Operator disentanglement

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(Received 24 March 1987)

We obtain the time-ordered form of the evolution operator for Hamiltonian linear combinations of the generators of the SU(1,1), SU(2), and h(4) groups. The relevance of the obtained results to physics problems such as the generation of non-Poissonian statistics in laser-plasma scattering and the pulse propagation in free-electron lasers is also briefly discussed.

I. INTRODUCTION

Optics and quantum optics are research fields in rapid evolution, and with mathematical tools are getting benefit from algebraic methods, which also played an important role in quantum field theory as well as in nuclear and elementary-particle physics.[1] Although these methods have been successfully and widely exploited and are well established, it may happen that the specific problem under study creates new mathematical difficulties not previously considered. In particular, operator-ordering theorems are getting more and more attention in connection with the growing interest for Lie-algebraic methods in optics.[2,3] Within this framework the necessity of evaluating the evolution of quantum states ruled by SU(2), SU(3), and SU(1,1) coherence-preserving Hamiltonians[3,4] imposed the rediscovery, the suitable rehandling, and the extension of operator-ordering techniques of the Magnus-type[5] and Wei-Norman–type[6] (for an extended review see Refs. 7 and 8). Together with these analytical techniques, symbolic computer codes have been developed[9] to deal with the sometimes awful algebra implicit in, e.g., the Magnus or Zassenhaus expansions.[7,8]

Coherent states have been reintroduced[10] in optics by Glauber more than 20 years ago[11] and the Heisenberg-Weyl or h(4) group with generators $(\hat{a},\hat{a}^{\dagger},\hat{1})$ is instrumental to their analysis. More recently, mostly in connection with the theory of reduced quantum fluctuations, coherent states of SU(2)- and SU(1,1)-type have also been extensively investigated.[12]

In principle, a necessary but not sufficient condition to preserve h(4), SU(2), or SU(1,1) coherence under time evolution is that the Hamiltonian operator, driving the state, be a linear combination of the generators of the relevant group. The problems associated to the evolution of the h(4), SU(2), and SU(1,1) states have been solved satisfactorily; the probability amplitudes characterizing the evolution have been studied in detail and have provided important information.[13]

Hamiltonian operators which underly a SU(1,1) $\oplus$h(4) or SU(2)$\oplus$h(4) group structure have been recently considered in the analysis both of the generation of non-Poissonian effects in a laser-plasma scattering[14,15] and of the pulse propagation in a free-electron laser. A Hamiltonian operator which is, e.g., a linear combination of the generators of both SU(1,1) and h(4) groups may not preserve both Glauber and Perolemov[16] coherence. It is, however, important to study the evolution of quantum states driven by a SU(2),SU(1,1) $\oplus$h(4) Hamiltonian to clarify, e.g., how much the statistical properties of an initially Glauber state have been modified by the interaction.

To this aim, an obliged step is the search for a proper ordered form of the evolution operator for the above-quoted Hamiltonians. For the sake of completeness we present two different techniques. The first, which only applies to time-independent Hamiltonians, uses a combination of the Wei-Norman algebraic procedure and the more conventional operator identities derived in Refs. 17 and 18. The second exploits the Wei-Norman method and holds even for time-dependent Hamiltonians.

II. THE ORDERING METHOD: TIME-INDEPENDENT HAMILTONIANS

Let us consider the Hamiltonian operator

$$\hat{H}=\omega_0\hat{k}_0+\Omega_1(\hat{k}_++\hat{k}_-)+\Omega_2(\hat{a}+\hat{a}^{\dagger}) , \tag{2.1}$$

where ω_0 and $\Omega_{1,2}$ are time-independent c numbers, $\hat{k}_0,\hat{k}_\pm$ are the generators of the SU(1,1) group realized as[19]

$$\hat{k}_+=\tfrac{1}{2}(\hat{a}^{\dagger})^2,\quad \hat{k}_-=\tfrac{1}{2}\hat{a}^2 ,$$
$$\hat{k}_0=\tfrac{1}{4}(\hat{a}^{\dagger}\hat{a}+\hat{a}\hat{a}^{\dagger}) , \tag{2.2}$$

and $\hat{a},\hat{a}^{\dagger}$ are harmonic-oscillator annihilation and creation operators. The relations of commutation obeyed by (2.2) are immediately inferred as

$$[\hat{k}_+,\hat{k}_-]=-2\hat{k}_0,\quad [\hat{k}_0,\hat{k}_\pm]=\pm\hat{k}_\pm . \tag{2.3}$$

Some physical problems which the Hamiltonian (2.1) is relevant to will be discussed in Sec. IV.

