Rotating oscillator—shifted $1/N$ expansion and supersymmetric considerations

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(Received 16 June 1987)

The rotating displaced oscillator problem has been treated by the shifted $1/N$ expansion method. Next, the Hamiltonian for this problem is framed in supersymmetric form for certain values of the parameter. Exact eigenvalues and eigenfunctions are obtained. The eigenvalues obtained from the shifted $1/N$ expansion method are compared with those obtained by numerical methods and supersymmetric exact values, and are found to be in satisfactory agreement both at low and high values of the coupling parameter.

I. INTRODUCTION

The rotating displaced oscillator problem, with the Schrödinger equation

$$-D^2\psi+\frac{l(l+1)}{r^2}\psi+\frac{(r-1)^2}{4\alpha^2}\psi=\left[\frac{\lambda+\frac{1}{2}}{\alpha}\right]\psi \tag{1}$$

and with $\psi(0)=\psi(\infty)=0$, has been known for many years in physics literature,[1-3] as it represents the simplest model of a rotating-vibrating molecule. In Eq. (1), α is the inverse of the coupling parameter assumed to satisfy $0<\alpha\leq 1$. In general, the exact analytical solution of (1) cannot be obtained but, as will be shown later on, for some particular values of the coupling parameter, the exact solution could be obtained from supersymmetric considerations. In recent years there has been a renewal of interest[4-8] in the problem and also some controversy.[5-8] Masson[8] has reviewed the history of the problem. Unless great care is exercised, formalism based on a three-term recurrence relation can lead to erroneous conclusions[5,7] about the eigenvalues λ of Eq. (1). Killingbeck[9] has shown that the use of three-term recurrence relations can lead to false eigenvalues.

The shifted $1/N$ expansion method proposed by Sukhatme and Imbo[10] and Imbo *et al.*[11] has proved to be a powerful method for obtaining the eigenvalues of spherically symmetry potentials.[11-15] It is nonperturbative in nature and thus can be used in problems where the coupling constant may not be small. The shifted $1/N$ expansion method overcomes the slow convergence of the large-N expansion method[16-19] by modifying the expression of the expansion parameter. In the large-N expansion, the expanding parameter is $1/k$, where $k=N+2l$, N being the spatial dimensions; in the shifted $1/N$ version, $1/k$ is changed to $1/\bar{k}$, where $\bar{k}=N+2l-a$, where a is a shift chosen by requiring agreement between the $1/\bar{k}$ expansion and the exact analytic results for the harmonic oscillator and the Coulomb potential.

In the present paper we shall employ the shifted $1/N$ expansion method to calculate the energy eigenvalues for the rotating displaced oscillator. Following Killingbeck,[9] we shall write the potential for this problem in a more general form,

$$V=V_1r+V_2r^2\ , \tag{2}$$

where the parameters V_1 and V_2 are related to α by

$$V_1=-2V_2=-1/2\alpha^2\ , \tag{3}$$

and the eigenenergy E is connected to λ and α by

$$E=\frac{\lambda+\frac{1}{2}}{\alpha}-\frac{1}{4\alpha^2}\ . \tag{4}$$

We have also framed the Hamiltonian for the rotating displaced oscillator problem in a supersymmetric form for certain values of the parameter and obtained exact eigenvalues and eigenfunctions which correspond to ground states of supersymmetric Hamiltonians.[20] In Sec. III this will be discussed in more detail.

II. ENERGY EIGENVALUE EXPRESSION

Imbo *et al.*[11] have described the procedure for determining the energy eigenvalues in the shifted $1/N$ expansion formalism. Hence, for the sake of brevity, we omit the intermediate steps and give here only the final expressions. We shall use the units in which $\hbar=2m=e=1$.

The effective potential in the shifted $1/N$ expansion is given by

$$V_{\rm eff}(r)=\frac{\hbar^2}{8mr^2}+\frac{V(r)}{\bar{k}^2}\ , \tag{5}$$

then it is assumed that $V(r)$ is sufficiently well behaved so that $V_{\rm eff}(r)$ has a minimum at $r=r_0$ and there are well-defined bound states.

For the potential (2), one obtains the following expression for $\bar{k}^2$:

$$\bar{k}^2=2V_1r_0^3+4V_2r_0^4\ , \tag{6}$$

and the position of the minimum r_0 is determined from

$$(2l+1)+(2n_r+1)\left[\frac{3V_1+8V_2r_0}{V_1+2V_2r_0}\right]^{1/2}=(2V_1r_0^3+4V_2r_0^4)^{1/2}\ , \tag{7}$$

where n_r is the radial quantum number.

