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Precision optical-frequency-difference measurements

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We demonstrate a new form of interferometry which is independent of Maxwell's equations and measures optical frequency differences rather than wavelength ratios. Stable laser and electro-optic FM sideband techniques make a direct measurement of the mode spacing of a high finesse cavity with subhertz precision (0.4 Hz rms) over a 50-THz region. Experiments indicate errors below 10^{-10} over a 50-THz bandwidth and that improvements of several orders of magnitude are possible. New variables for interferometry simplify the use of multilayer dielectric mirrors and are directly measurable *in situ*.

Precision interferometry plays an important role in optical fundamental constant experiments. All current measurements of the Rydberg in hydrogen,[1-4] for example, use an interferometer to measure atomic transition frequencies relative to an I_2-stabilized He-Ne laser frequency standard. Current interferometric techniques[5-7] derive their precision from the close agreement of the experimental apparatus with the predictions of Maxwell's equations [see Eq. (3) below]. Systematic errors arise from imperfect knowledge of cavity parameters required by the theory, for example, the radius of curvature or reflection phase shift of the mirrors. Such systematic effects limit the accuracy of present techniques to $\sim 3\times 10^{-11}$. Ultimately optical heterodyne and frequency synthesis techniques[8] may replace interferometry, but at present they are so difficult that they have been used only for the calibration of a few laser frequency standards.

In this Rapid Communication we demonstrate a new form of interferometry which is independent of Maxwell's equations and consequently free of the systematic errors mentioned above. It utilizes recently developed stable laser, heterodyne, and FM sideband techniques[9-14] to *directly* measure the frequency-dependent mode spacing of an interferometer with subhertz precision. In practice, its accuracy is limited only by the precision with which a fringe center can be found and it appears capable of several orders of magnitude better accuracy than current techniques. Unlike previous interferometers, it measures absolute optical frequency differences rather than wavelength ratios and can be calibrated relative to the primary time standard. This can lead, for example, to direct measurements of the Rydberg constant, independent of any optical frequency standard.

Our technique is made possible by the development of tunable lasers with linewidths $O(1\text{ Hz})$ by Hall and collaborators.[9,11] A direct measurement of the effective optical-mode spacing of an interferometer can be achieved by locking two such lasers to adjacent longitudinal orders and measuring their heterodyne beat on a frequency counter. By tuning the lasers across the frequency range of interest and recording the beat note as a function of optical frequency f, the frequency-dependent optical-mode spacing $\sigma(f)$ can be directly determined. Differences of many THz, for example between the Nth mode at $f(N)$ and the Mth at $f(M)$, can then be calculated by summing the measured spacings of all the modes between N and M, as given by the trivial identity

$$f(M)-f(N)=\sum_{K=N}^{M-1}\sigma(K) \ , \qquad (1)$$

where $\sigma(K)\equiv f(K+1)-f(K)$. Despite its simplicity, Eq. (1) has not played a key role in previous precision work because the available resolution (~ 1 MHz) did not permit direct measurements of σ with useful accuracy. Our use of Eq. (1) is basically different from previous work because our subhertz resolution permits precise, theory-independent measurements of the effective optical mode spacing. We can, therefore, treat the interferometer as a "black box" resonator whose mode spacings are repeatable and slowly varying but which need not agree with any theory. Equation (1), which is an identity, can then be used to compute the relative mode frequencies over a broad range without systematic error. By directly measuring $\sigma(f)$ we empirically calibrate the cavity without the need to understand its internal structure.