The group structure of (2.1), easily recognized as that of the semidirect sum SU(1,1)$\oplus$h(4), can be reduced to a

SU(1,1) structure by rescaling the creation and annihilation operators as

$$\hat{b}=\hat{a}+\hat{x}\ ,\qquad \hat{b}^{+}=\hat{a}^{\dagger}+\hat{x},\quad \hat{x}=\frac{2\Omega_2}{2\Omega_1+\omega_0}\hat{I}\ . \tag{2.4}$$

Therefore we can recast the Hamiltonian (2.1) in the form

$$\hat{H}=\Omega_1[\hat{\hat{k}}_{+}+\hat{\hat{k}}_{-}]+\omega_0\hat{\hat{k}}_0-\Omega_2\hat{x}\ , \tag{2.5}$$

where $\hat{\hat{k}}_{\pm},\hat{\hat{k}}_0$ are defined in the $\hat{b}$ basis according to (2.2). Using now the well-established methods to write the evolution operator, we get, in the $\hat{b}$ basis,[2-4]

$$\begin{aligned}&\hat{U}(t)=\hat{U}_0(t)\hat{U}_I(t)\ ,\\ &\hat{U}_0(t)=\exp[-i(\omega_0\hat{\hat{k}}_0-\Omega_2\hat{x})t]\hat{1}\ ,\\ &\hat{U}_I(t)=\exp[2h(t)\hat{\hat{k}}_0]\exp[g(t)\hat{\hat{k}}_{+}]\exp[-f(t)\hat{\hat{k}}_{-}]\hat{1}\ ,\end{aligned} \tag{2.6}$$

where the functions (h,g,f) are specified by the following system of differential equations:

$$\begin{aligned}&\dot{h}(t)=-i\bar{\Omega}_1(t)g(t)e^{2h(t)}\ ,\\ &\dot{g}(t)=-i\bar{\Omega}_1^{*}(t)e^{-2h(t)}-g(t)\dot{h}(t)\ ,\\ &\dot{f}(t)=i\bar{\Omega}_1(t)e^{2h(t)},\quad h(0)=f(0)=g(0)=0\\ &[\bar{\Omega}_1(t)=\Omega_1\exp(-i\omega_0 t)]\ .\end{aligned} \tag{2.7}$$

The final disentanglement is, however, required to express $\hat{U}(t)$ in the $\hat{a}$ basis.

The exponential operators containing $\hat{\hat{k}}_{\pm}$ are straightforwardly disentangled because problems of noncommutativity do not arise. On the other hand, the operator

$$\begin{aligned}\exp[2\bar{h}(t)\hat{\hat{k}}_0]&=\exp\{[\bar{h}(t)/2](2\hat{x}^2+1)\}\\ &\quad\times\exp\{\bar{h}(t)[\hat{a}^{\dagger}\hat{a}+x(\hat{a}^{\dagger}+\hat{a})]\}\ ,\end{aligned} \tag{2.8}$$

with $2\bar{h}(t)=2h-i\omega_0 t$, can be expressed as[19]

$$\begin{aligned}\exp[2\bar{h}(t)\hat{\hat{k}}_0]&=\exp[x^2(1-e^{-\bar{h}(t)})]\exp[2\bar{h}(t)\hat{k}_0]\\ &\quad\times\exp[-x(1-e^{\bar{h}(t)})\hat{a}]\\ &\quad\times\exp[x(1-e^{-\bar{h}(t)})\hat{a}^{\dagger}]\end{aligned} \tag{2.9}$$

by exploiting the Sack[18] and Sack-Weyl[19] identities.

