Quasilinear theory of the ordinary-mode electron-cyclotron resonance in plasmas

V. Arunasalam, P. C. Efthimion, J. C. Hosea, H. Hsuan, and G. Taylor
Plasma Physics Laboratory, Princeton University, P.O. Box 451, Princeton, New Jersey 08544-0451
(Received 3 June 1987; revised manuscript received 5 October 1987)

A coupled set of equations, one describing the time evolution of the ordinary-mode wave energy and the other describing the time evolution of the electron distribution function, is presented. The wave damping is mainly determined by $T_{\parallel}$, while the radiative equilibrium is mainly an equipartition with $T_{\perp}$. The time rate of change of $T_{\perp}$, $T_{\parallel}$, particle density (N_0), and current density ($J_{\parallel}$) are examined for finite-$k_{\parallel}$ electron-cyclotron-resonance heating of plasmas. The effects of collisional broadening and collisional damping are also examined. For blackbody absorbing conditions it is shown that the increase of $T_{\perp}$ with time in electron-cyclotron-resonance heating is exponential and not linear. From the quasilinear theory it is found that the Ohkawa steady-state current drive efficiency criterion is really a consequence of the conservation laws of energy, momentum, particle density, and the collisional relaxation of the current density.

I. INTRODUCTION

Recently, there has been renewed interest in electron-cyclotron-resonance heating (ECRH) of plasmas both in connection with the need for supplementary heating in magnetically confined devices and with ionospheric heating experiments. In the ordinary (O)-mode ECRH, the electric field $\mathbf{E}$ of the heating wave is parallel to the confining magnetic field $\mathbf{B}=B\hat{\mathbf{i}}_z$, and the heating is a consequence of the finite size of the electron Larmor orbits (ρ). Hence, intuitive expectations would suggest that the O-mode ECRH would only lead to a heating in the parallel direction (i.e., to an increase in the parallel electron temperature $T_{\parallel}$ since $\mathbf{E}\parallel\mathbf{B}$) with a heating rate that is determined mainly by $T_{\perp}$ (since this heating is due to ρ). However, in this paper we will show that the predictions of the quasilinear theory are in sharp contrast to the intuitive expectations. In particular, we will show that the perpendicular heating rate ($\partial T_{\perp}/\partial t$) is very much larger than the parallel heating rate ($\partial T_{\parallel}/\partial t$) and the wave damping is determined mainly by $T_{\parallel}$, while the radiative equilibrium is an equipartition mainly with $T_{\perp}$. Finally, we will obtain closed-form expressions for the time rate of change of $T_{\perp}$, $T_{\parallel}$, particle density (N_0), and current density ($J_{\parallel}$) for finite-$k_{\parallel}$ O-mode ECRH. The paper is organized as follows: In Sec. II we present the theory for a collisionless plasma. In Sec. III we consider the effects of collisions and the steady-state rf current drive and in Sec. IV we state our conclusions.

II. THEORY FOR A COLLISIONLESS PLASMA

We will begin by assuming Landau's quantized particle motion of the electrons in a uniform magnetic field[1,2] and we shall use the notations and the results found in Ref. 1. For O mode the transition probabilities of absorption j_A and emission j_E of Eq. (8) of Ref. 1 become

$$\left.\begin{matrix} j_A/N_{ks} \\ j_E/(N_{ks}+1) \end{matrix}\right\} = \left\{\begin{matrix} n+1 \\ n \end{matrix}\right\} \times \left[\frac{2\pi^2 q^2 k_\perp^2}{L^3 \mu\omega\omega_b}\right](v_z+\hbar k_z/2\mu)^2 \delta_{v_z',v_z+\hbar k_z/\mu} \times \delta(\omega-\omega_b-k_z(v_z+\hbar k_z/2\mu)) \begin{cases} \text{for } (n,v_z)\leftrightarrow(n+1,v_z') \\ \text{for } (n,v_z)\leftrightarrow(n-1,v_z') \end{cases} , \tag{1}$$

where the factor $\{{}^{n+1}_{n}\}$ means $(n+1)$ for transitions between the quantum states $|n,v_z\rangle$ and $|n+1,v_z'\rangle$, and n for transitions between the states $|n,v_z\rangle$ and $|n-1,v_z'\rangle$. Here $\omega_p^2=(4\pi N_0 q^2/\mu)$, $\omega_b=(qB/\mu c)$, q and μ are the charge and mass of the electron, respectively, and the electrons are contained in a box of volume L^3. In Eq. (1) the Kronecker δ indicates the conservation of z momentum and the Dirac δ function indicates the conservation of energy. On making use of Eq. (1), and Eq. (29) of Ref. 1, the classical limit of Eq. (13) of Ref. 1 becomes

$$\left[\frac{\partial\varepsilon_k}{\partial t}\right] = \int_{-\infty}^{\infty} dv_z \int_0^{\infty} dE_\perp \left[\frac{\pi k_\perp^2 \omega_p^2}{2\omega_b^2}\right](v_z^2 E_\perp)\left[\left[\frac{\omega_b}{\omega}\varepsilon_k\right][\delta(\omega-\omega_b-k_z v_z)QF(E_\perp,v_z)]+\delta(\omega-\omega_b-k_z v_z)F(E_\perp,v_z)\right] , \tag{2}$$

