Amplification of sound waves in an imploding plasma shell: Exact results

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In an extended model, a rigorous proof is given for sound-wave amplifications in an imploding plasma shell. It is shown that, in the absence of a massless free surface, the boundary conditions give the exact eigenvalues which determine the asymptotic solution to the problem.

In 1978 Book[1] first proposed the possibility of sound-wave amplifications (SWA) in an imploding spherical target driven by charged particles or a laser beam. This possibility arises from the adiabatic compression of a fluid layer of the target by a slowly increasing external pressure, which is transmitted in the form of shock waves. The external pressure compresses both traveling sound waves and the fluid layer which supports the waves. The occurrence of SWA (Ref. 1) was shown in a hydrodynamic model in which self-consistency is preserved by employing Sedov's similarity hypothesis.[2]

While the model has some unique properties, it has not been possible to carry out a rigorous analysis to prove the occurrence of SWA in an imploding shell.[3] Particularly vexing is the fact that, in spite of its simplicity, the model introduces a mathematical difficulty, which prevents an exact determination of the asymptotic behavior of perturbations. The difficulty arises from the massless free surface of the model and manifests itself as an end-point singularity in the eigenvalue equation. The boundary condition on the perturbations at the massless free surface is unknown; there is no density perturbation at the massless free surface because there is no mass. Thus, it appears that the notion of SWA has no physical basis and must be discarded.

This need not be the case. It is the purpose of this paper to show the occurrence of SWA in an extended model in which the massless free surface is absent. The present model removes the massless free surface by introducing a small, finite pressure at the inner surface. At the same time it preserves the self-consistency by employing Sedov's similarity hypothesis. The asymptotic behavior of the perturbations is obtained exactly by the use of both the initial-value treatment and the boundary conditions on the compressible perturbations.

We shall examine the role played by the boundary conditions in determining the asymptotic behavior of the perturbations and study the extent to which the asymptotic behavior can be specified by a given density profile. The exact calculation of the asymptotic behavior shows that if $\gamma < \frac{5}{3}$, SWA occurs for a finite value of the mode number l and is insensitive to the aspect ratio of the shell. Perhaps more important is the fact that amplitudes of SWA in the long-wavelength regime are larger than those in the short-wavelength regime.

As in previous treatments,[1,3] we consider an imploding spherical shell that obeys the following ideal fluid equations:

$$\rho\dot{\mathbf{v}} = -\nabla p \ , \qquad (1)$$

$$\dot{\rho} + \rho\nabla\cdot\mathbf{v} = 0 \ , \qquad (2)$$

$$(p/\rho^{\gamma})^{\cdot} = 0 \ , \qquad (3)$$

with $\gamma > 1$. Here, the Lagrangian time derivatives are denoted by dots. It is assumed that the plasma fluid is initially isentropic.

We follow Book[1] but take $\beta = p_+/p_-$ as an input parameter. Here, p_+ and p_- are the initial hydrodynamic pressures at the outer and inner surfaces, respectively. The pressure at the inner surface can be thought of as the pressure of a high-temperature, low-density gas at the core and is comparable to the pressure of equilibrium blackbody radiation.[4]

The simplest self-consistent description of the shell motion, Eqs. (1)–(3), can be obtained by introducing Sedov's hypothesis of uniform self-similar motion, which in the Lagrangian representation is given as

$$R(r_0,t) = r_0 f(t) \ , \qquad (4)$$

where R is the radial position of a fluid element at time t and r_0 is its initial position. By combining Eqs. (2) and (3), together with the hypothesis of self-similar motion, we find the time-dependent density and pressure as

$$\rho(r_0,t) = \rho_0(r_0) f^{-3}, \ \ p(r_0,t) = p_0(r_0) f^{-3\gamma} \ . \qquad (5)$$

