subject to the potential $V=V_+$, i.e., V specified by g_+. The rise of the characteristic with F is stronger than linear, strictly linear, or weaker than linear, depending on whether $\chi^2<\pi/4$, $=\pi/4$, or $>\pi/4$, respectively (Fig. 2). For $F\to V_0$ the $u(F)$ curves reach the values $u_0/\chi=u_s$.

The scaling function Δ_+ defined by Eq. (2.7) has the form

$$\Delta_+(\phi_+)=\pi[1+4/(\pi\phi_+^2)]^{1/2}=\pi/(F/V_0)\ , \tag{3.9}$$

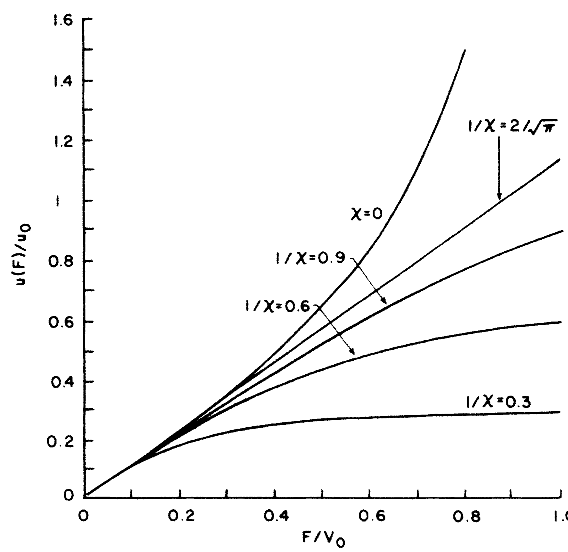

FIG. 2. Propagation velocity of driven kinks as a function of the force in a chain subject to the potential, V_+, Eq. (3.1). The $u(F)$ characteristic for the purely viscous chain ($\chi=0$) defines the characteristic function ϕ_+. Additional curves obtained with Eq. (2.5) for $1/\chi=0.9,0.6,0.3$ give the $u(F)$ characteristics for the subsonic kinks in the chain with inertia.

and the kink width is found from Eq. (2.8) as

$$\delta_+(u)=2\sqrt{\pi}\delta_0\left[1+\left[\frac{\pi}{4}-\chi^2\right](u/u_0)^2\right]^{1/2} . \quad (3.10)$$

It increases with u, is independent of u, or decreases with u, depending on whether $\chi^2<\pi/4$, $=\pi/4$, or $>\pi/4$, respectively (Fig. 3). The limiting values for the static kink and the sonic kink are $\delta_+(0)=2\sqrt{\pi}\delta_0$ and $\delta_+(u_s)=\pi\delta_0/\chi$, respectively.

B. Example (b)

For kinks in a chain with potential $V=V_-$, specified by g_-, we study the single-particle motion, Eq. (1.4), in a potential $U=V_+$ which has cusps at the local maxima $\theta_n=\pi(2n+1)$. A motion connecting the cusps at $-\pi$ and π exists for $F<V_0$ only in the case of underdamped motion of the particle, $\alpha^2>\pi/4$, and takes a finite time T. The motion describing a kink is given by

$$\theta(\zeta)=\theta_B-(\pi+\theta_B)\frac{\lambda_2 e^{-\lambda_1\zeta}-\lambda_1 e^{-\lambda_2\zeta}}{\lambda_2-\lambda_1}, \quad 0\le\zeta\le T , \quad (3.11a)$$

and the motion which describes the antikink is given by

$$\theta(\zeta)=\theta_B+(\pi-\theta_B)\frac{\lambda_2 e^{\lambda_1\zeta}-\lambda_1 e^{\lambda_2\zeta}}{\lambda_2-\lambda_1}, \quad 0\le\zeta\le T , \quad (3.11b)$$

where $\theta_B=-\pi F/V_0$ is the maximum of the potential V_- in the interval $-\pi\le\theta\le\pi$, and

$$\lambda_{1,2}=\{1\pm i[(4\alpha^2/\pi)-1]^{1/2}\}/(2\alpha^2)$$

with $\alpha=1/\phi_-(F/V_0)$. The time T follows from the requirement $\theta(T)=0$,

$$T=2\pi\alpha^2/[(4\alpha^2/\pi)-1]^{1/2} , \quad (3.12)$$

and the condition $\theta(T)=\pm\pi$ (for the kink and antikink, respectively) yields the function ϕ_-,

$$\phi_-(F/V_0)=\frac{4}{\sqrt{\pi}}\frac{\operatorname{arctanh}(F/V_0)}{[\pi^2+4\operatorname{arctanh}^2(F/V_0)]^{1/2}} \quad (3.13)$$

for $F<V_0$. It has zero-field slope $\phi'_-(0)=4/\pi^{3/2}$ and reaches the value $\phi_-=2/\sqrt{\pi}$ at $F/V_0=1$. For $F=V_0$ one has, in addition, a branch of overdamped motions ($\alpha^2<\pi/4$) starting at the upper cusp at finite time ($\zeta=0$) and reaching the lower cusp at $\zeta=\infty$, which are also given by Eq. (3.11) with $\theta_B=\pi$ and

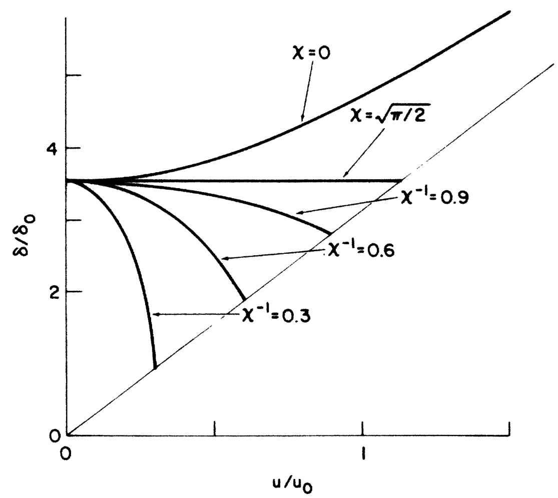

FIG. 3. Width of driven kinks in the potential V_+, Eq. (3.1), as a function of propagation velocity (dark lines) using Eq. (3.10). The $\chi=0$ curve determines the scaling function Δ_+, Eq. (3.9). The width of the kinks with velocity $u=u_s$ is given by the light line.