Tunable lasers with linewidths <1 Hz have been developed by Hall and collaborators using FM sideband techniques first applied to optics by Drever and Hall[9] and independently by Bjorklund.[10] Hall and co-workers have previously measured the beats of stable lasers locked to adjacent orders of the same cavity,[11] primarily to study stabilization techniques. This technique, while optimum for some applications, is difficult to realize since it requires locking and tuning two independent lasers with $\sim$1-Hz precision over a broad range. We have developed an alternative method, dual frequency modulation (DFM),[12] which resonates electro-optic FM sidebands, rather than separate lasers, with adjacent orders of the interferometer (see Fig. 1). Its primary advantage is that it cancels laser jitter and locking errors from the mode spacing measurement so that subhertz precision in σ can be achieved with a single laser of $\sim$100 Hz stability. Modulation techniques have previously been applied to interferometry by Bay, Luther, and White in a measurement of the speed of light.[13] The DFM method differs from the above work of Hall, Bjorklund, and Bay in that it senses the intermodulation products of two sequential electro-optic phase modulators.[12] Cutler has developed a

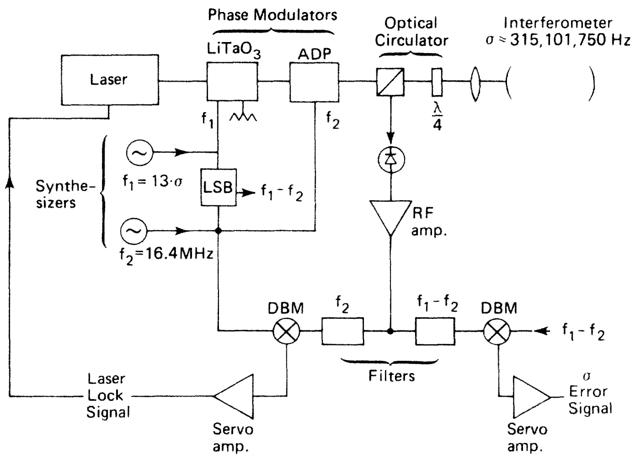

FIG. 1. Dual-frequency-modulation apparatus for electronically measuring the optical-mode spacing $\sigma(f)$ (see Ref. 12). FM sidebands at f_2 lock the tunable dye laser to the Nth order of the cavity while the f_1 sidebands at 4096 MHz simultaneously resonate with the $N+13$ and $N-13$ orders of the 315-MHz cavity. The mode spacing is measured by adjusting the f_1 synthesizer so that the "σ error signal" vanishes; then $\sigma(f)=f_1/13$.

different double-frequency modulation technique for the same purpose.[14] A variant of DFM, in which one sideband is generated by quantum noise rather than by a modulator, has also been used to observe squeezed states of light in fibers.[15]

To explore the accuracy of this technique we have measured the effective optical mode spacing $\sigma(f)$ of a precision interferometer at 149 frequencies over a 48 THz region around 473 THz (633 nm) (see Fig. 2). The cavity[16] used ultra-low-loss spherical mirrors of 50-cm radius spaced 47 cm apart in a high vacuum and had a linewidth full width at half maximum (FWHM) of 13.9 kHz and a finesse of 22700. For each measurement the dye-laser frequency was set with a commercial wavemeter and the f_1 synthesizer in Fig. 1 was adjusted so that the "σ error signal" vanished, indicating that $f_1=13\sigma(f)$. The data were taken in three overlapping runs with different dyes; 48 points with DCM, 82 points with Rhodamine-B, and 19 points with Rhodamine-6G. The data fit a parabola (solid line) $\sigma(f)=\sigma_0-k(f-f_c)^2$, where the best fit values are $\sigma_0=\sigma(f_c)=315,\ 101,\ 749.75\ \pm 0.2$ Hz, $k=4.9\pm 0.4\times 10^{-2}$ Hz/(THz)^2, and the coating center frequency $f_c=476.7\pm 0.6$ THz. The rms deviation of the

