Scaling structure of strange attractors

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The spectrum of scaling exponents of multifractal, hyperbolic strange attractors is quantitatively understood by realizing that each unstable periodic point contributes one scaling exponent to the spectrum. The calculation of the $f(\alpha)$ function can thus be reduced to counting periodic orbits.

The essential property of multifractals is that they have a spectrum of scaling exponents. Considering a covering of a multifractal with boxes of size l, one finds[1] that the measure in each nonempty box scales like

$$P(l) \sim l^{\alpha} , \tag{1}$$

with an α that takes on a range of values, $\alpha_{\min} < \alpha < \alpha_{\max}$. In contrast, self-similar fractal sets have a mass-scaling law $M(l) \sim Al^{D}$, with D being the fractal dimension, and where different boxes exhibit[2] at most a variation in the preexponential parameter A. A convenient way of characterizing a multifractal is[1,3] by the function $f(\alpha)$, which measures how many times $N(\alpha)\Delta\alpha$ one finds the scaling exponent α falling in an interval of size $\Delta\alpha$,

$$N(\alpha)\Delta\alpha \sim l^{-f(\alpha)}\Delta\alpha . \tag{2}$$

The values taken by the function $f(\alpha)$ have been interpreted[1] as the dimensions of the subsets with scaling exponents α. Clearly, the understanding of a multifractal calls for a quantitative evaluation of the function $f(\alpha)$.

Although multifractals appear in a large variety of physical problems of intense current interest, such as turbulence,[3,4] chaos in dynamical systems,[1,5,6] fractal growth patterns,[7] etc., quantitative understanding of the functions $f(\alpha)$ has been achieved to date only in very few cases.[5,6,8] Notable are the multifractal orbits that exist right at the borderline of chaos.[5,8] Heavy use of the fact that these are still time ordered has been made in developing a theory for their understanding. The aim of this paper is to suggest a way to understand in a quantitative fashion the $f(\alpha)$ function of multifractal strange attractors well in the chaotic region. The typical orbits lose their time ordering and new ideas are needed. The main new idea is that each and every unstable periodic orbit (and these are dense on the strange attractor) is responsible for one scaling exponent. By considering longer and longer periodic orbits we can hierarchically approach and converge to the $f(\alpha)$ function.

To see the relation between periodic orbits and scaling exponents, consider a typical time series $\{\mathbf{X}_i\}_{i=1}^{N}$ of a chaotic dynamical system. We shall limit the discussion to strange attractors of two-dimensional maps of the plane to itself, of the type of the Henon or the Lozi maps; thus $\mathbf{X}_i \in R^2$. For such attractors it is convenient to consider partial scaling laws[9,10] in the cardinal axes defined by the locally expanding and contracting directions, respectively. Denoting the unit vectors in the expanding and contracting directions for the n th point $\mathbf{X}_n$ in the time series by $e_1(n)$ and $e_2(n)$, respectively, we consider the meaurse in a box of size $l_1 \times l_2$ around that point. Denoting this measure by $P_n(l_1,l_2)$, we define[11]

$$P_n(l_1,l_2) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} E_n(\mathbf{X}_i - \mathbf{X}_n) , \tag{3a}$$

where

$$E_n(x) = \begin{cases} 0, & |\xi| > l_1 \text{ or } |\eta| > l_2 \\ 1, & |\xi| < l_1 \text{ and } |\eta| < l_2 \end{cases} \tag{3b}$$

and

$$X - X_n = \xi e_1(n) + \eta e_2(n) . \tag{3c}$$

The fundamental scaling assumption is that $P_n(l_1,l_2)$ scales like

$$P_n(l_1,l_2) \sim l_1^{\alpha_1} l_2^{\alpha_2} . \tag{4}$$

The relation to the usual isotropic scaling $P_n(l) \sim l^{\alpha}$ is obtained when $l_1 = l_2$, leading to $\alpha = \alpha_1 + \alpha_2$.

Since the unstable periodic points are dense on the attractor, every point $\mathbf{X}_n$ is close to some point belonging to an orbit of length, say, m. Thus $\mathbf{X}_{n+m} \approx \mathbf{X}_n$, and in light of Eq. (3a),

$$P_{n+m}(l_1,l_2) \approx P_n(l_1,l_2) . \tag{5}$$

On the other hand, after m iterations, the original box $l_1 \times l_2$ has been deformed to a box $l_1 \exp(\lambda_1^{(m)}) \times l_2 \exp(\lambda_2^{(m)})$, where $\lambda_1^{(m)}$ and $\lambda_2^{(m)}$ are the Lyapunov exponents of the m-order cycle. Using the preservation of probability,[11]

$$\begin{aligned} P_{n+m}(l_1,l_2) &= P_n(l_1 e^{-\lambda_1^{(m)}}, l_2 e^{-\lambda_2^{(m)}}) \\ &= l_1^{\alpha_1} l_2^{\alpha_2} e^{-(\lambda_1^{(m)}\alpha_1 + \lambda_2^{(m)}\alpha_2)} , \end{aligned} \tag{6}$$

where Eq. (5) has been used in the last step. Using now Eqs. (4)–(6), we conclude that

$$\lambda_1^{(m)}\alpha_1 + \lambda_2^{(m)}\alpha_2 = 0 \tag{7}$$

for any cycle of any length m.