$$\lambda_{1,2}=[1\pm(1-4\alpha^2/\pi)^{1/2}]/(2\alpha^2) .$$

Thus, at $F/V_0=1$, the function ϕ_- becomes indeterminate, taking on any value between $2/\sqrt{\pi}$ and ∞. The solution for $\alpha=0$ is given by Eq. (2.17) with $g=g_-$. The function ϕ_- is shown in Fig. 4 ($\chi=0$ curve).

The $u(F)$ characteristic of the kinks in a chain subject to the potential $V=V_-$, i.e., V specified by g_-, is determined from the function ϕ_- by Eq. (2.5),

$$u=\pm\frac{4u_0}{\sqrt{\pi}}\frac{\operatorname{arctanh}(F/V_0)}{[\pi^2+4(1+4\chi^2/\pi)\operatorname{arctanh}^2(F/V_0)]^{1/2}} , \quad (3.14)$$

and is shown in Fig. 4. For $F\to V_0$, the $u(F)$ curves reach the values

$$u^*=(2u_0/\sqrt{\pi})(1+4\chi^2/\pi)^{1/2}<u_s .$$

For the scaling function Δ_-, one finds

$$\Delta_-(\phi_-)=\begin{cases}\dfrac{2\sqrt{\pi}}{\phi_-}\dfrac{\exp(Q^{-1}\arctan Q)}{1+\tanh[\pi/(2Q)]}, & \phi_-<2/\sqrt{\pi}\\[2ex] \dfrac{\sqrt{\pi}}{\phi_-}\exp(Q^{-1}\operatorname{arctanh} Q), & \phi_->2/\sqrt{\pi}\end{cases} \quad (3.15)$$

with $Q=|1-4/(\pi\phi_-^2)|^{1/2}$. This yields a kink width

$$\delta_-(u)=\begin{cases}2\sqrt{\pi}\delta_0[1-(u/u_s)^2]^{1/2}\dfrac{\exp(Q^{-1}\arctan Q)}{1+\tanh[\pi/(2Q)]}, & |u|<u^*\\[2ex] \sqrt{\pi}\delta_0[1-(u/u_s)^2]^{1/2}\exp(Q^{-1}\operatorname{arctanh} Q), & |u|>u^* ,\end{cases} \quad (3.16)$$

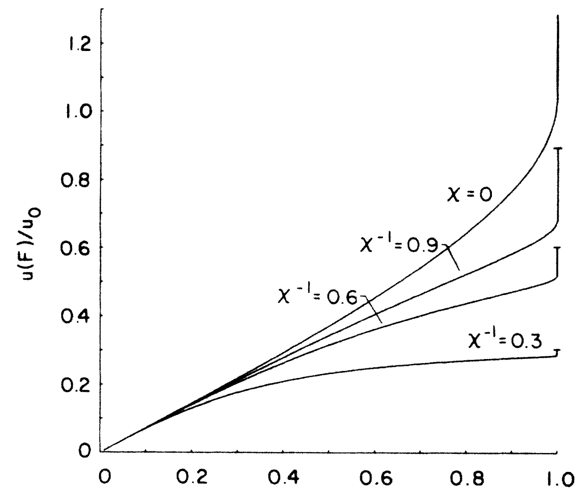

FIG. 4. Propagation velocity of driven kinks as a function of the force in a chain subject to the potential V_-, Eq. (3.2). The $u(F)$ curve for the purely viscous chain ($\mathcal{X}=0$) defines the characteristic function ϕ_-. The $\mathcal{X}=0$ curve is tangent to the vertical line $F=V_0$ at $\phi_-=2/\sqrt{\pi}$. The curves for $1/\mathcal{X}=0.9,0.6,0.3$ give the $u(F)$ characteristics for the subsonic kinks in the chain with inertia.

where Q has to be expressed in terms of u,

$$Q=\frac{2}{\sqrt{\pi}}\frac{u_0}{|u|}\,|1-(u/u^*)^2|^{1/2}\ . \tag{3.17}$$

The kink width δ_- as a function of u is shown in Fig. 5. The limiting values are $\delta_-(0)=2\sqrt{\pi}\delta_0$, $\delta_-(u^*)=\frac{1}{2}\pi e\delta_0/(\frac{1}{4}\pi+\mathcal{X}^2)^{1/2}=\frac{1}{2}\pi e\delta_0 u^*/u_0$, and $\delta_-(u_s)=\pi\delta_0/\mathcal{X}=\pi\delta_0 u_s/u_0$.

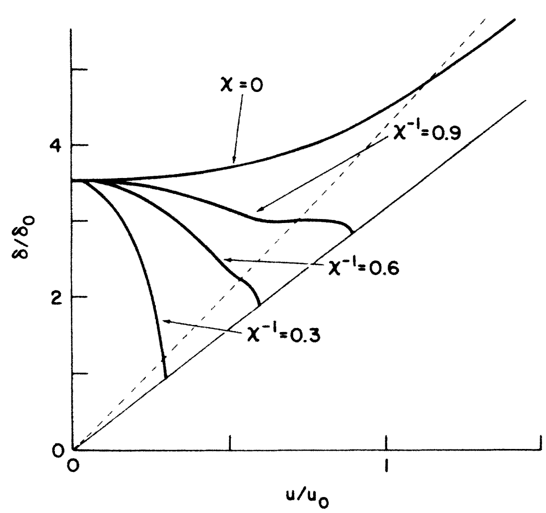

FIG. 5. Width of driven kinks in the potential V_-, Eq. (3.2), as a function of propagation velocity (dark lines) using Eq. (3.16). The $\mathcal{X}=0$ curve determines the scaling function Δ_-, Eq. (3.15). The width of the limiting kinks with velocity $u=u_s$ is given by the light line. The region between the dashed line and the light line corresponds to kinks at $F=V_0$ with algebraically decaying tails.