where the wave energy $\varepsilon_k=N_{ks}\hbar\omega$, $E_\perp=\mu v_\perp^2/2$, and the linear differential operator $Q=[(\partial/\partial E_\perp)+(k_z/\mu\omega_b)(\partial/\partial v_z)]$. Equation (13) of Ref. 1 is the energy balance equation and Eq. (29) of Ref. 1 gives the prescription for taking the classical limit of the quantum variables. Similarly, by applying the principle of detailed balance for the transition probabilities per unit volume of emission and absorption between the pairs of quantum states, ($|n,v_z\rangle$; $|n+1,v_z'\rangle$) and ($|n,v_z\rangle$; $|n-1,v_z''\rangle$), and then taking the classical limit, after a certain amount of algebra one can show[3] that the time evolution of the electron distribution function $F(E_\perp,v_z)$ may be written

$$\partial F(E_\perp,v_z)/\partial t = \sum_{\mathbf{k}} (2\pi^2 q^2 k_\perp^2 v_z^2/L^3\mu\omega\omega_b)\big((\omega_b\varepsilon_k/\omega)\{(1+E_\perp Q)[\delta(\omega-\omega_b-k_z v_z)QF(E_\perp,v_z)]\} + \{(1+E_\perp Q)[\delta(\omega-\omega_b-k_z v_z)F(E_\perp,v_z)]\}\big) . \quad (3)$$

For a collisionless plasma the set of Eqs. (2) and (3) is the coupled pair of classical quasilinear equations, one describing the time evolution of the ordinary-mode wave energy and the other describing the time evolution of the electron distribution function. In Eq. (2) the first term which is proportional to ε_k yields the growth or damping rate of the wave and is a consequence of a balance between induced emission and absorption; while the second term which is independent of ε_k is a consequence of spontaneous emission. It is seen that Eq. (3) is a Fokker-Planck equation whose first and second terms account for "diffusion" and "dynamical friction," respectively. The velocity-space diffusion term is proportional to the wave energy ε_k and is again a consequence of a balance between stimulated emission and absorption. On the other hand, the dynamical friction term [i.e., the term which is independent of ε_k on the right-hand side of Eq. (3)] is a consequence of spontaneous emission.[4]

If we now assume that $F(E_\perp,v_z)$ is a Maxwell-Boltzmann distribution function, and $\varepsilon_k=0$ at a time $t=0$, then the solution of Eq. (2) is $\varepsilon_k=\varepsilon_k^{(0)}[1-\exp(-2\gamma_k t)]$, where

$$\varepsilon_k^{(0)}=(\omega k_B T_\perp/\omega_b)/[1+(T_\perp/T_\parallel)(\omega-\omega_b)/\omega_b]\approx k_B T_\perp$$

for $\omega\approx\omega_b$, the damping rate

$$2\gamma_k\approx(\pi^{1/2}D\omega\omega_p^2 v_R^2/2c^2\omega_b k_z v_\parallel) \times\{1+(T_\perp/T_\parallel)[(\omega-\omega_b)/\omega_b]\}\exp[-v_R^2/v_\parallel^2] , \quad (4)$$

$v_R=(\omega-\omega_b)/k_z$, $v_\parallel^2=(2k_B T_\parallel/\mu)$, and $D\approx(c^2k_\perp^2/\omega^2)\approx(1-\omega_p^2/\omega^2)$ for $k_\perp \gg k_\parallel$. In the limit $k_z\to 0$,

$$2\gamma_k\approx(k_B T_\parallel/\mu)(\pi D\omega_p^2/2c^2)\delta(\omega-\omega_b) .$$

It is therefore interesting to find that although the wave damping $2\gamma_k$ is determined mainly by $T_\parallel$, the radiative equilibrium between the electrons and the O-mode radiation field is an equipartition mainly with $T_\perp$. At equilibrium the energy per mode $\varepsilon_k^{(0)}\approx k_B T_\perp$ as a result of the equipartition theorem.[5] That is, the balance between induced emission and absorption is determined mainly by $T_\parallel$, while the global balance between spontaneous emission, induced emission, and absorption is determined mainly by $T_\perp$. This result is somewhat in contrast to intuitive expectations.

We now wish to calculate the time rate of change of $T_\perp$, $T_\parallel$, particle density (N_0), and current density ($J_\parallel$) for finite $k_\parallel$ ECRH. These are given by

$$\frac{\partial}{\partial t}\begin{Bmatrix}\langle E_\perp\rangle\\ \langle E_\parallel\rangle\\ N_0\\ J_\parallel\end{Bmatrix}=\frac{\partial}{\partial t}\begin{Bmatrix}k_B T_\perp\\ \frac{1}{2}k_B T_\parallel\\ N_0\\ J_\parallel\end{Bmatrix}=\int_{-\infty}^{\infty}dv_z\int_0^{\infty}dE_\perp\times\begin{Bmatrix}E_\perp\\ \frac{1}{2}\mu v_z^2\\ N_0\\ N_0 q v_z\end{Bmatrix}\times\frac{\partial}{\partial t}F(E_\perp,v_z) . \quad (5)$$