Substitution of Eq. (5) into Eq. (1) and use of Eq. (4) yield the equations for the unperturbed shell motion,

$$\ddot{f}(t) f^{3\gamma-2}(t) = -\frac{1}{\rho_0 r}\frac{d}{dr}[\rho_0^{\gamma}(r)] = -\frac{1}{t_c^2} = -1 \ , \qquad (6)$$

where the subscript in r is dropped. Here t_c is the constant of separation of variables and has the dimension of time. It is scaled to unity for convenience. If we assume the fluid is initially isentropic, a trial density profile satisfying Eq. (6) can be constructed as

$$\rho_0(r) = [\rho_+^{\gamma-1}(r^2 - r_-^2)/(r_+^2 - r_-^2) + \rho_-^{\gamma-1}(r_+^2 - r^2)/(r_+^2 - r_-^2)]^{1/(\gamma-1)} \ , \qquad (7)$$

where r_+ and r_- are the outer and inner radii of the shell; $r_- \le r \le r_+$, $\rho_0(r_+)=\rho_+$, and $\rho_0(r_-)=\rho_-$. The corresponding initial pressure profile is then given by $p_0(r)=a\rho_0^\gamma(r)$ with $p_0(r_+)=p_+$ and $p_0(r_-)=p_-$. In addition, the uniqueness of the solution of Eq. (6) demands that ρ_+ and ρ_- must satisfy

$$\left[\frac{(\gamma-1)}{2}\right](r_+^2-r_-^2)/(c_+^2-c_-^2)=t_c^2=1\ , \tag{8}$$

where $c_+^2=\gamma p_+/\rho_+=a\gamma\rho_+^{\gamma-1}$ and $c_-^2=\gamma p_-/\rho_-=a\gamma\rho_-^{\gamma-1}$. Equation (8) can be viewed as a subsidiary condition on ρ. A direct substitution of Eq. (7) into Eq. (6) with Eq. (8) shows that Eq. (7) is the unique solution of Eq. (6) with $\rho_0(r_\pm)=\rho_\pm$.

We see immediately that in contrast to the previous analysis,[1,3] the present model has well-defined, physically observable surfaces on both sides of the shell. Perhaps more importantly, we may now find appropriate boundary conditions on perturbations at the free surfaces. The point is that for an imploding (or expanding) shell with a massless free surface there is no appropriate boundary condition on perturbations for the simple reason that there is no mass to be perturbed at the free surface.

Next, the time-dependent part of Eq. (6) has the first integral

$$\dot f=[(2/\alpha)(f^{-\alpha}-1)]^{1/2}\ , \tag{9}$$

where the initial values $f(0)=1$ and $\dot f(0)=0$ have been used. Here $\alpha=3(\gamma-1)$. For $\gamma=\frac{5}{3}$, Eq. (9) can be integrated once again to give

$$f(t)=(1-t^2)^{1/2}\ . \tag{10}$$

For other values of γ, Eq. (6) must be integrated numerically and the results differ considerably from Eq. (10) for other γ values.

In order to study the stability of an imploding shell, we now introduce the Lagrangian displacement $\xi(r,t)$ and linearize the equations of motion. Straightforward but tedious algebra gives the first-order equation in ξ as

$$f^{(3\gamma-1)}\ddot{\xi}=[(\gamma-1)/2](r^2-r_1^2)\nabla\sigma+(\gamma-1)\sigma\mathbf{r}+\mathbf{r}\times\omega+(\mathbf{r}\cdot\nabla)\xi\ , \tag{11}$$

or

$$f^{(3\gamma-1)}\ddot{\xi}=\nabla\left[\left[\frac{\gamma-1}{2}\right](r^2-r_1^2)\nabla\cdot\xi+\xi\cdot\mathbf{r}\right]-\xi\cdot\nabla\mathbf{r}\ , \tag{12}$$

where

$$\sigma=\nabla\cdot\xi\ ,$$

$$\omega=\nabla\times\xi\ ,$$

$$r_1^2=(c_+^2r_-^2-c_-^2r_+^2)/(c_+^2-c_-^2)\ ,$$

and

$$\rho(\mathbf{r}+\xi)=\rho_0(r)(1-\nabla\cdot\xi)\ ,$$

which is derived from Eq. (2), has been used.