C. Example (c)

In addition to the symmetry $\cos(-\theta)=\cos\theta$, this example has the property that $\cos(\theta+\pi)=-\cos\theta$, i.e., a translation of half the period yields the periodic potential turned upside down. Therefore, the potential U can be obtained from $V=V_0(1-\cos\theta)-F\theta$ by changing the sign of the force and translating the field by π. The function ϕ_g has been computed numerically in Ref. 15. The slope at $F=0$ is $\phi_g'(0)=\pi/4$. ϕ_g increases monotonically with increasing field to a value[2,13,15] $\phi^*\cong1.19$ at $F=V_0$ and diverges on the vertical line $F=V_0$ to infinity. The corresponding $u(F)$ curves following from Eq. (2.5) are shown in Fig. 6. For $F\rightarrow V_0$ they reach the values $u^*=u_0\phi^*/[1+(\mathcal{X}\phi^*)^2]^{1/2}<u_s$.

The limiting solution, Eq. (2.17), at $F=V_0$ and $\alpha^2=0$, corresponding to the sonic kink for the sinusoidal potential, is given by[16]

$$\theta_0(\zeta)=-\tfrac{1}{2}\pi\pm2\arctan\zeta\ , \tag{3.18}$$

for the kink and antikink, respectively, where $\zeta=-(V_0/\gamma u_s)\xi$. It varies algebraically both for $\zeta\rightarrow+\infty$ and $\zeta\rightarrow-\infty$, because both maxima of $U(\theta)$, connected by the kink and antikink, disappear simultaneously for $F=V_0$. Solutions in the neighborhood of $\theta_0(\zeta)$, i.e., for small α^2 and $F=V_0$, can be obtained by perturbation theory. With the ansatz $\theta(\zeta)=\theta_0(\zeta)+\Delta\theta(\zeta)$, where $\Delta\theta(\zeta)$ is a small correction, Eq. (2.1), to linear order in $\Delta\theta$, becomes

$$d\Delta\theta/d\zeta+[2\zeta/(1+\zeta^2)]\Delta\theta=\alpha^2[4\zeta/(1+\zeta^2)^2]\ . \tag{3.19}$$

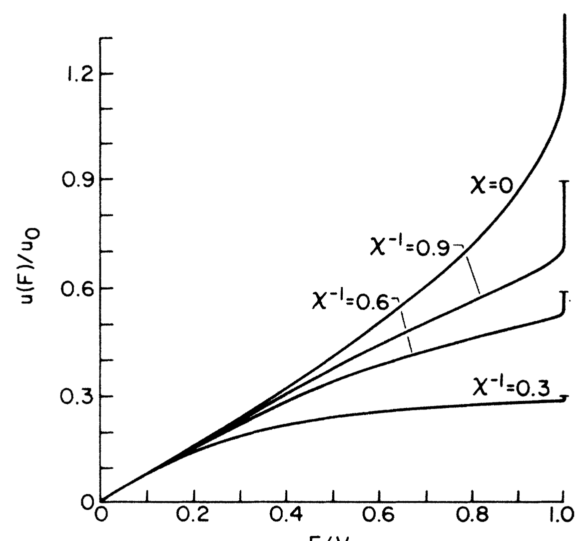

FIG. 6. Propagation velocity of driven kinks as a function of the force in a chain subject to the potential specified by the sinusoidal function, Eq. (3.3). The $\mathcal{X}=0$ curve of the purely viscous chain defines the characteristic function ϕ_g (from Ref. 15). It is tangential to the vertical line $F=V_0$ at $\phi=1.19$. Additional curves for $1/\mathcal{X}=0.9,0.6,0.3$ give the $u(F)$ characteristics for the subsonic kinks in the chain with inertia.

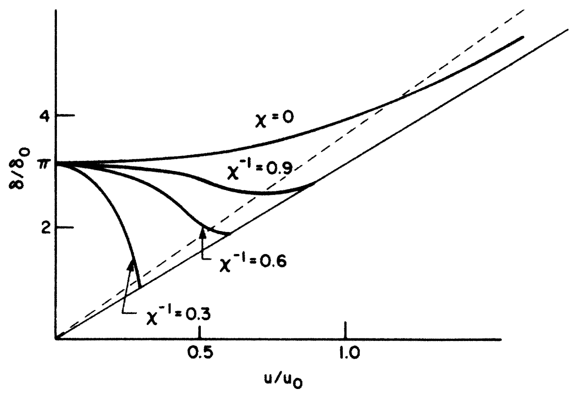

FIG. 7. Width of driven kinks in the sinusoidal potential as a function of propagation velocity (dark lines). The curve labeled $\mathcal{X}=0$ determines the scaling function $\Delta_g(\phi)$ and the other curves are generated by Eq. (2.9). The width of the limiting kinks with velocity $u=u_s$ is given by the light line. The region between the dashed line and the light line corresponds to kinks at $F=V_0$ with algebraically decaying tails.

Equation (3.19) can be integrated and yields

$$\Delta\theta(\zeta)=2\frac{u_s^2-u^2}{u_0^2}\frac{\ln(1+\zeta^2)}{1+\zeta^2}+\frac{\text{const}}{1+\zeta^2} . \tag{3.20}$$

Here we have used Eq. (2.3), $\alpha^2=(u_0^2-\mathcal{X}^2u^2)/u^2=(u_s^2-u^2)/u_0^2$. The last term is proportional to $d\theta_0(\zeta+\zeta_0)/d\zeta_0\,|_{\zeta_0=0}$ and thus represents a shift of the time origin or a translation of the kink in space, i.e., the Goldstone mode (see also Sec. IV). Note that the limiting kink with velocity $u=u_s$ is antisymmetric in ζ, whereas the kinks with $u<u_s$ also acquire a symmetrical part, leading to a different shape of the leading and trailing edge of these kinks.