To solve the general problem of strange attractors we need another relationship between α_1 and α_2. Lacking this, we limit the discussion now to hyperbolic attractors, where the measure is absolutely continuous[10] in the expanding direction and $\alpha_1 = 1$. Such cases contain the trivial generalized baker transformation and the nontrivial Lozi attractors.[12] In these cases we find that each cycle contributes one value to spectrum of α's, with multiplicity m. This value is

$$\alpha = \alpha_1 + \alpha_2 = 1 - \lambda_1^{(m)}/\lambda_2^{(m)} . \tag{8}$$

All that remains therefore is to locate the periodic orbits, calculate their stabilities, and count how many times the value of α falls intervals of size $\Delta\alpha$.

To demonstrate these ideas we consider two cases: (i) the Baker map; (ii) the Lozi map. For the Baker map we can do everything analytically. The map is given by

$$\begin{aligned} &(x,y) \to (\mu_a x, y/\eta), \quad y < \eta \\ &(x,y) \to [1/2 + \mu_b x, (y-\eta)/(1-\eta)], \quad y > \eta \end{aligned} \tag{9}$$

with $\mu_a, \mu_b < 1/2$, $\eta < 1$. Using symbolic dynamics with $\chi(x,y) = 1$ for $y > \eta$ and $\chi(x,y) = 0$ for $y < \eta$, we find that every itinerary is allowed, and in particular there are 2^m points belonging to periodic orbits of length m. The Lyapunov exponents of an m-cycle depend only on the number of 0's and 1's in its itinerary. Denoting the number of 0's by n, we find

$$e^{\lambda_1^{(m)}} = \eta^{-n}(1-\eta)^{-(m-n)}, \quad e^{\lambda_2^{(m)}} = \mu_a^n \mu_b^{m-n} , \tag{10}$$

with an associated α value given by Eq. (8). Evidently, the number of times that each such value of α is obtained is $\binom{m}{n}$. The length scale associated with each m-cycle is precisely $e^{\lambda_2^{(m)}}$, and therefore

$$e^{-\lambda_2^{(m)} f_2(\alpha_2)} = \begin{bmatrix} m \\ n \end{bmatrix} , \tag{11}$$

where $f(\alpha) = f(1+\alpha_2) = 1 + f_2(\alpha_2)$. Equations (10) and (11) together with Eq. (8) yield an $f(\alpha)$ function which is identical to the one obtained in Ref. 1.

The Lozi map $(x',y') = L(x,y)$ is given by[12]

$$(x,y) \to (1 - a\,|x| + by, x) . \tag{12}$$

The values $a = 1.7$, $b = 0.5$ yield the strange attractor demonstrated by Lozi. We have computed an $f(\alpha)$ function for this map directly from a long time series by the usual direct methods of pair-counting (also known as Grassberger-Procaccia) algorithms. These entail calculations of the generalized dimensions[13] D_q, and evaluations of $f(\alpha)$ via the Legendre-transform relations[1] $\tau(q) = (q-1)D_q$, $\alpha = \partial\tau(q)/\partial q$, and $f(\alpha) = q\,\partial\tau(q)/\partial q - \tau(q)$. The result is shown in Fig. 1. In spite of using extremely long time series ($\sim 10^6$ per data point) and repeating attempts of improvements, we could not decide whether this function is converged and whether its termination at $f > 1$ on the right-hand branch is an artifact. As we shall see, the present treatment resolves these difficulties.

Good symbolic dynamics for the Lozi map is obtained by defining $\chi(x,y) = 0$ for $x < 0$ and $\chi(x,y) = 1$ for $x > 0$. Contrary to the previous example, not all the 2^m periodic points are allowed here and numerical schemes for finding the periodic orbits are needed. One way of finding the periodic orbits and their stabilities is obtained by extracting them straight from the chaotic time series as suggested in Ref. 14. A faster way for this case is obtained by looking at all the 2^m strings $\chi_1, \ldots, \chi_m$ and solving for the mth-order cycle the equation $L_{\chi m} \circ \cdots \circ L_{\chi 1}(x,y) = (x,y)$, where $L_\chi = (1 \mp ax + by, y)$, with the minus sign for $\chi = 1$ and the plus sign for $\chi = 0$. Iterating the solution m times one checks now whether the orbit has the correct itinerary and whether it belongs to the attracting region.[10] If it does, the orbit is kept and if it does not the solution is discarded; there is no periodic orbit with such an itinerary. Calculations based on these two methods yielded essentially identical results.