Here the angular brackets refer to an average, and we have written $\langle E_\perp\rangle=\langle\mu v_\perp^2/2\rangle=\langle\mu(v_x^2+v_y^2)/2\rangle=k_B T_\perp$ and $\langle E_\parallel\rangle=\langle\mu v_z^2/2\rangle=k_B T_\parallel/2$ since the perpendicular motion has two degrees of freedom, while the parallel motion has only one; the average energy per degree of freedom is thus $k_B T/2$ as a result of the equipartition theorem. On making use of Eq. (3), Eq. (5) yields

$$\frac{\partial}{\partial t}\begin{Bmatrix}k_B T_\perp\\ \frac{1}{2}k_B T_\parallel\\ N_0\\ J_\parallel\end{Bmatrix}=\sum_{\mathbf{k}}\begin{Bmatrix}(v_\parallel^2/\omega_b N_0)[(\omega_b v_R/k_z v_\parallel^2)+2T_\perp/T_\parallel]\\ (2v_\parallel^2/\omega_b N_0)v_R^2/v_\parallel^2\\ 2\mu\omega_b\\ (3qv_\parallel/\mu\omega_b)v_R/v_\parallel\end{Bmatrix}\times\frac{Y_k}{L^3} , \quad (6)$$

where the terms in the curly braces on both sides of Eq. (6) have the similar one-to-one correspondence as in Eq. (5),

$$Y_k=\left\{\frac{\pi k_\perp^2\omega_p^2 v_R}{2\omega^2}\right\}\left[\varepsilon_k\left\{F_\parallel(v_z)-\frac{k_B T_\perp k_z}{\mu\omega_b}\frac{\partial F_\parallel(v_z)}{\partial v_z}\right\}-\frac{\omega k_B T_\perp}{\omega_b}F_\parallel(v_z)\right]_{v_z=v_R} , \quad (7)$$

and $F_{\parallel}(v_z)=\int_0^\infty dE_\perp F(E_\perp, v_z)$. The somewhat lengthy and tedious algebra leading to Eqs. (6) and (7) only makes use of integration by parts over $dE_\perp$ and over dv_z; and the constant terms resulting from these integrations vanish at $E_\perp=0$ and $E_\perp=\infty$ and at $v_z=\pm\infty$. Similarly Eq. (2) can be rewritten as

$$\partial \varepsilon_k/\partial t = -(v_{\parallel}^2/\omega_b)(\omega v_R/k_z v_{\parallel}^2) Y_k \ . \tag{8}$$

If we now assume that $F_{\parallel}(v_z)$ is a Maxwell-Boltzmann distribution function so that

$$F_{\parallel}(v_z)=(1/\pi^{1/2}v_{\parallel})\exp(-v_z^2/v_{\parallel}^2) \ ,$$

then $\partial F_{\parallel}/\partial v_z = -(2v_z/v_{\parallel})F_{\parallel}$ and Y_k of Eq. (7) become

$$Y_k \approx (\pi D\omega_p^2 v_R/2c^2)\{ \varepsilon_k[1+(T_\perp/T_{\parallel})(\omega-\omega_b)/\omega_b] - (\omega k_B T_\perp/\omega_b)\} F_{\parallel}(v_z=v_R) \ , \tag{9}$$

where $D\approx(c^2k_\perp^2/\omega^2)\approx(1-\omega_p^2/\omega^2)$ for $k_\perp \gg k_{\parallel}$. In Eqs. (7) and (9) the term that is proportional to ε_k is a consequence of a balance between induced emission and absorption, while the term that is independent of ε_k is a consequence of spontaneous emission. It may be noted that $Y_k=0$ of Eq. (9) yields the previously discussed radiative equilibrium solution $\varepsilon_k^{(0)}\approx k_B T_\perp$ for $\omega\approx\omega_b$.

In the usual formulation of the classical quasilinear plasma kinetic theory (CQPKT) (Refs. 6 and 7) based on the Vlasov-Maxwell equations, it is extremely difficult to obtain the dynamical friction term of Eq. (3). That is, in CQPKT one assumes that the wave energy ε_k is much greater than $k_B T_\perp$; and consequently, one can neglect the dynamical friction term in Eq. (3) and the spontaneous emission terms of Eqs. (5)–(9). Then Y_k of Eq. (9) becomes

$$Y_k' = A_k(v_R/v_{\parallel}\pi^{1/2})[1+(T_\perp/T_{\parallel})(\omega-\omega_b)/\omega_b] \times\exp(-v_R^2/v_{\parallel}^2) \ , \tag{10}$$

where $A_k=(\varepsilon_k \pi D\omega_p^2/2c^2)$. In Fig. 1 we have plotted the dimensionless functions

$$[(\omega_b v_R/k_z v_{\parallel}^2)+(2T_\perp/T_{\parallel})](Y_k'/A_k)=Z_k(T_\perp) \ ,$$