The function $\Delta_g(\phi)$ is obtained from the numerically determined maximum slope $d\theta/d\zeta\,|_{\max}$ by Eq. (2.6), and is shown as the curve labeled $\mathcal{X}=0$ in Fig. 7. The other curves in this figure giving the kink width as a function of velocity for various values of the parameter $\mathcal{X}$ are generated from $\Delta_g(\phi)$ by Eq. (2.9). From the exactly known shapes of the static kink and of the sonic kink (3.18), one obtains the limiting values $\delta(0)=\pi\delta_0$ and $\delta(u_s)=\pi\delta_0/\mathcal{X}$. Minima in the kink width as a function of the propagation velocity have also been found by Ferrigno and Pace[21] but the scaling behavior is not addressed.

IV. STABILITY OF KINKS

We now investigate which parts of the $u(F)$ characteristic correspond to stable kink solutions. First, we transform Eq. (1.1) into a frame moving with the velocity u of the solitary TW, $\theta_T(x-ut)$, and introduce dimensionless coordinates

$$\zeta=(V_0/|\eta|)(x-ut),\quad s=(V_0/\gamma)t . \tag{4.1}$$

Then Eq. (1) takes the form

$$\mathcal{X}^2(\partial^2\theta/\partial s^2\mp 2\partial^2\theta/\partial s\partial\zeta)-\alpha^2\partial^2\theta/\partial\zeta^2+\partial\theta/\partial s\mp\partial\theta/\partial\zeta$$
$$=-g'(\theta)+F/V_0,\quad u>0,\ u<0 , \tag{4.2}$$

with the parameters $\mathcal{X}^2=IV_0/\gamma^2$ and $\alpha^2=\kappa V_0/(u\gamma)^2-\mathcal{X}^2$ introduced previously.

To test the stability of the kink solution $\theta_T(\zeta)$, we put $\theta(\zeta,s)=\theta_T(\zeta)+\delta\theta(\zeta,s)$ and study the time evolution of the perturbation $\delta\theta$ to linear order. Since the kink is stationary in the moving frame, it is sufficient to consider perturbations of the form $\delta\theta(\zeta,s)=\phi_\lambda(\zeta)\exp(\lambda s)$. Linearizing Eq. (4.2) with respect to $\delta\theta$ yields the generalized eigenvalue equation

$$L(\lambda)\phi_\lambda=\lambda(1+\mathcal{X}^2\lambda)\phi_\lambda , \tag{4.3}$$

where the linear operator $L(\lambda)$ is given by

$$L(\lambda)=\alpha^2d^2/d\zeta^2\pm(1+2\mathcal{X}^2\lambda)d/d\zeta-g''(\theta_T) ,$$
$$u>0,\ u<0 . \tag{4.4}$$

The kink $\theta_T(\zeta)$ breaks the translational invariance of Eq. (1.1). With $\theta_T(\zeta)$ also any translated kink $\theta_T(\zeta+\zeta_0)$ is a solution of Eq. (1.3). The infinitesimal translation $\delta\theta=(d\theta_T/d\zeta)\delta\zeta_0$ restores the broken symmetry and is an eigenmode of Eq. (4.3) with eigenvalue $\lambda=0$ (Goldstone mode). A kink $\theta_T(\zeta)$ is stable if all eigenmodes of Eq. (4.3) besides the Goldstone mode have eigenvalues with $\text{Re}\lambda<0$.

For definiteness, we assume in the following, $u>0$. Further, the analysis is carried out for the case that the restoring force $g''(\theta)$ at the more stable uniform state $\theta=\theta_2(F)$ is higher than that at the less stable state $\theta_1(F)$,

$$g''(\theta_1)<g''(\theta_2)\ \text{ for } V(\theta_1)>V(\theta_2) ,$$
$$V_0g'(\theta_{1,2})=F . \tag{4.5}$$

At the send of this section, we indicate the changes which occur if this condition is violated. We further assume $F<V_0$, which guarantees an exponential decay of the kink tails $\theta_T(\zeta)\to\theta_{1,2}$ for $\zeta\to\pm\infty$.

A. The purely viscous kink

For the purely viscous case [case (a)] $\mathcal{X}=0$, one obtains the ordinary eigenvalue problem

$$L^{(0)}\phi^{(0)}=\lambda^{(0)}\phi^{(0)} , \tag{4.6}$$

where

$$L^{(0)}=\alpha^2d^2/d\zeta^2+d/d\zeta-g''(\theta_T) . \tag{4.7}$$

The transformation

$$\phi^{(0)}(\zeta)=\psi(\zeta)\exp(-\zeta/2\alpha^2) \tag{4.8}$$

yields

$$\hat{L}\psi=\lambda^{(0)}\psi \tag{4.9}$$

with the Hermitian operator

$$\hat{L}=\alpha^2d^2/d\zeta^2-[1/4\alpha^2+g''(\theta_T)] . \tag{4.10}$$

This has the form of a Schrödinger equation with mass $\hbar^2/(2\alpha^2)$, potential $1/4\alpha^2+g''(\theta_T)$, and energy eigenvalue $-\lambda^{(0)}$. Since $\theta_T(\zeta)$ is monotonous, the Goldstone mode $(d\theta_T/d\zeta)\exp(\zeta/2\alpha^2)$ is nodeless, and represents, therefore, the ground state. For $\zeta \to \pm\infty$, $\theta_T(\zeta) \to \theta_{1,2}$, $g''(\theta_{1,2}) = g''_{1,2}$, the Schrödinger potential approaches the values $1/4\alpha^2 + g''_{1,2}$. Thus the bound states and the scattering states of $\hat{L}$ have the following two eigenvalue spectra: discrete spectrum,

$$-(1/4\alpha^2 + g''_1) < \lambda^{(0)} \le 0 ,$$

and continuum,

$$-\infty < \lambda^{(0)} \le -(1/4\alpha^2 + g''_1) , \tag{4.11}$$

respectively.