Finally, from (2.6) and (2.9) after a simple rearrangement of the various terms, we end up with the following ordered form for $\hat{U}(t)$:

$$\begin{aligned}\hat{U}(t)=&\exp[g_6(t)\hat{1}]\exp[2\bar{h}(t)\hat{k}_0]\exp[g(t)\hat{k}_{+}]\\ &\times\exp[-f(t)\hat{k}_{-}]\exp[g_4(t)\hat{a}^{\dagger}]\exp[-g_5(t)\hat{a}]\ ,\end{aligned} \tag{2.10}$$

which solves the problem of the time evolution under a SU(1,1)⊕h(4) Hamiltonian. The functions $(\bar{h},g,f)$ and (g_4,g_5,g_6) are explicitly given by

$$\begin{aligned}&\bar{h}(t)=-\ln[1+\alpha^2(t)]-i\eta(t),\quad \alpha(t)=\Omega_1\frac{\sin(Q/2)t}{Q/2}\ ,\\ &g(t)=-i\alpha(t)[1+\alpha^2(t)]^{1/2}\exp[i\eta(t)]\ ,\\ &f(t)=i\alpha(t)[1+\alpha^2(t)]^{-1/2}\exp[-i\eta(t)]\ ,\\ &g_4(t)=\Omega_2\left[\frac{\sin(Qt/4)}{Q/4}\right]\\ &\quad\times\left[1-\frac{4\Omega_1}{Q^2}(\omega_0-2\Omega_1)\sin^2\frac{Qt}{4}\right]^{1/2}\\ &\quad\times\exp\left\{-itg^{-1}\left[\left[\frac{\omega_0+2\Omega_1}{\omega_0-2\Omega_1}\right]^{1/2}\cot\left[\frac{Qt}{4}\right]\right]\right\}\ ,\end{aligned} \tag{2.11}$$

$$\begin{aligned}&g_5(t)=g_4^{*}(t)\ ,\\ &g_6(t)=-\frac{1}{2}\left[\Omega_2\frac{\sin(Qt/4)}{Q/4}\right]^2\\ &\quad\times\left[1-4\frac{Q_1}{Q^2}(\omega-2\Omega_1)\sin^2\frac{Qt}{4}\right]\\ &\quad+i2\frac{\Omega_2^2}{Q^2}(\omega_0-2\Omega_1)\left[t-2/Q\sin\frac{Qt}{2}\right]\ ,\\ &\eta(t)=\arctan[(\omega_0/Q)\tan(Qt/2)],\quad Q=(\omega_0^2-4\Omega_1^2)^{1/2}\ .\end{aligned}$$

III. THE ORDERING METHOD: THE TIME-DEPENDENT CASE

In this section we consider the time-dependent counterpart of (2.1), i.e., the Hamiltonian operator

$$\hat{H}=\sum_{i=1}^{2}\left[\frac{\omega_i(t)}{2}\hat{a}_i^{\dagger}\hat{a}_i+\Omega_i(t)\hat{a}_i+\text{H.c.}\right]+\Gamma(t)\hat{k}_{-}+\Gamma^{*}(t)\hat{k}_{+}\ , \tag{3.1}$$

where $\omega_i(t)$ and $\Omega(t)$, $\Gamma(t)$ are nonsingular functions of the time real and complex, respectively.

The subscripts 1,2 label independent harmonic oscillators and the SU(1,1) generators are realized as

$$\begin{aligned}&\hat{k}_{+}=\hat{a}_1^{\dagger}\hat{a}_2^{\dagger},\quad \hat{k}_{-}=\hat{a}_1\hat{a}_2\ ,\\ &\hat{k}_0=\tfrac{1}{2}(\hat{a}_1^{\dagger}\hat{a}_1+\hat{a}_2\hat{a}_2^{\dagger})\ .\end{aligned} \tag{3.2}$$