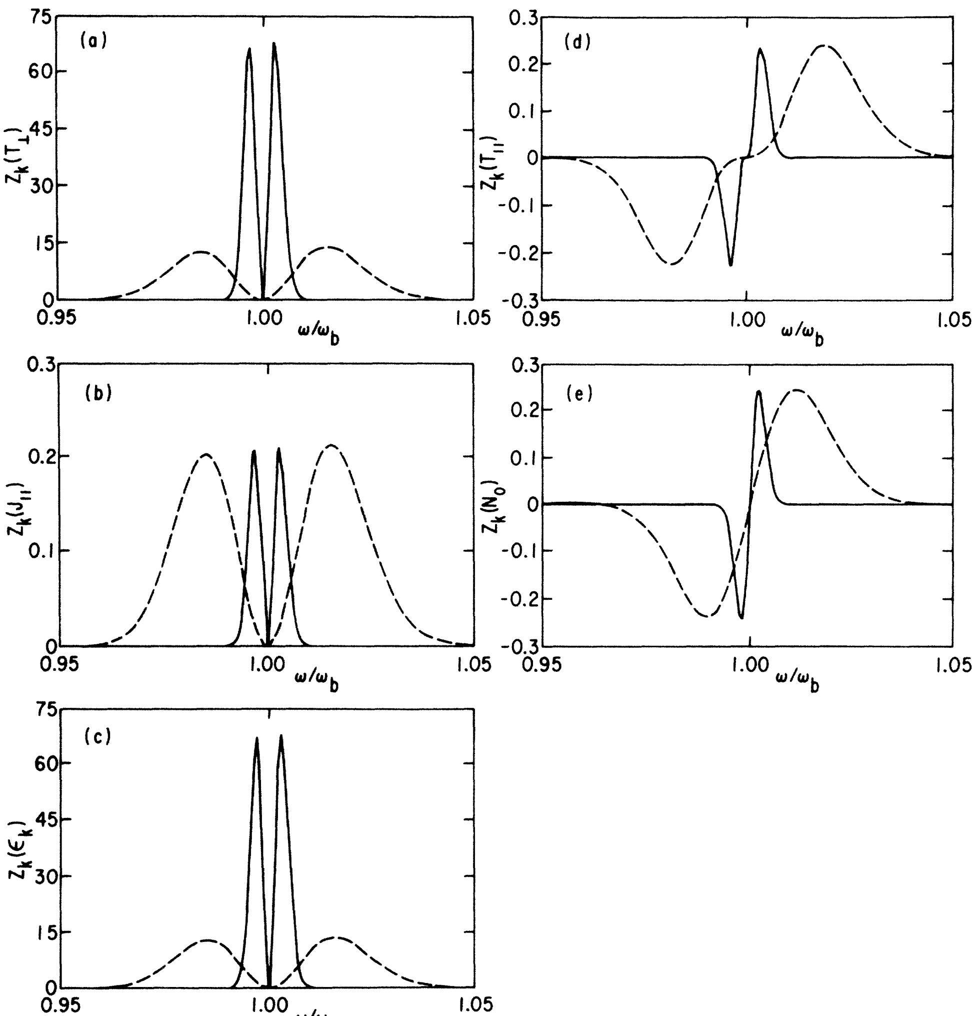

FIG. 1. Plots of the dimensionless functions $Z_k(T_\perp)$, $Z_k(T_{\parallel})$, $Z_k(N_0)$, $Z_k(J_{\parallel})$, and $Z_k(\varepsilon_k)$ as a function of ω/ω_b for two different angles of propagation: ——, $\theta=2°$; — — —, $\theta=10°$. (a) $Z_k(T_\perp)$, (b) $Z_k(J_{\parallel})$, (c) $Z_k(\varepsilon_k)$, (d) $Z_k(T_{\parallel})$, and (e) $Z_k(N_0)$. Conditions are $k_B T_\perp=3$ keV, $k_B T_{\parallel}=2$ keV, and $\omega_b/2\pi=9\times10^{10}$ Hz.

$$(v_R^2/v_\parallel^2)(Y_k'/A_k)=Z_k(T_\parallel)\ ,$$

$$(Y_k'/A_k)=Z_k(N_0)\ ,$$

$$(v_R/v_\parallel)(Y_k'/A_k)=Z_k(J_\parallel)\ ,$$

and

$$(\omega v_R/k_z v_\parallel^2)(Y_k'/A_k)=Z_k(\varepsilon_k)$$

for two different angles of propagation. In this figure these functions $Z_k(T_\perp)$, $Z_k(T_\parallel)$, $Z_k(N_0)$, $Z_k(J_\parallel)$, and $Z_k(\varepsilon_k)$ illustrate the behavior of $\partial T_\perp/\partial t$, $\partial T_\parallel/\partial t$, $\partial N_0/\partial t$, $\partial J_\parallel/\partial t$, and $\partial \varepsilon_k/\partial t$, respectively, as a function of ω/ω_b for a given value of **k** [see Eqs. (6) and (8)]. It is interesting to note from this figure that the Doppler effect due to the finite value of k_z does not yield the usually expected Gaussian broadening as found in the literature.[5] However, it leads to a somewhat symmetric Doppler splitting for $\partial T_\perp/\partial t$, $\partial J_\parallel/\partial t$, and $\partial\varepsilon_k/\partial t$ as a function of ω/ω_b, and to an antisymmetric splitting for $\partial T_\parallel/\partial t$ and $\partial N_0/\partial t$ as a function of ω/ω_b. For tokamak plasmas the confining magnetic field $B\propto R^{-1}$, where R is the major radius of the torus. For these plasmas the Doppler splitting in frequency will map over as a splitting in major radius R around the resonance zone $R\approx R_b$, where $\omega\approx\omega_b$.