To compare Eq. (11) with that of Book, it is first useful to rewrite r_1^2 with the aid of Eq. (8), which gives

$$r_1^2=r_-^2(\beta^{(\gamma-1)/\gamma}-r_+^2/r_-^2)(\beta^{(\gamma-1)/\gamma}-1)^{-1}\ , \tag{13}$$

where $\beta=p_+/p_-$. One notices at once that in the limit $\beta\to\infty$ (i.e., $p_-\to 0$), $r_1\to r_-$ and that Eq. (11) reduces to that obtained by Book.

In order to study SWA, we first seek a solution, corresponding to compressible perturbations, to Eq. (11), subject to boundary conditions $\nabla\cdot\xi=0$ at $r=r_\pm$. It can be shown that the homogeneous equation in σ satisfies the following coupled equations:

$$\frac{(\gamma-1)}{2}(r^2-r_1^2)\left[\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{dW_n}{dr}\right]-\frac{l(l+1)}{r^2}W_n\right]+(2\gamma-1)r\frac{dW_n}{dr}+(3\gamma-2+\mu_n)W_n=0 \tag{14}$$

and

$$f^{3\gamma-1}\ddot{S}+\mu_nS(t)=0\ , \tag{15}$$

where

$$\sigma(r,t)=S(t)W_n(r)Y_{lm}(\vartheta,\varphi)$$

and μ_n is the constant of separation of variables.

In the following, we examine the analytic structure of Eq. (14) to determine whether it is possible to implement the boundary conditions $\nabla\cdot\xi=0$ at $r_\pm$, which is equivalent to $W_n(r_\pm)=0$, and thus to determine the eigenvalues μ_n. It is first helpful to transform Eq. (14) to the hypergeometric differential equation,

$$x(1-x)\frac{d^2Z_n}{dx^2}+\left[1\pm(l+\tfrac{1}{2})-\left[1\pm(l+\tfrac{1}{2})+\frac{2\gamma-1}{\gamma-1}\right]x\right]\frac{dZ_n}{dx}-\frac{1}{2(\gamma-1)}\{[1\pm(l+\tfrac{1}{2})](2\gamma-1)-\tfrac{1}{2}+\mu_n\}Z_n=0\ , \tag{16}$$

where $x=r^2/r_1^2$ and $Z_n=r^{\mp(l+1/2)+1/2}W_n(r)$. This equation has regular singular points at $x=0$, 1, and ∞, and has the general solution near the singular point $x=1$ as[5]

$$Z_n(x)=e_1\,{}_2F_1(a_1,b_1,c_1;1-x)+e_2(1-x)^{-\gamma/(\gamma-1)}\,{}_2F_1(a_2,b_2,c_2;1-x)\ . \tag{17}$$

Here,

$$a_1=(\alpha_\pm+id)/2\ ,$$

$$b_1=(\alpha_\pm-id)/2,\ c_1=(2\gamma-1)/2\ ,$$

$$\alpha_\pm=1\pm(l+\tfrac{1}{2})+\gamma/(\gamma-1)\ ,$$

and

$$d=(2\mu_n+4\gamma-3)/(\gamma-1)-(l+\tfrac{1}{2})^2 -(2\gamma-1)^2/(\gamma-1)^2 ;$$

$$a_2=(\beta_\pm+id)/2 ,$$

$$b_2=(\beta_\pm-id)/2 ,$$

$$c_2=-1/(\gamma-1) ,$$

$$\beta_\pm=\pm(l+\tfrac{1}{2})-1/(\gamma-1) ,$$

and e_1 and e_2 are arbitrary constants.

For a finite value of $\beta=p_+/p_-$, $x>1$ at $r=r_\pm$ and thus the arbitrary constants e_1 and e_2 can be determined by the boundary conditions $W(r_\pm)=0$. However, in the limit $\beta\rightarrow\infty$, $x=1$ at $r=r_-$, which is a singular point. Thus in this limit, Eq. (14) does not have a general solution which is finite at $r=r_-$.