It is convenient, in this case, to implement the use of the interaction Hamiltonian which reads

$$\hat{H}_I=\sum_{i=1}^{2}(\bar{\Omega}_i\hat{a}_i+\text{H.c.})+\bar{\Gamma}\hat{k}_{-}+\bar{\Gamma}^{*}\hat{k}_{+}\ , \tag{3.3}$$

[the unperturbed Hamiltonian is $\hat{H}_0=\sum_{i=1}^{2}\omega_i(t)\hat{a}_i^{\dagger}\hat{a}_i$ and the corresponding evolution operator reads $\hat{U}_0=\prod_{j=1}^{2}\exp(-i\int_0^t\omega_j(t')dt'\hat{a}_j^{\dagger}a_j)$] with $\bar{\Omega}$ and $\bar{\Gamma}$ defined by

$$\bar{\Omega}_i=\Omega_i \exp\left[-i\int_0^t \omega_i(t')dt'\right] ,$$
$$\bar{\Gamma}=\Gamma \exp\left[-i\int_0^t [\omega_1(t')+\omega_2(t')]dt'\right] . \tag{3.4}$$

According to the Wei-Norman suggestion[6] we factorize $\hat{U}_I$ as

$$\hat{U}_I=\hat{U}_S\hat{U}_R . \tag{3.5}$$

The operators $\hat{U}_S$ and $\hat{U}_R$ are relevant to the *unperturbed* SU(1,1) part of $\hat{H}_I$ and to the "Hamiltonian"

$$\hat{H}'_R=\hat{U}_S^{-1}\hat{H}_R\hat{U}_S , \tag{3.6}$$

with

$$\hat{H}_R=\sum_{i=i}^{2}(\Omega_i\hat{a}_i+\text{H.c.}) , \tag{3.7}$$

respectively.

Consequently $\hat{U}_S$ obeys the equation of motion

$$i\frac{\partial}{\partial t}\hat{U}_S=(\bar{\Gamma}^*\hat{k}_+ +\bar{\Gamma}\hat{k}_-)\hat{U}_S \tag{3.8}$$

whose solution is immediately written down in the form

$$\hat{U}_S=\exp[2h(t)\hat{k}_0]\exp[g(t)\hat{k}_\pm]\exp[-f(t)\hat{k}_-] , \tag{3.9}$$

h, f, and g being defined by the system (2.7) with $\bar{\Omega}_1(t)$ replaced by $\bar{\Gamma}(t)$.

Correspondingly the Schrödinger equation for $\hat{U}_R$ reads

$$i\frac{\partial}{\partial t}\hat{U}_R=[\gamma_1(t)\hat{a}_1+\gamma_2(t)\hat{a}_2+\text{H.c.}]\hat{U}_R ,$$
$$\hat{U}_R(0)=\hat{I} , \tag{3.10}$$

where γ_1 and γ_2 are given by

$$\begin{bmatrix}\gamma_1\\ \gamma_2\end{bmatrix}=\begin{bmatrix}\bar{\Omega}_1 & \bar{\Omega}_1^*\\ \bar{\Omega}_1^* & \bar{\Omega}_2\end{bmatrix}\begin{bmatrix}H^*\\ F\end{bmatrix}\begin{matrix}(H\equiv e^{-h})\\ (F\equiv fe^{-h}) .\end{matrix} \tag{3.11}$$

Therefore the operator $\hat{U}_R$ can be easily specialized as

$$\hat{U}_R=\prod_{j=1}^{2}\exp\left[i\,\text{Im}\int_0^t\gamma_j^*\beta_j dt'\right]\exp(-\tfrac{1}{2}|\beta_j|^2)$$
$$\times\exp(-i\beta_j^*\hat{a}_j^\dagger)\exp(-i\beta_j\hat{a}_j)$$
$$\left[\beta_j=\int_0^t\gamma_j(t')dt'\right] , \tag{3.12}$$

thus yielding the solution to the problem.

The SU(2)⊕h(4) case will be analyzed in the Appendix.

IV. CONCLUSIONS

Hamiltonians of the type (2.1) play an important role in the generation of squeezed states via the laser-plasma scattering.[14] Within this framework it could be interesting to investigate the variations induced in the statistical properties of an initially Glauber coherent state undergoing an interaction governed by (2.1) (see also Ref. 15).