In Eq. (6) the sum over **k** goes over essentially[3] as an integral over $d\omega$. Thus for radiative equilibrium the antisymmetric functions in Eq. (6) will integrate to zero. That is, for the antisymmetric terms of Eq. (6), whatever the particles gained from or lost to the waves for $\omega<\omega_b$ are again lost to or gained from the waves, respectively, for $\omega>\omega_b$. Consequently, from Eqs. (6) and (8) one can show that

$$\sum_{\mathbf{k}}(\partial\varepsilon_k/\partial t)=-[\partial(L^3N_0k_BT_\perp)/\partial t]\ ,$$

$$\sum_{\mathbf{k}}(k_z/\omega)(\partial\varepsilon_k/\partial t)$$
$$=-[(L^3\mu/q)(\partial J_\parallel/\partial t)-(L^3\mu v_R)(\partial N_0/\partial t)]\ , \tag{11}$$

$$(\partial N_0/\partial t)=0\ ,$$

$$[\partial(L^3N_0k_BT_\parallel)/\partial t]=0\ .$$

The first three expressions in Eq. (11) represents the conservation laws of energy, momentum, and particle number density, respectively. The last relation represents the fact that an equilibrium distribution of waves does not spend any of its energy to increase the particle's parallel temperature. This clearly shows that the coupled set of Eqs. (2) and (3) is self-consistent, and the heating is primarily in the perpendicular direction. Indeed from Eq. (6) it is relatively easy to see[8] that $(\partial T_\perp/\partial t)/(\partial T_\parallel/\partial t)\propto[\omega_b/(\omega-\omega_b)]\gg1$. That is, the O-mode ECRH would lead to a heating mainly in the perpendicular direction. This result is consistent with the conservation of parallel canonical momentum.[9] Since the parallel z coordinate is cyclic or ignorable[10] in the Hamiltonian $H=(\mathbf{p}-q\,\mathbf{A}/c)^2/2\mu$ (where $\mathbf{A}=\mathbf{A}^{(0)}+\mathbf{A}^{(1)}$, $\mathbf{A}^{(0)}$ is the vector potential of the applied magnetic field, and $\mathbf{A}^{(1)}$ is the vector potential of the O-mode electromagnetic field), the parallel canonical momentum $p_z=\mu v_z+qA_z(x_\perp,t)/c$ is a constant of motion. Since $A_z^{(0)}$ is a constant and $A_z^{(1)}$ is an oscillating bound function, v_z must be oscillating and bound as well indicating that on the average electrons do not gain parallel energy. Energy is fed to the particular motion by the $qv_zB_\perp^{(1)}/c$ force, where $B_\perp^{(1)}$ is the wave magnetic field. This indicates that the perpendicular heating rate is mainly determined by $T_\parallel$. A similar reasoning is given in Ref. 9 assuming that the parallel canonical momentum is conserved. Here we justify this assumption by pointing out that the parallel coordinate is cyclic in the total Hamiltonian.

Let us now examine the limiting case of Eq. (6) for exactly perpendicular propagation. In the limit $k_z\to0$,

$$Z_k(T_\perp)=[(\omega_bv_R/k_zv_\parallel^2)+(2T_\perp/T_\parallel)](Y_k'/A_k)$$
$$\to(\omega_b/2)\delta(\omega-\omega_b)\ ,$$

$$Z_k(T_\parallel)=(v_R^2/v_\parallel^2)(Y_k'/A_k)\to0\ ,$$

$$Z_k(N_0)=(Y_k'/A_k)\to0,\ Z_k(J_\parallel)=(v_R/v_\parallel)(Y_k'/A_k)\to0\ ,$$

and

$$Z_k(\varepsilon_k)=(\omega v_R/k_zv_\parallel^2)(Y_k'/A_k)\to(\omega/2)\delta(\omega-\omega_b)\ .$$

Here we have used the fact that the Dirac δ function satisfies the relations $x\delta(x)=0$ and $x\delta'(x)=-\delta(x)$. This limiting case illustrates the reason for grouping the terms in the form shown in Eqs. (6) and (8). It may again be noted that $Z_k(T_\perp)=-Z_k(\varepsilon_k)$ is a reflection of the conservation laws of energy when $k_z\to0$. In the conventional collisionless hot-plasma quasilinear theory the only mechanism responsible for broadening and/or splitting the resonances is the longitudinal Doppler effect. Since for $k_z\to0$ the longitudinal Doppler shift vanishes, it is apparent why this theory predicts a δ-function resonance. A more exact theory should incorporate other mechanisms[11] that may be responsible for resonance broadening and/or splitting.

III. THE EFFECTS OF COLLISIONS AND STEADY-STATE CURRENT DRIVE

Thus far we have considered a collisionless plasma. That is, in Eqs. (2) and (3) we have neglected the effects of electron-ion collisions. Collisions basically have a twofold effect on these equations. First, a collision can give rise to a broadening of the cyclotron resonance. If one wishes to take this into account then one should replace $\delta(\omega-\omega_b-k_zv_z)$ by

$$(\nu/2\pi)/[(\omega-\omega_b-k_zv_z)^2+\nu^2/2]\ ,$$

where $\nu(v_\perp,v_z)$ is the collision frequency. Second, in Eq. (2) it can give rise to a collisional damping which must be added to the right-hand side of this equation. One can show that for O mode with $\mathbf{E}\parallel\mathbf{B}$, the collisional damping may be written

$$\left[\frac{\partial \varepsilon_k}{\partial t}\right]_\nu \approx \int_{-\infty}^{\infty} dv_z \int_0^{\infty} 2\pi v_\perp dv_\perp \left[\frac{\omega_p^2 \varepsilon_k \nu(v_\perp, v_z)}{(\omega - k_z v_z)^2 + \nu^2(v_\perp, v_z)}\right] F(v_\perp, v_z) \ . \tag{12}$$