However, because of the exponential factor in Eq. (4.8), only those eigenfunctions of $\hat{L}$ which for $\zeta \to -\infty$ decay more strongly than $\exp(-|\zeta|/2\alpha^2)$ yield eigenfunctions $\phi^{(0)}$ of $L^{(0)}$. On the other hand, there exist eigenfunctions of $L^{(0)}$ corresponding to solutions of $\hat{L}\psi = \lambda^{(0)}\psi$ which increase exponentially for $\zeta \to +\infty$.

In order to gain insight into this situation, we study the asymptotic behavior of the solutions of Eqs. (4.6) and (4.9),

$$\begin{aligned} \phi^{(0)}(\zeta) &\sim \exp(\kappa_{1,2}\zeta), \quad \zeta \to \pm\infty , \\ \psi^{(0)}(\zeta) &\sim \exp(\hat{\kappa}_{1,2}\zeta), \quad \zeta \to \pm\infty , \end{aligned} \tag{4.12}$$

where

$$\hat{\kappa}_{1,2} = \kappa_{1,2} + 1/2\alpha^2 \tag{4.13}$$

on account of Eq. (4.8). $\hat{\kappa}_{1,2}$ is related to $\lambda^{(0)}$ by

$$\alpha^2\hat{\kappa}^2_{1,2} - (1/4\alpha^2 + g''_{1,2}) = \lambda^{(0)} . \tag{4.14}$$

The solution $\phi^{(0)}(\zeta)$ is bounded if

$$\mathrm{Re}\kappa_1 \le 0, \quad \mathrm{Re}\kappa_2 \ge 0 . \tag{4.15}$$

Thus an eigenstate yields a bounded $\phi^{(0)}(\zeta)$ only if $\hat{\kappa}_2 \ge 1/2\alpha^2$, i.e., if

$$-g''_2 \le \lambda^{(0)} . \tag{4.16}$$

If $g''_2 - g''_1 \le 1/4\alpha^2$, this condition is satisfied for bound states of $\hat{L}$ only; if $g''_2 - g''_1 \ge 1/4\alpha^2$, it is satisfied in addition by scattering states with wave numbers $q_1 = \mathrm{Im}\hat{\kappa}_1$, $\alpha^2 q_1^2 \le g''_2 - g''_1 - 1/4\alpha^2$.

On the other hand, unbounded solutions of $\hat{L}\psi = \lambda^{(0)}\psi$ giving rise to bounded $\phi^{(0)}(\zeta)$ require

$$0 < \mathrm{Re}\hat{\kappa}_1 \le 1/2\alpha^2 \le \mathrm{Re}\hat{\kappa}_2 . \tag{4.17}$$

If these conditions are used in Eq. (4.14), one obtains after elimination of $\mathrm{Im}\hat{\kappa}_{1,2}$

$$-g''_2 \le \mathrm{Re}\lambda^{(0)} + \alpha^2(\mathrm{Im}\lambda^{(0)})^2 \le -g''_1 . \tag{4.18}$$

For any $\lambda^{(0)}$ different from an eigenvalue of $\hat{L}$, there are *two* linearly independent unbounded solutions of Eq. (4.9) with asymptotic behavior described by $(\hat{\kappa}_1, \hat{\kappa}_2)$ and $(-\hat{\kappa}_1, -\hat{\kappa}_2)$. From this fact it follows that to any eigenvalue parameter $\lambda^{(0)}$ satisfying Eq. (4.18) there exists a bounded $\phi^{(0)}$ mode. For $\lambda^{(0)}$ values inside the region bounded by the two parabola, the modes are localized; for $\lambda^{(0)}$ values on the right (left) boundary, the modes are extended for $\zeta \to +\infty(-\infty)$. The complete spectrum of Eq. (4.6) is shown in Fig. 8(a) for the case $g''_2 - g''_1 \le 1/4\alpha^2$, and in Fig. 8(b) for the case $g''_2 - g''_1 \ge 1/4\alpha^2$.