The evolution of states ruled by a Hamiltonian of the type (2.1) occurs according to

$$\Psi(t)=\sum_{n=0}^{\infty}\exp\left[-\frac{|\alpha_0|^2}{2}\right]\frac{\alpha_0^n}{\sqrt{n!}}\sum_{l=-n}^{\infty}C_l(t)|n+l\rangle , \tag{4.1}$$

where $|\alpha_0|^2$ is the initial average number of photon, and the time-dependent probability amplitudes $C_l(t)$ expressed by the matrix elements

$$C_l(t)=\langle n+l|\hat{U}(t)|n\rangle \tag{4.2}$$

explicitly read

$$C_l(t)=\exp\left[g_6(t)+\frac{\bar{h}(t)}{2}[1+2(n+l)]\right]\sqrt{n!(n+l)!}g_4^l(t)\left[\sum_{q=0}^{[n/2]}\frac{(-1)^q}{q!}\left[\frac{f(t)g(t)}{4}\right]^q\right.$$
$$\left.\times\left[\sum_{m=-[l/2]}^{q}\left[\frac{2g_4^2(t)}{g(t)}\right]^m\frac{1}{(q-m)!}\frac{L_n^{l+2m}[g_4(t)g_5(t)]}{(n+l-2q+2m)!}\right]\right] . \tag{4.3}$$

The symbol $[\alpha]$ denotes the largest integer less or equal to α and $L_n^l(x)$ are the generalized Laguerre polynomials.

The state Ψ possesses the features of a Glauber and a SU(1,1) coherent state, but it is neither. It can be proven indeed that by setting $\Omega_1=0$, (4.3) reduces to

$$C_l(t)|_{\Omega_1=0}=\left[\frac{n!}{(n+l)!}\right]^{1/2}[\alpha(t)]^l\exp\left[-i\frac{\omega_0 t}{4}(1+2n)\right]\exp\left[+\frac{i\omega_0}{4}\int_0^t|\alpha(t')|^2dt'\right]e^{-|\alpha(t)|^2/2}L_n^l|\alpha(t)|^2$$

$$\left[\alpha(t)=-i\left[\Omega_2\frac{\sin(\omega_0 t/4)}{\omega_0/4}\right]e^{+i\omega_0 t/4}\right] , \tag{4.4}$$

which yields a Glauber coherent state at any time t. On the other hand, with $\Omega_2=0$, (4.3) becomes

$$C_l(t)\,|_{\Omega_2=0}=\left[\frac{g(t)}{2}\right]^{l/2}\exp\{\tfrac{1}{2}[1+2(n+l)]\bar{h}(t)\}\sqrt{n!(n+l)!}H_n^l[g(t)f(t)]$$

$$\left[H_n^l(x)=\sum_{m=0}^{[n/2]}\frac{(-1)^m(x/2)^m}{2^m m!(n-2m)!(l/2+m)}\right], \tag{4.5}$$

which for an initially vacuum field yields SU(1,1) coherent states according to the results of Ref. 20.

A further important problem which can be treated within the framework of the present formalism is the evolution of the optical field in a free-electron laser (FEL). In a FEL a bunch of ultrarelativistic electrons is injected into a N-period magnetic undulator, where it undergoes transverse oscillations and emits radiation at fixed wavelength λ. Such radiation, stored in an optical cavity, copropagates into the magnet with a new bunch and becomes amplified by the mechanism of stimulated bremsstrahlung.[21] The electron beam, if furnished by a radio-frequency accelerator, consists of a series of bunches with a time distance fixed by the period of the radio frequency and with a longitudinal length σ_z fixed by the phase stable angle.[22] In this configuration a mode-locked laser field naturally develops, and if the electron-bunch longitudinal length is much longer than the slippage distance $N\lambda$, the evolution of the laser electric field Ψ is governed by an equation of the type[22,23]