Neglecting $k_z v_z$ in Eq. (12) and for $\omega \approx \omega_b >> \bar{\nu}$, the approximate form of Eq. (12) may be written

$$\left[\frac{\partial \varepsilon_k}{\partial t}\right]_\nu \approx \left[\frac{\omega_p^2}{\omega_b^2}\right] \bar{\nu} \varepsilon_k \ , \tag{13}$$

where $\bar{\nu}$ is the average value of the collision frequency. It should be noted that this term is proportional to ε_k and hence it has to be compared with the damping term due to absorption minus induced emission in Eq. (2). In the limit $k_z \to 0$, the approximate form of this cyclotron damping may be written

$$\begin{aligned} 2\gamma_k \varepsilon_k &\approx (1 - \omega_p^2/\omega^2)\left[1 + \frac{T_\perp}{T_\parallel}\left[\frac{\omega - \omega_b}{\omega_b}\right]\right] \\ &\quad \times \left[\frac{(\pi \omega_p^2 v_\parallel^2 \varepsilon_k / 4c^2)(\bar{\nu}/2\pi)}{(\omega - \omega_b)^2 + \bar{\nu}^2/2}\right] \\ &\approx (1 - \omega_p^2/\omega^2)(\omega_p^2 v_\parallel^2 / 4c^2 \bar{\nu}) \varepsilon_k \quad \text{for } \omega \approx \omega_b \ . \end{aligned} \tag{14}$$

For $\omega \approx \omega_b >> \omega_p$, it is seen from Eqs. (13) and (14) that when $v_\parallel / c >> 2\bar{\nu}/\omega_b$, $2\gamma_k \varepsilon_k >> (\partial \varepsilon_k / \partial t)_\nu$. That is, when $v_\parallel / c >> \bar{\nu}/\omega_b$ one can neglect the collisional damping in Eq. (2). For the conditions of Fig. 1, $v_\parallel / c \approx 2 \times 2 \times 10^3 / (5 \times 10^5) \approx 8 \times 10^{-3}$ and $\bar{\nu}/\omega_b \approx (6 \times 10^3 / (2\pi \times 9 \times 10^{10}) \approx 10^{-8}$ for an electron density $n = 3 \times 10^{13}$ cm^{-3}. Hence, for normal tokamak discharges one can neglect the collision damping contribution to Eq. (2) and the radiative equilibrium is indeed a balance between absorption, induced emission, and spontaneous emission (i.e., a balance between the Einstein A and B coefficients).

Similarly, one has to add the rate of change of the electron distribution function due to collisions $[\partial F(E_\perp, v_z)/\partial t]_\nu$ to the right-hand side of Eq. (3). However, this term is independent of ε_k and consequently does not affect the heating rates of Fig. 1. Usually, this term is larger than the dynamical friction term of Eq. (3). If one wishes to solve the much more complex problem of the final steady-state values $T_\perp$, $T_\parallel$, N_0, and $J_\parallel$ in rf plasma heating, one must balance this term against the velocity-space quasilinear diffusion term of Eq. (3). In this paper we have not attempted to solve for this steady state. This is an extremely difficult problem to solve analytically and has to be done numerically with the aid of computers. For numerical solutions the needed $[\partial F(v_\perp, v_z)/\partial t]_\nu$ are those of Chandrasekhar and Spitzer.[12]

We now wish to examine the efficiency of rf-driven plasma current in O-mode ECRH. This efficiency, by which the practicality of a reactor incorporating rf-driven steady-state current may be assessed, is $J_\parallel / P$, the amount of parallel current generated per power dissipated. In ECRH this steady-state current $J_\parallel$ is produced by the cyclotron damping of the waves traveling in only one direction, either parallel (or antiparallel) to the confining magnetic field. The waves have net parallel (or antiparallel) momentum, which, upon being absorbed by electrons traveling with the wave parallel phase velocity, exerts a force that drives the electric current. Usually the Ohmic current decays because of the momentum loss between electrons and ions by Coulomb collisions[12] $\nu = \omega_p^4 \ln\Lambda / 2\pi N_0 v^3$. If opposite forces are applied to electrons and ions separately and their strengths are equal to the momentum loss rate, the current will be maintained. The required force per unit volume is $qN_0 \eta J_\parallel$, where $\eta = (\mu\nu / N_0 q^2) = (4\pi\nu / \omega_p^2)$ is the plasmas resistivity. Hence the wave power P required to maintain a steady-state current $J_\parallel$ is given by

$$P = (\omega / k_z) q N_0 \eta J_\parallel = (\omega / k_z)(\nu\mu / q) J_\parallel \ . \tag{15}$$

This is the Ohkawa[13] steady-state current drive efficiency criterion.