Next we can show that Eq. (14) reduces to a Sturm-Liouville problem and that the solutions to Eq. (14) form a complete set of orthonormal functions,

$$\int_{r_-}^{r_+} dr\, r^2(r^2-r_1^2)^{\gamma/(\gamma-1)}W_n(r)W_{n'}(r)=c_n\delta_{nn'} , \qquad (18)$$

where $W_n(r)$ is the eigenfunction corresponding to an eigenvalue μ_n and c_n is the normalization constant. The eigenvalue μ_n in Eq. (14) is the number for which Eq. (14) possesses nontrivial solutions subject to the boundary conditions $W_n(r_\pm)=0$ and is given as $\mu_n=[(\gamma-1)/2]\mu_n'-(3\gamma-2)$, where

$$\mu_n'=\frac{1}{c_n}\int_{r_-}^{r_+} dr\, r^2(r^2-r_1^2)^{(2\gamma-1)/(\gamma-1)} \times\left[\left[\frac{dW_n}{dr}\right]^2+\frac{l(l+1)}{r^2}W_n^2\right] . \qquad (19)$$

An important feature of Eq. (19) is the explicit end-point singularity in the limit $\beta\rightarrow\infty$, for which W_n becomes singular at $r=r_-$. As a result, the stability analysis based on a model with a massless free surface breaks down since the eigenvalues are indeterminate. This applies also to the expanding plasma shell in which the outer surface is taken as a massless free surface.[6,7] It is notable that the peculiar physics of a massless free surface manifests itself in the form of a mathematical singularity. This explains why the massless free surface is an ill-defined surface, and it may not be a physically realizable entity.

The integral representation of the eigenvalue is sufficient for most elementary problems, but it is not suitable for W_n given in Eq. (17). However, it is clear from Eq. (16) that, for a finite β value, the eigenvalues can be determined exactly by solving Eq. (16) numerically as an eigenvalue problem. Figure 1 shows that the smallest eigenvalue increases with the mode number. This result is quite different from that of the approximate calculation for a shell with a massless free surface.[3]

For a given μ_n, the solution of Eq. (15) can be expressed in terms of the hypergeometric functions [see Eqs. (20) and (21) of Ref. 1] and the asymptotic values are given by

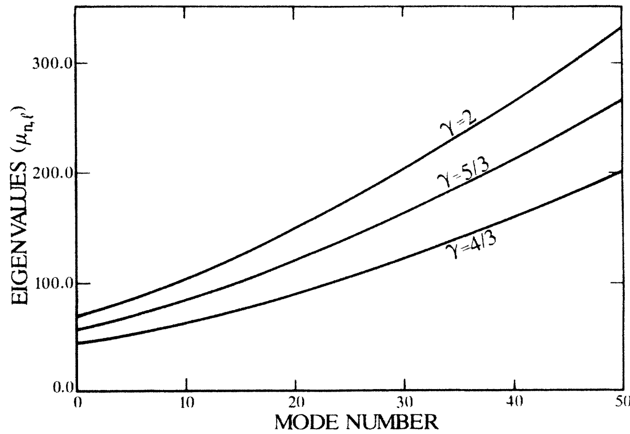

FIG. 1. Smallest eigenvalues μ for $\gamma=\frac{4}{3}$, $\frac{5}{3}$, and 2 plotted against the mode number. Here $\beta=1.0\times10^2$ and the aspect ratio is 10.0.

$$\lim_{t\rightarrow1}[\mathcal{F}(l,\gamma,t)/f(t)]\simeq a_{l,\mu_n}f^{(\alpha-2+id)/4}+\text{c.c.} , \qquad (20)$$

$$\lim_{t\rightarrow1}[\mathcal{G}(l,\gamma,t)/f(t)]\simeq b_{l,\mu_n}f^{(\alpha-2+id)/4}+\text{c.c.} , \qquad (21)$$

where $\mathcal{F}(t)$ and $\mathcal{G}(t)$ are the linearly independent solutions of Eq. (15). Here

$$a_{l,\mu_n}=\Gamma(\tfrac{1}{2})\Gamma[-id/(2\alpha)] \times\{\Gamma[\tfrac{1}{4}+(2-id)/(4\alpha)]\Gamma[\tfrac{1}{4}-(2+id)/(4\alpha)]\}^{-1} , \qquad (22)$$

$$b_{l,\mu_n}=i\Gamma(\tfrac{3}{2})\Gamma[-id/(2\alpha)] \times\{\Gamma[\tfrac{3}{4}+(2-id)/(4\alpha)]\Gamma[\tfrac{3}{4}-(2+id)/(4\alpha)]\}^{-1} , \qquad (23)$$

where $d=[8\alpha\mu_n-(\alpha+2)^2]^{1/2}$ and $\alpha=3(\gamma-1)$.