$$i\frac{\partial\Psi}{\partial\tau}=\hat{H}\Psi\,,$$

$$\hat{H}=i[\delta\hat{I}+\Omega(\hat{a}-\hat{a}^\dagger)+\omega\hat{k}_0+\Omega_1(\hat{k}_++\hat{k}_-)] \tag{4.6}$$

where δ, Ω, ω, and Ω_1 are complex numbers and the operators $\hat{a},\hat{a}^\dagger$ and $\hat{k}_\pm,\hat{k}_0$ are expressed as linear and nonlinear combinations, respectively, of the dimensionless longitudinal coordinate y and of the derivative with respect to it. The non-Hermitian nature of the "Hamiltonian" (4.6) is due to the gain or loss mechanism intrinsic to the FEL process [for a more detailed discussion on the role played by the various parameters in (4.6) see Refs. 22 and 23]. Expanding Ψ in terms of harmonic-oscillator eigenfunctions

$$\Psi(y,\tau)=\sum_{n=0}^{\infty}C_n(\tau)u_n(y)\,,$$
$$u_n(y)=\left[\frac{1}{\pi^{1/2}2^n n!}\right]^{1/2}e^{-y^2/2}H_n(y)\,, \tag{4.7}$$

where H_n are the Hermite polynomials, and skipping the details of the computation involving only algebraic troubles we get for $C_n(\tau)$ $[C_n(\tau_0)=\delta_{n,0}]$ the expression[24]

$$C_n(\tau)=e^{\delta(\tau-\tau_0)}R^{-1/2}I[Z(\tau)]^n\frac{1}{2^{n/2}\sqrt{n!}}H_n[X(\tau)]\,, \tag{4.8}$$

where we have defined

$$Z(\tau)=\left[-\frac{G}{R}\right]^{1/2},\quad X(\tau)=\left[-\frac{P^2}{G}\right]^{1/2},\quad R(\tau)=\cosh[Q/2(\tau-\tau_0)]-\omega/Q\sinh[Q/2(\tau-\tau_0)]\,,$$

$$\begin{aligned}I(\tau)=\exp\Bigg[&2\frac{\Omega^2}{Q^2}(\omega+2\Omega_1)(\tau-\tau_0)-\frac{2\Omega^2}{Q^2}\Omega_1^2(\omega+2\Omega_1)(\tau-\tau_0)^2-\frac{1}{[Q/2(\tau-\tau_0)]^2}\sinh^2[Q/2(\tau-\tau_0)]\\&+\frac{\Omega^2}{2Q^2}(\omega+2\Omega_1)^2(\tau-\tau_0)^2\frac{1}{[Q/4(\tau-\tau_0)]^2}\sinh^2[Q/4(\tau-\tau_0)]\\&-\frac{2\Omega^2}{Q^2}(\omega+2\Omega_1)(\tau-\tau_0)\frac{1}{Q/2(\tau-\tau_0)}\sinh[Q/2(\tau-\tau_0)]\Bigg]\,,\end{aligned} \tag{4.9}$$

$$P(\tau)=\left[-\Omega(\tau-\tau_0)\frac{1}{Q/2(\tau-\tau_0)}\sinh[Q/2(\tau-\tau_0)]+\left[\frac{\omega+2\Omega_1}{4}\right]\Omega(\tau-\tau_0)^2\frac{1}{[Q/4(\tau-\tau_0)]^2}\sin^2[Q/4(\tau-\tau_0)]\right],$$

$$G(\tau)=\Omega_1(\tau-\tau_0)\frac{1}{[Q/2(\tau-\tau_0)]^2}\sinh^2[Q/2(\tau-\tau_0)],\quad Q\equiv(\omega^2-4\Omega_1^2)^{1/2},\quad \frac{1}{x}\sinh(x)=\frac{\sinh(x)}{x}\,.$$