In quasilinear theory this steady-state current $J_\parallel$ is a consequence of a balance between the first velocity moments of $(\partial F / \partial t)$ of Eq. (3) due to the wave particle interaction and the Fokker-Planck collision operator $(\partial F / \partial t)_\nu$. Since the collisional relaxation of the plasma current $(\partial J_\parallel / \partial t)_\nu = -\nu J_\parallel$, it is seen from Eq. (11) that for O-mode rf heating

$$\begin{aligned} \sum_{\mathbf{k}} (k_z/\omega)(\partial \varepsilon_k / \partial t) &= -(k_z/\omega)[\partial(L^3 N_0 k_B T_\perp)/\partial t] \\ &= (L^3 \mu / q) \nu J_\parallel \ . \end{aligned} \tag{16}$$

Now the rf power dissipated per unit volume is given by

$$P = -\frac{1}{L^3} \sum_{\mathbf{k}} (\partial \varepsilon_k / \partial t) = \partial(N_0 k_B T_\perp)/\partial t \ , \tag{17}$$

where we have made use of the first relation of Eq. (11). Thus from Eqs. (16) and (17) we get $P = (\omega / k_z)(\nu\mu / q) J_\parallel$, which is precisely the Ohkawa steady-state current drive efficiency criterion of Eq. (15). This exact agreement of our results with that of Ohkawa not only shows the self-consistency of our coupled pair of quasilinear equations (2) and (3) but also shows for the first time that this current drive efficiency criterion is really a consequence of the conservation laws of energy, momentum, particle density of Eq. (11), and the collisional relaxation of the current density.

We now wish to show that if the absorbing resonant plasma layer is optically thick to the incident rf power, then the initial increase in the perpendicular temperature in the absorbing volume is exponential and not linear in time. Neglecting the spontaneous emission term in Eq. (2), $(\partial \varepsilon_k / \partial t) = -2\gamma_k \varepsilon_k$, where the damping rate $2\gamma_k$ is given by Eq. (4). Also

$$\sum_{\mathbf{k}} [\ ] \to \left[\frac{L}{2\pi c}\right]^3 \int d\Omega_k \int d\omega \, D^{3/2} \omega^2 [\ ] \ , \tag{18}$$

where $d\Omega_k$ is the element of solid angle. But $2\gamma_k = v_g 2\,\mathrm{Im}k$, where Im stands for the imaginary part and the group velocity $v_g = (\partial\omega/\partial\,\mathrm{Re}k_\perp) \approx c(1-\omega_p^2/\omega^2)^{1/2}$, since the real part of the O-mode dielectric coefficient $D \approx (c\,\mathrm{Re}k_\perp/\omega)^2 \approx (1-\omega_p^2/\omega^2)$. Here Re stands for the real part. For $k_z \to 0$, γ_k and Imk are both proportional to $\delta(\omega-\omega_b)$ and for a tokamak plasma $\omega_b \propto R^{-1}$ implying $d\omega = -d\omega_b = (\omega_b/R)dR$, where R is the major radius of the torus. Thus, on making use of Eq. (18), Eq. (17) becomes

$$\begin{aligned} P &= -\frac{1}{L^3}\sum_{\mathbf{k}}(\partial\varepsilon_k/\partial t) \\ &= \frac{1}{L^3}\sum_{\mathbf{k}} 2\gamma_k\varepsilon_k \\ &= \left[\frac{1}{2\pi c}\right]^3 \int d\Omega_k \int d\omega\,\omega^2(1-\omega_p^2/\omega^2)^{3/2} 2\gamma_k\varepsilon_k \\ &= (\omega_b^3/2\pi^2 Rc^2)(1-\omega_p^2/\omega_b^2)^2\tau\varepsilon_k\,, \end{aligned} \tag{19}$$

where the optical depth[14]

$$\tau = \int 2\,\mathrm{Im}k\,dR = \int (2\gamma_k/v_g)dR = (1-\omega_p^2/\omega_b^2)^{1/2}(\pi R\,\omega_p^2 k_B T_\parallel/2\omega mc^3)\,. \tag{20}$$

Here we have made use of Eq. (4). The power P absorbed on a single transit of the absorption region is $P = P_0[1-\exp(-\tau)]$, where P_0 is the incident ECRH power. If the wall reflection coefficient r is large, then

$$P = P_0[1-\exp(-\tau)]/[1-r\exp(-\tau)]\,,$$

and multiple transits will enhance the ECR absorption considerably. For $\tau > 2$, the system absorbs the incident power like a blackbody and $P \approx P_0$. In this case, within the absorbing volume the absorbed radiation field reaches thermodynamic equilibrium with the plasma electrons in a time of order $1/2\gamma_k$ regardless of the collisional relaxation of the electron distribution function, and $\varepsilon_k \approx k_B T_\perp$ as a result of the equipartition theorem.[5] Thus from Eqs. (17) and (19) we get

$$(P/N_0 k_B T_\perp) = (1/k_B T_\perp)[\partial(k_B T_\perp)/\partial t] \approx (\omega_b^3\tau/2\pi^2 N_0 Rc^2)(1-\omega_p^2/\omega_b^2)^2\,. \tag{21}$$

The solution of Eq. (21) for $[P_0/N_0 k_B T_\perp(t)] \lesssim 1$ is

$$T_\perp(t) = T_\perp(t=0)\exp[(\omega_b^3\tau/2\pi^2 N_0 Rc^2)(1-\omega_p^2/\omega_b^2)^2 t]\,. \tag{22}$$

This localized heating should, of course, cease at a time t_0 such that $P_0 \approx N_0 k_B T_\perp(t=t_0)$. That is, for $\tau > 2$ and $t_0 > (1/2\gamma_k)$, $T_\perp(t \lesssim t_0)$ of the absorbing electrons can at most increase exponentially in time as given by Eq. (22). However, the absorbing volume ΔV is only a small part of the total tokamak plasma volume $V \approx 2\pi^2 R_0 a^2$, where a is the plasma minor radius and R_0 is the central plasma major radius. $\Delta V \approx \Delta R\,A$, where A is the launching antenna vertical spot size area on the resonant layer of width ΔR along the major radius. If Doppler width is the dominant absorption line broadening, then $\Delta R \approx (R/\omega_b)(k_z v_T)$, where v_T is the thermal velocity. In general, the energy will flow out of the absorbing volume as determined by the local value of the transport coefficients, and one must do a complete power balance study to obtain the exact form of $T_\perp(t)$. Thus $T_\perp(t)$ of Eq. (22) is an upper bound.