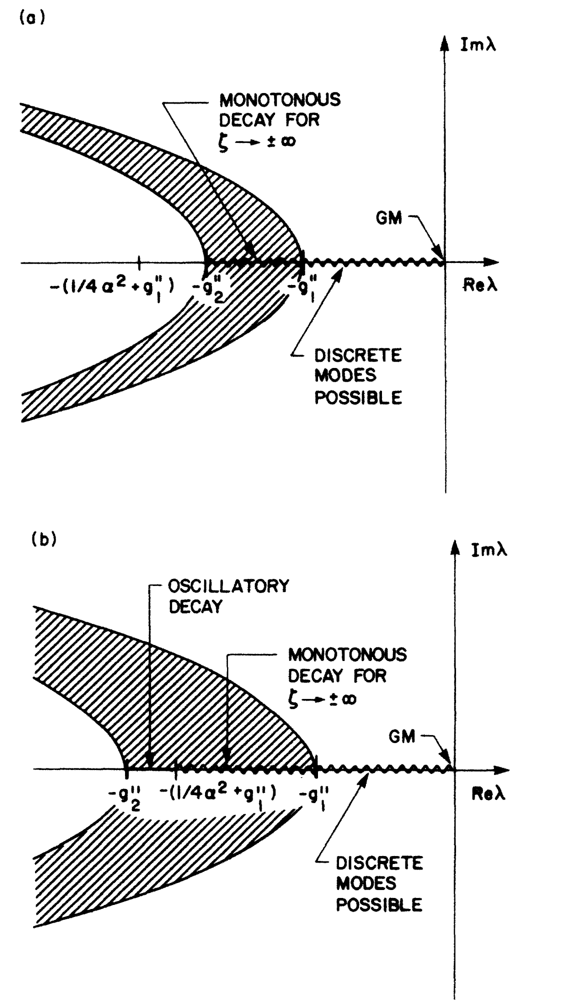

FIG. 8. Eigenvalues of a driven kink in the purely viscous chain, Eq. (4.6), for (a) $g''_2 - g''_1 < 1/4\alpha^2$ and (b) $g''_2 - g''_1 > 1/4\alpha^2$. Discrete eigenvalues (wavy lines), localized modes with monotonous decay for $t \to \infty$ and for $\zeta \to \pm\infty$; continuum (dark line), monotonous decay for $t \to \infty$ and $\zeta \to -\infty$, oscillatory decay for $\zeta \to +\infty$; shaded area, localized modes with oscillatory decay for $t \to \infty$ and $\zeta \to \pm\infty$; boundary of shaded area, modes extended at $\zeta \to +\infty$ (outer boundary) and $\zeta \to -\infty$ (inner boundary).

The eigenvalue problem for a purely viscous kink in a sinusoidal potential was briefly discussed by Büttiker and Landauer.[15] In this case the restoring forces of the two states θ_1 and θ_2 are equal, $g''_1 = g''_2 = g''$. The ϕ modes corresponding to unbounded $\psi(\zeta)$ are extended for $\zeta \to \pm\infty$ and form one-dimensional sets described by the eigenvalues λ on the parabola $\mathrm{Re}\lambda + \alpha^2(\mathrm{Im}\lambda)^2 = g''$. For the specific case of a purely viscous kink in a θ^4 potential the eigenvalue problem was studied by Schlögl, Escher, and Berry.[22] Except for the fact that they overlooked the possibility of complex $\lambda^{(0)}$, their results are compatible with ours.

B. The kink with inertia

For a chain with inertia, $\chi \neq 0$, the transformation

$$\phi(\zeta)=\psi(\zeta)\exp[-(1+2\chi^2\lambda)\zeta/2\alpha^2] \quad (4.19)$$

yields

$$\hat{L}\psi=\hat{\lambda}\psi , \quad (4.20)$$

with the same Schrödinger operator as in Eq. (4.10) and an eigenvalue parameter $\hat{\lambda}$ which is related to λ by

$$\hat{\lambda}=(1+\chi^2/\alpha^2)\lambda(1+\chi^2\lambda) . \quad (4.21)$$

Solving for λ yields

$$\lambda^{(\sigma)}=-(1-\sigma\sqrt{1+\hat{\lambda}/\lambda_\chi})/2\chi^2 , \quad (4.22)$$

with $\sigma=\pm 1$ and $\lambda_\chi=\frac{1}{4}(\chi^{-2}+\alpha^{-2})$. Thus, each solution of the Schrödinger equation, Eq. (4.20), gives rise to two solutions $\phi^{(\sigma)}(\zeta)$ of the original equation (4.3). In particular, the two ϕ modes corresponding to the ground state ψ_0 of $\hat{L}$ with $\hat{\lambda}_0=0$ have a simple physical interpretation; one $(\sigma=+1)$ is the Goldstone mode $\phi_0^{(+)}(\zeta)\propto d\theta_T/d\zeta$ with $\lambda_0^{(+)}=0$, the other $(\sigma=-1)$ is the "universal inertia mode" $\phi_0^{(-)}(\zeta)=\phi_0^{(+)}(\zeta)\exp(-\zeta/\alpha^2)$ with $\lambda_0^{(-)}=-1/\chi^2$, which has been established by Magyari[23] and which describes the relaxation of a perturbation of the kink velocity u. Below, we show that this mode has the strongest relaxation rate, i.e., all other modes have $\text{Re}\lambda > -1/\chi^2$.

However, again, not all eigenstates of $\hat{L}$ yield bounded solutions $\phi^{(\sigma)}(\zeta)$, and on the other hand, there exist ϕ modes corresponding to exponentially increasing $\psi(\zeta)$. An asymptotic analysis similar to the one presented in Sec. IV A, where Eqs. (4.13) and (4.14) are now replaced by

$$\hat{\kappa}_{1,2}=\kappa_{1,2}^{(\sigma)}+(1+2\chi^2\lambda^{(\sigma)})/2\alpha^2 \quad (4.23)$$

and

$$\alpha^2\hat{\kappa}_{1,2}^2-(1/4\alpha^2+g_{1,2}'') =(1+\chi^2/\alpha^2)\lambda^{(\sigma)}(1+\chi^2\lambda^{(\sigma)}) \quad (4.24)$$

yields the following results. The behavior depends in an essential way on the parameter $\chi^2=IV_0/\gamma^2$, which is a measure of the inertia effects.

1. *Strongly damped kink*

In the case of strong damping [case (b)] $4\chi^2 g_1'' < 4\chi^2 g_2'' < 1$, only those states of $\hat{L}$ yield $\phi^{(\pm)}$ modes which have eigenvalues $\hat{\lambda}$ satisfying either

$$-(1+\chi^2/\alpha^2)g_2'' \le \hat{\lambda}, \quad \sigma=+1 , \quad (4.25a)$$

$$-(1+\chi^2/\alpha^2)g_1'' \le \hat{\lambda}, \quad \sigma=-1 , \quad (4.25b)$$

$$\hat{\lambda} < -\lambda_\chi, \quad \sigma=\pm 1 . \quad (4.26)$$

In the first case, Eqs. (4.25a) and (4.25b), the $\phi^{(\pm)}$ modes have *real* eigenvalues in the intervals

$$\lambda_2^{(+)} \le \lambda^{(+)} \le 0 , \quad (4.27a)$$

$$-1/\chi^2 \le \lambda^{(-)} \le \lambda_1^{(-)} , \quad (4.27b)$$

where

$$\lambda_{1,2}^{(\pm)}=-[1\mp(1-4\chi^2 g_{1,2}'')^{1/2}]/2\chi^2 . \quad (4.28)$$