It is clear from Eqs. (20)–(23) that in the limit $t\rightarrow t_c$ (i.e., $f(t)\rightarrow0$), the limits diverge for all values of l if $\alpha<2$ (or equivalently, $\gamma<\frac{5}{3}$). Moreover, Fig. 2 shows

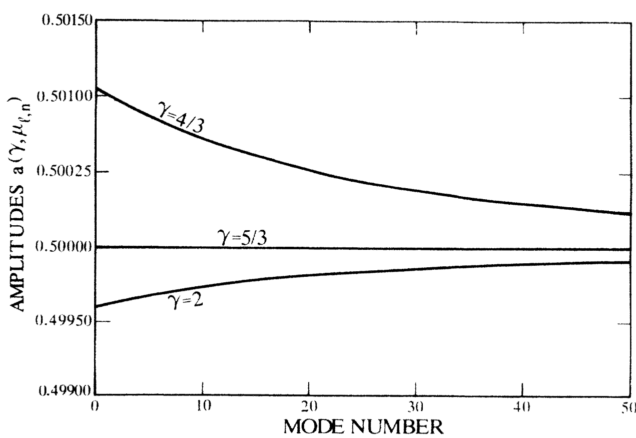

FIG. 2. Absolute values of $a(\gamma,\mu_{l,n})$ for $\gamma=\frac{4}{3}$, $\frac{5}{3}$, and 2 plotted against the mode number l. Here $\mu_{l,n}$ is the smallest eigenvalue for a given l with $\beta=1.0\times10^2$ and the aspect ratio is 10.0.

that the amplitudes of amplified sound waves in the long-wavelength regime are larger than those in the short-wavelength regime. This implies that the occurrence of SWA in the long-wavelength regime will persist even in the presence of viscosity or other dissipative processes. However, this prescription for determining the stability criteria of traveling sound waves begs the question. Namely, the limits in Eqs. (20) and (21) should diverge for finite values of $\mathcal{F}(t)$ and $\mathcal{G}(t)$, the time-dependent parts of $\sigma(\mathbf{r},t)$, since $f(t)\rightarrow 0$ as $t\rightarrow t_c$. To put it another way, the limits do not necessarily suggest the occurrence of SWA.

Book argued in favor of these limits as the stability criteria by demonstrating that a model-independent estimate, based on standing sound waves, is in agreement with the limits. However, the point of this discussion is that a standing sound wave cannot exist because of a pressure gradient across the fluid layer. In addition, there still remains the question of whether SWA can disrupt a uniform implosion.

To clarify the behavior of sound waves in an imploding plasma shell, we have carried out numerical simulations using a one-dimensional Lagrangian code[8] which is written specifically for this purpose. It is found that SWA occurs for a reasonable value of γ (i.e., $1<\gamma<2$), which supports the essential correctness of the stability criteria, but that it does not disrupt a uniform implosion. However, we have observed an instability driven by SWA at the inner surface. This instability can provide a decisive test for the occurrence of SWA, because it can, in principle, be observed in experiments.

The conclusion is that, in an imploding spherical shell, SWA can occur in the long-wavelength regime provided that the external pressure matches with the prescribed time-dependent pressure profile given in the present model.

Finally, we note that for $\beta > 1.0\times 10^4$, it was not possible to obtain exact eigenvalues. This can be understood by noting that the contribution from the singular point is no longer negligible for this value of β [i.e., $r_1\rightarrow r_-$, see Eq. (13)].

We conclude with a remark about the analysis of Book.[1] Although his model contains a massless free surface, his conclusion on SWA is in agreement with ours.

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