Finally, using (4.7), (4.8), and the sum rules of the Hermite functions we can recast (4.7) in the closed form:

$$\Psi(y,\tau)=e^{\delta(\tau-\tau_0)}I[\Phi(\tau)]^{1/2}\exp\left[\frac{X^2Z^2}{1+Z^2}\right]\exp\left[-\frac{1+Z^2}{2(1-Z^2)}\left[y-\frac{2XZ}{1+Z^2}\right]^2\right], \tag{4.10}$$

$$\Phi(\tau)=1/\sqrt{\Pi}\frac{e^{\omega/2(\tau-\tau_0)}}{R(1-Z^2)}\,.$$

The above expression yields the space-time evolution of the FEL optical field in the long-bunch approximation and displays also some features which have been previously analyzed only numerically. For instance it predicts that the position of the maximum of the wave packet is a function of the time given by

$$y_0(\tau)=\frac{2XZ}{1+Z^2}=\sqrt{2}\Omega\left[\frac{d}{d\tau}\ln P\right]^{-1} . \quad (4.11)$$

According to (4.11) the packet moves back at a speed

$$\dot{y}_0(\tau)=\sqrt{2}\Omega\left[\frac{d}{d\tau}\ln P\right]^{-2}\frac{d^2}{d\tau^2}(\ln P) , \quad (4.12)$$

thus taking into account the so-called lethargic effect which plays a crucial role in the design of an FEL.[24] For a more detailed discussion of these problems the reader is referred to Ref. 24.

APPENDIX

In this appendix we complete the analysis of the disentanglement theorems by discussing the case of an SU(2)⊕h(4) Hamiltonian, namely,

$$\hat{H}=\sum_{j=1}^{2}\left[\frac{\omega_j(t)}{2}\hat{a}_j^{\dagger}\hat{a}_j+\Omega_j(t)\hat{a}_j+\text{H.c.}\right] + \Gamma\hat{a}_1^{\dagger}\hat{a}_2+\Gamma^*\hat{a}_2^{\dagger}\hat{a}_1 . \quad (A1)$$

We proceed as in Sec. IV, introducing the interaction Hamiltonian

$$\hat{H}_I=\bar{\Omega}_1\hat{a}_1+\bar{\Omega}_2\hat{a}_2+\bar{\Gamma}\hat{a}_2\hat{a}_1^{\dagger}+\text{H.c.} , \quad (A2)$$

where

$$\bar{\Omega}_j=\Omega_j\exp\left[-i\int_0^t\omega_j(t')dt'\right] ,$$
$$\bar{\Gamma}=\Gamma\exp\left[-i\int_0^t[\omega_2(t')-\omega_1(t')]dt'\right] . \quad (A3)$$

The SU(2)⊕h(4) structure of (A2) can be easily recognized, recalling that the operators[25]

$$\hat{J}_+=\hat{a}_1\hat{a}_2^{\dagger},\ \hat{J}_-=\hat{a}_2\hat{a}_1^{\dagger},\ \hat{J}_0=\tfrac{1}{2}(\hat{a}_1^{\dagger}\hat{a}_1-\hat{a}_2^{\dagger}\hat{a}_2) \quad (A4)$$

obey the angular momentum commutation rules

$$[\hat{J}_+,\hat{J}_-]=2\hat{J}_0,\ [\hat{J}_0,\hat{J}_\pm]=\pm\hat{J}_\pm . \quad (A5)$$

According to the recipe of Sec. IV we get

$$\hat{U}_I=\hat{U}_S\hat{U}_R , \quad (A6)$$

where

$$\hat{U}_S=\exp(2h\hat{J}_0)\exp(g\hat{J}_+)\exp(-f\hat{J}_-) \quad (A7)$$

and the functions (h,g,f) obey the set of differential equations

$$\dot{h}=i\bar{\Gamma}ge^{2h} ,$$
$$\dot{g}=-i\bar{\Gamma}^*e^{-2h}-\dot{h}g , \quad (A8)$$
$$\dot{f}=i\bar{\Gamma}e^{2h},\ f(0)=h(0)=g(0)=0 .$$

Finally, $\hat{U}_R$ is in the same form as in (3.12), with the only difference that the γ functions are now given by

$$\gamma_1=\bar{\Omega}_1H+\bar{\Omega}_2F^* ,$$
$$\gamma_2=\bar{\Omega}_2H-\bar{\Omega}_1F . \quad (A9)$$

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