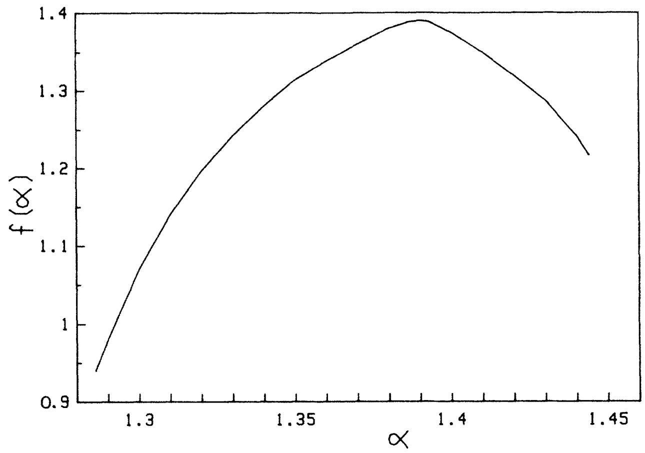

FIG. 1. The $f(\alpha)$ function for the Lozi map as obtained from direct pair-counting algorithms on a long chaotic time series. Although *a posteriori* (cf. Fig. 2), it appears to yield α values in the correct range, it is clearly not converged, especially in the wings.

Having located all the cycles up to order m, their contributions to the α spectrum are calculated from Eq. (8). Next, we bin all these α values in bins of size $\Delta\alpha$ (on the order of 0.01). The value of f is calculated as follows: The typical length scale associated with a cycle of length m is $e^{\lambda_2^{(m)}}$. Thus the length scale $l(\alpha)$ associated with any α bin is precisely this one. If only contributions from lower-order cycles are found in a given bin at this point, say, of highest order $k < m$, the appropriate length scale is estimated as

$$l(\alpha) = \exp[\lambda_2(k) m/k] .$$

Finally, we get $f(\alpha) = -\ln N(\alpha)/\ln l(\alpha)$. The results of this calculation are shown in Fig. 2.

The $f(\alpha)$ curve converges well with increasing m. It is very clear that the function in Fig. 1 is not yet converged. The lack of convergence in the wings is because the two

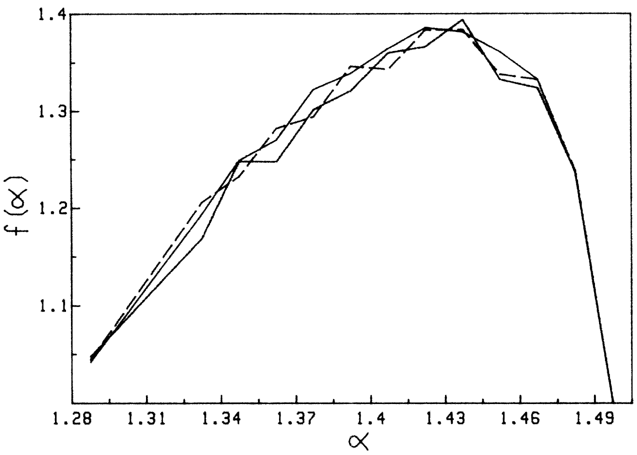

FIG. 2. $f(\alpha)$ functions obtained from all unstable cycles up to order 15 (dashed-dotted line), 16 (dashed line), and 17 (solid line). The calculation converges well, and in fact reasonable results are found also when cycles up to order 13 or 14 are used. The α-bin size is 0.015, and points were plotted in the middle of the bins and joined.

points, $\alpha_{2,\max}=0.492$ and $\alpha_{2,\min}=0.286$, are contributed by the periods of order 1 (i.e., the fixed point) and order 2, respectively, and these represent very atypical scaling exponents to this attractor. Most of the time one encounters α values for f near the maximum of $f(\alpha)$, and it is difficult to converge in the wings. The general shape of the curve in Fig. 1 seem shifted towards lower values of α. We believe that this stems from the fact that in pair-counting algorithms one does not consider nonisotropic scaling. With isotropic scaling, the one-dimensional direction dominates the scaling exponents and pulls them towards $\alpha=1$. The present approach does not suffer from the difficulties. Notice, however, that $f_2(\alpha_{2,\min})$ should be zero [(i.e., $f(\alpha_{\min})=1$)], but since $N(\alpha_{\min})=2$, it converges very slowly to zero like $\ln 2/m$.

In summary, it appears that we can understand the scaling properties of at least hyperbolic strange attractors. Nonhyperbolic cases such as the Henon map may call for further new ideas, and in particular further relationships between α_1 and α_2 (and in general between d values of α_i and d-dimensional systems) are needed.

Note added in proof. Since this paper has been submitted, the second relationship between α_1 and α_2 has been found (see Ref. 15). Thus calculations for Hénon-type maps become feasible as well

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