The final expression for the eigenvalues is as follows:

$$E=\frac{\bar{k}^2}{r_0^2}\left[\frac{1}{4}+\frac{V_1r_0^3+V_2r_0^4}{2V_1r_0^3+4V_2r_0^4}+\frac{\beta^{(1)}}{\bar{k}^2}+\frac{\beta^{(2)}}{\bar{k}^3}+O\left[\frac{1}{\bar{k}^4}\right]\right] \tag{8}$$

The quantities $\beta^{(1)}$ and $\beta^{(2)}$ appearing in the corrections to the leading order of the energy expansion are

$$\beta^{(1)}=\tfrac{1}{4}(1-a)(3-a)+(1+2n_r)\tilde{\varepsilon}_2+3(1+2n_r+2n_r^2)\tilde{\varepsilon}_4$$
$$-\frac{1}{\omega}[\tilde{\varepsilon}_1^2+6(1+2n_r)\tilde{\varepsilon}_1\tilde{\varepsilon}_3+(11+30n_r+30n_r^2)\tilde{\varepsilon}_3^2]\ , \tag{9}$$

$$\beta^{(2)}=(1+2n_r)\tilde{\delta}_2+3(1+2n_r+2n_r^2)\tilde{\delta}_4+5(3+8n_r+6n_r^2+4n_r^3)\tilde{\delta}_6$$
$$-\omega^{-1}[(1+2n_r)\tilde{\varepsilon}_2^2+12(1+2n_r+2n_r^2)\tilde{\varepsilon}_2\tilde{\varepsilon}_4+2(21+59n_r+51n_r^2+34n_r^3)\tilde{\varepsilon}_4^2+2\tilde{\varepsilon}_1\tilde{\delta}_1$$
$$+6(1+2n_r)\tilde{\varepsilon}_1\tilde{\delta}_3+30(1+2n_r+2n_r^2)\tilde{\varepsilon}_1\tilde{\delta}_5+6(1+2n_r)\tilde{\varepsilon}_3\tilde{\delta}_1+2(11+30n_r+30n_r^2)\tilde{\varepsilon}_3\tilde{\delta}_3$$
$$+10(13+40n_r+42n_r^2+28n_r^3)\tilde{\varepsilon}_3\tilde{\delta}_5]$$
$$+\omega^{-2}[4\tilde{\varepsilon}_1^2\tilde{\varepsilon}_2+36(1+2n_r)\tilde{\varepsilon}_1\tilde{\varepsilon}_2\tilde{\varepsilon}_3+8(11+30n_r+30n_r^2)\tilde{\varepsilon}_2\tilde{\varepsilon}_3^2+24(1+2n_r)\tilde{\varepsilon}_1^2\tilde{\varepsilon}_4$$
$$+8(31+78n_r+78n_r^2)\tilde{\varepsilon}_1\tilde{\varepsilon}_3\tilde{\varepsilon}_4+12(57+189n_r+225n_r^2+150n_r^3)\tilde{\varepsilon}_3^2\tilde{\varepsilon}_4]$$
$$-\omega^{-3}[8\tilde{\varepsilon}_1^3\tilde{\varepsilon}_3+108(1+2n_r)\tilde{\varepsilon}_1^2\tilde{\varepsilon}_3^2+48(11+30n_r+30n_r^2)\tilde{\varepsilon}_1\tilde{\varepsilon}_3^3+30(31+109n_r+141n_r^2+94n_r^3)\tilde{\varepsilon}_3^4]\ , \tag{10}$$

TABLE I. Energy eigenvalues obtained from the shifted $1/N$ expansion method are compared to those obtained by Fröman *et al.*[6] by a numerical solution of the Schrödinger equation.

α	V_1	V_2	n_r	l	E (This paper)	E (Fröman *et al.*)
0.01	−5000	2500	1	1	−2348.9	−2347.8
			2	1	−2252.6	−2247.7
0.03	−555.555	277.778	0	0	−261.23	−261.11
			1	0	−229.90	−227.78
			2	0	−199.61	−194.44
			3	0	−168.49	−161.09
			4	0	−135.78	−127.67
			5	0	−101.34	−94.002
0.05	−200	100	0	0	−90.127	−89.998
			0	1	−87.716	−87.68
			1	0	−71.368	−69.974
			1	1	−67.288	−66.72
			2	0	−51.928	−49.798
			2	1	−46.050	−45.18
			3	0	−30.902	−29.12
			4	0	−8.337	−7.476
			5	0	15.54	+15.40
0.1	−50	25	0	0	−19.962	−19.925
			0	1	−17.463	−17.46
			1	0	−9.651	−9.42
			1	1	−5.793	−5.74
			2	0	1.950	1.98
			2	1	6.775	6.75
			3	0	14.634	14.37
			4	0	28.11	27.59
			5	0	42.17	41.44

in which

$$\tilde{\varepsilon}_j=\varepsilon_j/\omega^{j/2},\quad \tilde{\delta}_j=\delta_j/\omega^{j/2},$$

$$\omega=\left[\frac{3V_1+8V_2r_0}{V_1+2V_2}\right]^{1/2},\quad a=2-(2n_r+1)\omega,$$

$$\delta_1=-\tfrac{2}{3}\delta_2=-(1-a)(3-a)/2,$$

$$\delta_3=-\tfrac{4}{5}\delta_4=2\varepsilon_1=-\tfrac{4}{3}\varepsilon_2=2(2-a),$$

$$\varepsilon_3=-1,\quad \varepsilon_4=\tfrac{5}{4},\quad \delta_5=-\tfrac{3}{2},\quad \delta_6=\tfrac{7}{4}.$$