If $(1+\chi^2/\alpha^2)g_2''-g_1'' < 1/4\alpha^2$, the condition, Eq. (4.25a), is satisfied for bound states of $\hat{L}$ only; if $(1+\chi^2/\alpha^2)g_2''-g_1'' > 1/4\alpha^2$, it is satisfied in addition by scattering states with wave numbers $q_1=\text{Im}\hat{\kappa}_1$, $\alpha^2 q_1^2 \le (1+\chi^2/\alpha^2)g_2''-g_1''-1/4\alpha^2$. In the latter case we have localized ϕ modes with oscillatory decay for $\zeta\to+\infty$ in the interval $\lambda_2^{(+)} < \lambda^{(+)} \le \lambda_0^{(+)}$, where $\lambda_0^{(+)}$ is determined by Eq. (4.22) taken at the continuum limit $\hat{\lambda}=-(g_1''+1/4\alpha^2)$ and is thus given by

$$\lambda_0^{(+)}=-\{1-[(1-4\chi^2 g_1'')/(1+\chi^2/\alpha^2)]^{1/2}\}/2\chi^2 . \quad (4.29)$$

The condition, Eq. (4.25b), is satisfied for bound states of $\hat{L}$ only in the region $4\chi^2 g_2'' < 1$.

In the case of Eq. (4.26), which is satisfied for scattering states only, the $\phi^{(\pm)}$ modes are extended and have complex eigenvalues covering the whole axis

$$\text{Re}\lambda^{(\pm)}=-1/2\chi^2, \quad -\infty < \text{Im}\lambda^{(\pm)} < \infty . \quad (4.30)$$

Unbounded solutions of $\hat{L}\psi=\hat{\lambda}\psi$ giving rise to bounded $\phi^{(\pm)}(\zeta)$ require [see Eq. (4.24)]

$$\text{Re}\lambda > -1/2\chi^2, \quad 0 < \text{Re}\hat{\kappa}_1 \le (1+2\chi^2\text{Re}\lambda)/\alpha^2 \le \text{Re}\hat{\kappa}_2 . \quad (4.31)$$

If these conditions are used in Eq. (4.24) one obtains after elimination of $\text{Im}\hat{\kappa}_{1,2}$

$$1-4\chi^2 g_2'' \le (1+2\chi^2\text{Re}\lambda)^2+4\chi^2(\alpha^2+\chi^2)(\text{Im}\lambda)^2 \le 1-4\chi^2 g_1'' , \quad (4.32)$$

with $\text{Re}\lambda > -1/2\chi^2$. Again, to any eigenvalue parameter satisfying Eq. (4.32), there exists a bounded ϕ mode, which is localized for λ values inside the region bounded by the two ellipses and the line $\text{Re}\lambda=-1/2\chi^2$ and extended for λ values on the boundary. The complete spectrum for the strongly damped kink in a chain with inertia is shown in Fig. 9(a) for the case $(1+\chi^2/\alpha^2)g_2''-g_1'' < 1/4\alpha^2$ and in Fig. 9(b) for the case $(1+\chi^2/\alpha^2)g_2''-g_1'' > 1/4\alpha^2$.

2. *Intermediately damped kink*

In the case of intermediate damping [case (c)] $4\chi^2 g_1'' < 1 < 4\chi^2 g_2''$, Eqs. (4.25a) and (4.27a) are replaced by

$$-\lambda_\chi \le \hat{\lambda} ,$$

$$-1/2\chi^2 \le \lambda^{(+)} ,$$

i.e., *all* eigenstates of $\hat{L}$ give rise to $\phi^{(+)}$ modes. Further, the lower bound in Eq. (4.30) is replaced by zero [Fig. 10(a)].

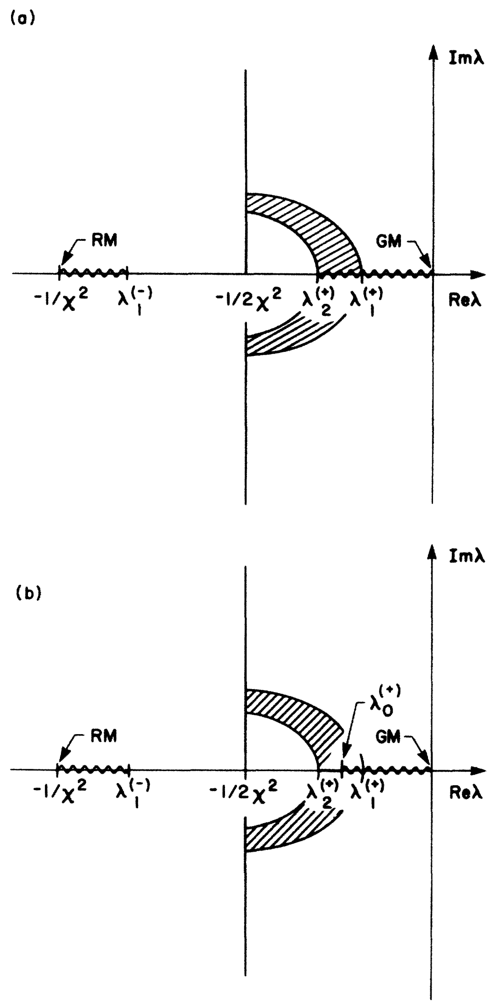

FIG. 9. Eigenvalue spectrum of a strongly damped kink in a chain with inertia, $(4\chi^2 g_1'' < 4\chi^2 g_2'' < 1)$. (a) is for $(1+\chi^2/\alpha^2)g_2'' - g_1'' < 1/4\alpha^2$ and (b) is for $(1+\chi^2/\alpha^2)g_2'' - g_1'' > 1/4\alpha^2$. The shaded area between the two half ellipses corresponds to a continuum of localized modes. In addition to the Goldstone mode (GM), the location of the universal relaxation mode (RM), discussed in Ref. 20, is also indicated.