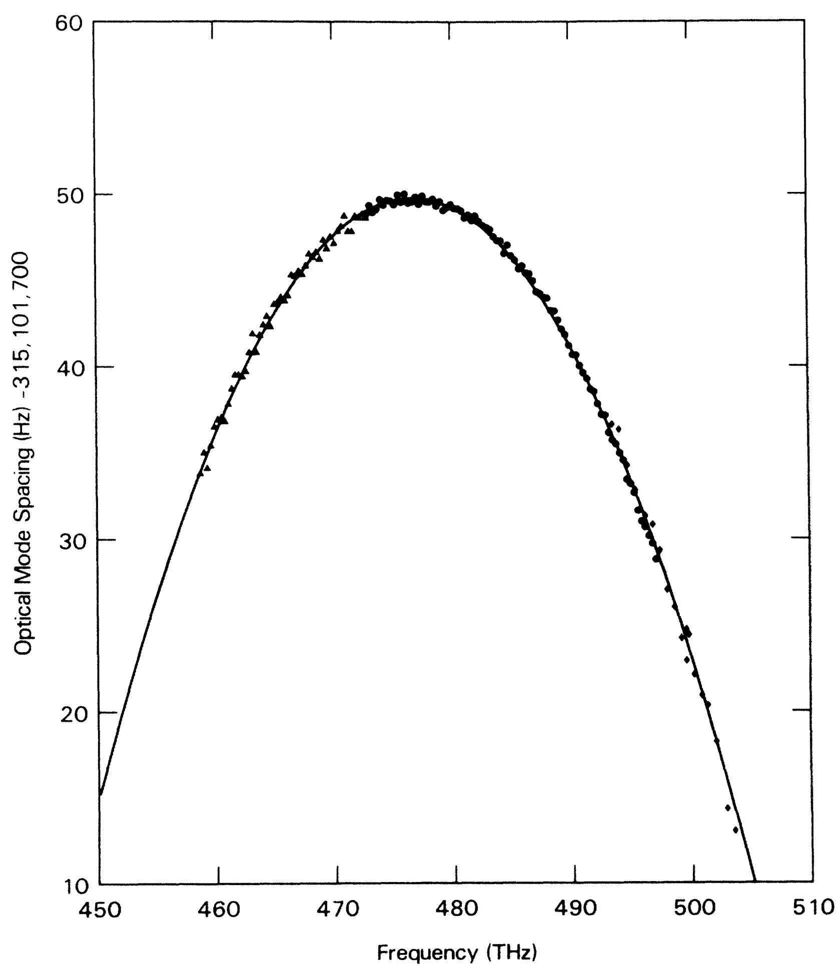

FIG. 2. A plot of 149 measurements of the optical-mode spacing $\sigma(f)$ over 48 THz around the mirror center frequency f_c. Note that $\sigma(f)$ has zero slope at f_c and varies quadratically by less than 40 Hz from its central value of 315, 101, 749.8 Hz. (Triangles stand for DCM dye, dots stand for Rh B, and diamonds stand for Rh 6G.) The rms deviation of the data from the parabola is ~0.4 Hz, indicating that the apparatus of Fig. 1 measures $\sigma(f)$ with subhertz precision.

149 points from the parabola is 0.4 Hz per point. The quality of the fit indicates first that the DFM apparatus has achieved subhertz precision in σ over a 50-THz interval and second that these mirrors lead to a mode spacing that is so slowly varying that the frequency of any mode in this range may be interpolated with negligible error.

Substituting this quadratic fit into Eq. (1) and approximating the sum by an integral, we find

$$f(M)-f(N)=\sigma_0(M-N)-\tfrac{1}{3}k\sigma_0^2(M-N)^3\ , \quad (2)$$

where we have assumed for simplicity that $f(N)\sim f_c$. Equation (2) calibrates the relative frequencies of all cavity modes over a 50 THz region with an accuracy limited only by the errors placed on σ_0 and k by the data of Fig. 2. The ± 0.2 Hz error in σ_0 contributes a maximum error of only 16 kHz for a ± 25 THz frequency difference. Similarly, the $\pm 8\%$ error on k creates a 55 kHz uncertainty at ± 25 THz, but this drops as the third power of the frequency difference. These errors are less than the current 72 kHz uncertainty in the I_2 stabilized He-Ne optical frequency standard.[8]