Fröman *et al.*[6] have obtained eigenvalues for (1) for certain values of α. In Table I we have compared the eigenvalues obtained from the shifted $1/N$ expansion method with those obtained by Fröman *et al.*[6]

III. SUPERSYMMETRIC CHARACTER OF THE ROTATING OSCILLATOR

Before casting the rotating oscillator in supersymmetric form, let us recall briefly some salient features of supersymmetric quantum mechanics (SUSYQM) in one dimension.[20,21] In one dimension the Hamiltonian of SUSYQM is given by

$$H^S=\{Q^\dagger,Q\}=\begin{bmatrix}H_+ & 0\\ 0 & H_-\end{bmatrix}, \tag{11}$$

where

$$H_\pm=-\frac{d^2}{dx^2}+V_\pm(x), \tag{12}$$

$$V_\pm(x)=W^2(x)\pm W'(x), \tag{13}$$

$$W'(x)=\frac{dW}{dx}.$$

$W(x)$ is called the superpotential and $Q,Q^\dagger$ are the supercharges, whose explicit forms are given below:

$$Q=(p+iW)\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}, \tag{14}$$

$$Q^\dagger=(p-iW)\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}. \tag{15}$$

The relations obeyed by $Q,Q^\dagger$ and H^S are the following:

$$[H^S,Q]=[H^S,Q^\dagger]=0,\qquad Q^2=Q^{\dagger 2}=0. \tag{16}$$

The eigenstates of H^S are of the form

$$\phi^n(x)=\begin{bmatrix}\phi_+^n(x)\\ \phi_-^n(x)\end{bmatrix}. \tag{17}$$

If supersymmetry is unbroken the ground-state energy is zero and the ground-state wave functions are of the form $\begin{pmatrix}\phi_+^0(x)\\ 0\end{pmatrix}$ or $\begin{pmatrix}0\\ \phi_-^n(x)\end{pmatrix}$. The choice will depend on the normalizability of $\phi_\pm^0(x)$. Now if $|\psi\rangle$ is a ground state then

$$Q|\psi\rangle=Q^\dagger|\psi\rangle=0. \tag{18}$$

From (14) and (15) it can easily be seen that

$$\phi_\pm^0(x)=\exp\left[\pm\int^x W(t)dt\right]. \tag{19}$$

Now we come to the rotating oscillator. To show that its Hamiltonian can be cast into the form (1), we chose

$$W(r)=\sqrt{V_2}(r-1)-\frac{2}{1+gr}+\frac{c}{r}, \tag{20}$$

then it can be easily seen that $V_-(r)$ can be written in the form

$$V_-(r)=V_2(r-1)^2+\left[\frac{2\sqrt{V_2}+2g\sqrt{V_2}+2gc}{1+gr}\right]-\frac{2c(g+\sqrt{V_2})}{r}+\frac{c(c+1)}{r^2}+\sqrt{V_2}(2c-3). \tag{21}$$

Therefore, the effective potential appearing in the Schrödinger equation corresponding to (2) and that appearing in the radial Schrödinger equation corresponding to (21) are, respectively, given by

$$V^{\text{eff}}(r)=V_2r^2+V_1r+l(l+1)/r^2 \tag{22}$$

and

$$V_-^{\text{eff}}(r)=V_2r^2-2V_2r+\frac{(2\sqrt{V_2}+2g\sqrt{V_2}+2gc)}{1+gr}-\frac{2c(g+\sqrt{V_2})}{r}+\frac{c(c+1)}{r^2}. \tag{23}$$

For $V_1=-2V_2$, i.e., for the rotating oscillator, (22) and (23) can be identified provided we take

$$g=-\sqrt{V_2}, \tag{24}$$

$$c=-l-1, \tag{25}$$

[the negative value of c is chosen to ensure the normalizability of $\exp(-\int^x W(t)dt)$] and

$$V_2=\frac{1}{(l+2)^2}. \tag{26}$$

The relation between the corresponding energy eigenvalues is [E_R^n is the energy as given in (4)]

$$E_R^n+V_2+\sqrt{V_2}(2c-3)=E_-^n. \tag{27}$$

Now the ground-state wave function corresponding to the Bosonic sector ($-$) is found to be

$$\phi_-^0(r)\sim\exp\left[-\int^x W(t)dt\right]=Ar^{l+1}(1+gr)e^{-\sqrt{V_2}(r-1)^2}, \tag{28}$$

where A is a normalization constant. It is clear that

$$\lim_{r\to\infty}\phi_-^0(r)=\lim_{r\to 0}\phi_-^0(r)=0. \tag{29}$$

TABLE II. Comparison with exact supersymmetric eigenvalues. $n_r=1$ state, $V_2=1/(l+2)^2$.