3. *Weakly damped kink*

In the case of weak damping [case (d)] $1 < 4\chi^2 g_1'' < 4\chi^2 g_2''$, *all* eigenstates of $\hat{L}$ give rise to $\phi^{(+)}$ *and* $\phi^{(-)}$ modes, and there exist *no* ϕ modes corresponding to unbounded $\psi(\zeta)$. Bound states of $\hat{L}$ yield $\phi^{(\pm)}$ modes with eigenvalues

$$-1/\chi^2 \le \lambda^{(-)} \le 1/2\chi^2 \le \lambda^{(+)} \le 0 \tag{4.33}$$

and

$$\mathrm{Re}\lambda^{(\pm)} = -1/2\chi^2 ,$$
$$|\,\mathrm{Im}\lambda^{(\pm)}\,| < (1/2\chi^2)[(4\chi^2 g_1'' - 1)/(1+\chi^2/\alpha^2)]^{1/2} . \tag{4.34}$$

The scattering states yield

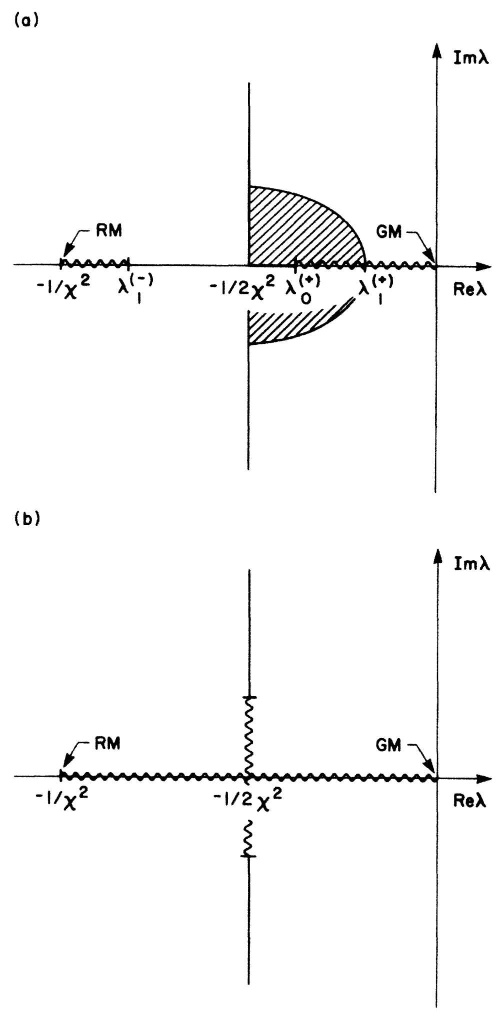

FIG. 10. (a) Eigenvalue spectrum of a kink in the intermediately damped chain $(4\chi^2 g_1'' < 1 < 4\chi^2 g_2'')$. (b) Eigenvalue spectrum of a kink in the weakly damped chain $(1 < 4\chi^2 g_1'' < 4\chi^2 g_2'')$.

$$\mathrm{Re}\lambda^{(\pm)} = -1/2\chi^2 ,$$
$$|\,\mathrm{Im}\lambda^{(\pm)}\,| \ge (1/2\chi^2)[(4\chi^2 g_1'' - 1)/(1+\chi^2/\alpha^2)]^{1/2} . \tag{4.35}$$

The spectrum for this case is shown in Fig. 10(b).

The presence of a two-dimensional set of eigenvalues in the cases (a), (b), and (c) will give rise to a very complicated time dependence of an initial perturbation. However, since $\mathrm{Re}\lambda < 0$ for all ϕ modes, we arrive at the important conclusion that kinks connecting two stable uniform states $\theta_{1,2}$ are stable in the whole range $0 \le F < V_0$.

If $g_2'' < g_1''$, i.e., if the more stable uniform state has a *smaller* restoring force than the less stable state, the spectrum is obtained by interchanging g_1'' and g_2'', changing the sign of σ, and a reflection at the line $\mathrm{Re}\lambda = -1/2\chi^2$.

If $g_1'' = g_2'' = g''$, i.e., if the restoring forces of the two states θ_1 and θ_2 are equal, the ϕ modes corresponding to unbounded $\psi(\zeta)$ are extended for $\zeta \to \pm\infty$ and form one-dimensional sets described by eigenvalues λ on the parabola

$$\mathrm{Re}\lambda + \alpha^2(\mathrm{Im}\lambda)^2 = -g'' \tag{4.36}$$

in case (a), and by the *full* ellipse

$$(1+2\chi^2\mathrm{Re}\lambda)^2 + 4\chi^2(\alpha^2+\chi^2)(\mathrm{Im}\lambda)^2 = 1-4\chi^2 g'' \tag{4.37}$$

in case (b). The situation $g_1'' = g_2''$ occurs in particular in the case of a periodic potential $V(\theta)$, which has been studied for $\chi^2 \neq 0$ by Burkov and Lifsic.[24] Our results for this specific case are in agreement with their conclusions.

The kink velocity u and the kink width δ of a kink in the chain with inertia can be obtained through a scaling procedure if these quantities are known for the purely viscous kink. In contrast, for the eigenvalue spectrum a scaling procedure exists only for those eigenvalues of the kink with inertia which satisfy Eqs. (4.25a) and (4.25b) or Eq. (4.26). In this case the eigenvalues $\lambda^{(\sigma)}$ of the kink with inertia can be obtained from the eigenvalues $\lambda^{(0)}$ by using Eq. (4.22) with $\hat{\lambda} = \lambda^{(0)}$.

We emphasize that the stability of the kinks in the whole region of existence rests heavily on the assumption that the medium is described by a single-component field θ and on the restriction to one space coordinate. For a Hamiltonian kink described by a two-component field, an instability was found to occur against a noncollinear mode,[25] and for a uniformly driven domain wall an instability occurs against a branch of waves of wall displacements with the wave vector in the plane of the wall.[26]

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