IV. CONCLUSIONS

In conclusion we have presented a coupled pair of quasilinear equations for the O-mode electron-cyclotron resonance in plasmas. One of these equations describes the time evolution of the O-mode wave energy and the other describes the time evolution of the electron distribution function. We have shown that this coupled pair of equations is self-consistent by proving that these equations satisfy the conservation laws of energy, momentum, and particle density. It is found that although the wave damping is determined mainly by $T_\parallel$, the radiative equilibrium is an equipartition mainly with $T_\perp$. In O-mode ECRH the heating is primarily in the perpendicular direction even through $\mathbf{E}\|\mathbf{B}$. We have also presented closed-form expressions for the time rate of change of $T_\perp$, $T_\parallel$, N_0, $J_\parallel$, and ε_k. It is found that the Doppler effect due to the finite value of k_z does not yield the usually expected Gaussian broadening. However, it leads to a symmetric Doppler splitting for $\partial T_\perp/\partial t$, $\partial J_\parallel/\partial t$, and $\partial\varepsilon_k/\partial t$, and to an antisymmetric splitting for $\partial T_\parallel/\partial t$ and $\partial N_0/\partial t$ as a function of ω/ω_b. All these results are somewhat in contrast to intuitive expectations.

For normal tokamak discharges we have shown that the collisional damping of the O mode is negligible in comparison to the usual cyclotron damping. In quasilinear theory it is found that the Ohkawa steady-state current drive efficiency criterion is really a consequence of the conservation laws of energy, momentum, particle density, and the collisional relaxation of the current density. Finally, we have shown that if the absorbing resonant layer in ECRH is optically thick the increase in $T_\perp$ in the absorbing plasma volume is exponential and not linear in time.

ACKNOWLEDGMENTS

This work was supported by the U. S. Department of Energy Contract No. DE-AC02-76CH03073. We thank K. Bol, T. K. Chu, R. W. Motley, T. H. Stix, N. J. Fisch, and King-Lap Wong for useful comments.

[1]V. Arunasalam, J. Math. Phys. **10**, 1305 (1969).

[2]L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Addison-Wesley, Reading, MA, 1958).

[3]V. Arunasalam, Phys. Rev. **149**, 102 (1966); Am. J. Phys. **36**, 601 (1968). This Phys. Rev. paper treats only the extraordinary mode with **E**⊥**B**.

[4]V. N. Tsytovich, *Nonlinear Effects in Plasmas* (Plenum, New York, 1970); *An Introduction to the Theory of Plasma Turbulence* (Pergamon, New York, 1972); *Theory of Turbulent Plasmas* (Consultants Bureau, New York, 1977).

[5]G. Bekefi, *Radiation Processes in Plasmas* (Wiley, New York, 1966).

[6]R. Z. Sagdeev and A. A. Galeev, *Nonlinear Plasma Theory* (Benjamin, New York, 1969). For example, the dynamical friction term is absent in Eq. (11-33) of this reference.

[7]T. H. Stix, Nucl. Fusion **15**, 737 (1975); V. L. Yakimenko, Zh. Eksp. Teor. Fiz. **44**, 1534 (1963) [Sov. Phys.—JETP **17**, 1032 (1963)]; J. Rowlands, V. D. Shapiro, and V. I. Shevchenko, *ibid.* **50**, 979 (1966) [**23**, 651 (1966)]; A. A. Vedenov, J. Nucl. Energy, Part C **5**, 169 (1963).

[8]J. Busnardo-Neto, J. Dawson, T. Kamimura, and A. T. Lin, Phys. Rev. Lett. **36**, 28 (1976); V. Arunasalam, *ibid.* **37**, 746 (1976).

[9]T. M. Antonson, Jr. and W. M. Manheimer, Phys. Fluids **21**, 2295 (1978); K. R. Chu and B. Hui, *ibid.* **26**, 69 (1983).

[10]H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, MA, 1957).

[11]V. Arunasalam, E. B. Meservey, M. N. Gurnee, and R. C. Davidson, Phys. Fluids **11**, 1076 (1968).

[12]L. Spitzer, Jr., *The Physics of Fully Ionized Gases,* 2nd ed. (Interscience, New York, 1962).

[13]T. Ohkawa, Nucl. Fusion **10**, 185 (1970); N. J. Fisch, Phys. Rev. Lett. **41**, 873 (1978); C. F. F. Karney and N. J. Fisch, Nucl. Fusion **21**, 1549 (1981); N. J. Fisch, Rev. Mod. Phys. **59**, 175 (1987).

[14]V. Arunasalam, P. C. Efthimion, J. C. Hosea, H. Hsuan, and G. Taylor, Phys. Rev. A **36**, 3909 (1987).