l	V_2	E (Shifted $1/N$ expansion)	E (Supersymmetric)
1	$\frac{1}{9}$	2.223 39	2.222 22
2	$\frac{1}{16}$	2.187 84	2.187 50
3	$\frac{1}{25}$	2.160 13	2.160 00
4	$\frac{1}{36}$	2.138 94	2.138 89
5	$\frac{1}{49}$	2.122 48	2.122 45

TABLE III. Comparison with exact supersymmetric eigenvalue. $n_r=2$ state, $V_2=(9+4l)^2/[(l+2)^2(l+3)^2]$.

l	V_2	E (Shifted $1/N$ expansion)	E (Supersymmetric)
1	1.173 61	8.594 92	8.576 39
2	0.722 50	8.634 58	8.627 50
3	0.490	8.613 18	8.610 00
4	0.354 31	8.575 86	8.574 26
5	0.210 07	8.536 27	8.535 39

Hence $\phi_-^0(r)$ is an acceptable ground state, and in this case

$$E_-^n=0 .$$

Hence from (27) we have

$$E_R=\sqrt{V_2}(5+2l)-V_2=\frac{5+2l}{l+2}-\frac{1}{(l+2)^2} . \tag{30}$$

A comparison with pure oscillator ensures that (30) gives $n_r=1$ excited state when $V_2=1/(l+2)^2$. Now the general ansatz for W is

$$W=\sqrt{V_2}(r-1)+\sum_{i=1}^{N}\frac{g_i}{1+g_i r}+\frac{c}{r} . \tag{31}$$

The g_i's can be so chosen as to make (22) and (23) identical (with $V_1=-2V_2$). As $n_r=1$ case has been shown in detail above we just state the results for $n_r=2$ here for future comparison with $1/N$ shifted expansion result. We have

$$W(r)=\sqrt{V_2}(r-1)-\frac{g_1}{1+g_1 r}-\frac{g_2}{1+g_2 r}+\frac{c}{r} , \tag{32}$$

where

$$(g_1,g_2)=\frac{1}{2}\left[-\sqrt{V_2}\pm\left[V_2-\frac{4\sqrt{V_2}}{3+l}\right]^{1/2}\right] , \tag{33}$$

$$V_2=\frac{(9+4l)^2}{(l+2)^2(l+3)^2} , \tag{34}$$

$$\phi_0^-(r)=C_0 r^{l+1}e^{-\sqrt{V_2}(r-1)^2/2}(1+g_1 r)(1+g_2 r) , \tag{35}$$

and

$$E=\sqrt{V_2}(7+2l)-V_2 , \tag{36}$$

C_0 being a normalization constant. Again if we compare (36) with the results of pure oscillator, it can be seen that (36) gives the $n_r=2$ excited states for all values of l. In Tables II and III a comparison has been made between the exact supersymmetric values with the energy values obtained from the shifted $1/N$ expansion method.

IV. DISCUSSION

In this paper the shifted $1/N$ expansion method has been applied to find the energy eigenvalues of the rotating displaced oscillator. Since this method is applicable to any spherically symmetric potential it will not give rise to any problems such as those encountered in the use of three-term recurrence relations.[7–9] The numerical results obtained are in good agreement (Table I) with the published results[6] for small values of α. As is expected, the agreement is better for nonzero values of l. Furthermore, it has been shown that the rotating oscillator has a supersymmetric character for certain values of the coupling parameter. This property has been exploited successfully to find exact energy values and eigenstates of the rotating oscillator Hamiltonians which correspond to the ground state of SUSY Hamiltonians. Thus we have checked the accuracy of the $1/N$ method in another way. The energy values given by the shifted $1/N$ method match the exact SUSY values extremely well especially for values of $l\geq 2$. This occurs (Tables II and III) when V_2 is small, i.e., α is large. Thus SUSY provides a check for the numerical calculation of the energy eigenvalues of the rotating oscillator by any method. Also we find that the shifted $1/N$ expansion method is able to provide energy eigenvalues which are of reasonable accuracy both at low and high values of the coupling parameter.

ACKNOWLEDGMENT

This work was supported in part by a research grant from the Natural Sciences and Engineering Research Council of Canada to one of the authors (Y.P.V.).

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