The accuracy of Eq. (2) is limited almost entirely by the precision with which the center of a cavity resonance can be found. Since we have not used Maxwell's equations and have made no assumptions about the internal structure of the cavity, the usual systematic errors due to mirror phase shift and diffraction do not arise. Simple methods exist for improving the resolution, and consequently the accuracy, of our technique by several orders of magnitude. First, since resolution is proportional to cavity linewidth, it can be improved by factors of 10–100 simply by using longer cavities. Second, the heterodyne beat frequency can also be increased by factors of 10–100. 1 Hz locking accuracy at a 100 GHz beat frequency corresponds to 5×10^{-13} accuracy over a 50 THz region. Applying both approaches simultaneously yields improvements of 10^2–10^4, so that interferometry might ultimately be limited only by the accuracy of the cesium beam standard itself. Another advantage of our technique is that it calibrates the interferometer without perturbing it. In the previous method of virtual mirrors,[5,6] the cavity must be disassembled and realigned at two different mirror separations to cancel mirror phase shift. Thus, its accuracy is limited by the repeatability of a complex alignment procedure.

Optical frequency differences measured using Eq. (1) are incoherent; all information about the relative phase is lost. In the sense they differ from direct synthesis[8] and heterodyne methods.

We now discuss interferometer theory to show how the parabolic variation of the mode spacing observed in Fig. 2 arises from the phase shift of multilayer dielectric mirrors. This theory does not influence the accuracy of the above measurements, but has other applications. For example, *absolute* optical frequency measurements using the DFM technique become possible if the mirror phase shift and diffraction correction are known. The method of virtual mirrors[5,6] can benefit from better understanding of mirror phase shift. Also, in applications where less than the ultimate accuracy is required, understanding interferometer behavior can greatly simplify the apparatus. Lichten[7] has recently reviewed interferometry with multilayer dielectric mirrors. Mirror phase shift has been calculated by numerically modeling the transmission spectrum and compared to experiments on short cavities. He has also derived an expression for the mode spacing in terms of these experimental and theoretical phase shifts and shown that its variation is small.

Our own approach to the theory is to define new variables and expand them in a power series in frequency. Applying Maxwell's equations to an interferometer yields the well-known relation[6]

$$f(N)=\frac{c}{2L}\left[N-\frac{\phi(f)}{\pi}-\frac{\phi_D}{\pi}\right]\ , \quad (3)$$

where L is the mechanical distance between the mirror's surfaces, N is the longitudinal order, ϕ_D is the diffraction phase shift, and $\phi(f)$ is the mirror phase shift. All previous forms of interferometry depend for their accuracy on the apparatus obeying Eq. (3). This is limited in part by the use of the variables L and $\phi(f)$, which are defined relative to the mirrors *mechanical* surfaces. We, therefore, transform Eq. (3) to eliminate the distinction between where the spacer L ends and the mirror surface begins and express $f(N)$ only in terms of directly and accurately measurable optical quantities. We multiply both numerator and denominator of Eq. (3) by $1+(c/2\pi L)\partial\phi/\partial f$ and derive a new interferometer relation

$$f(N)=s(f)\left[N-\frac{\phi_0(f)}{\pi}-\frac{\phi_D}{\pi}\right]\ , \quad (4)$$

where $c/2L$ is replaced by an effective optical mode spacing

$$s(f)=\frac{c}{2L+(c/\pi)\partial\phi/\partial f}\ , \quad (5)$$

and $\phi(f)$ replaced by

$$\phi_0(f)=\phi(f)-f\frac{\partial\phi}{\partial f}\ . \quad (6)$$

Equation (5) has been given previously[17] but Eqs. (4) and (6) are new. Equation (5) represents the spacing between adjacent cavity modes as influenced by mirror phase shift. A detailed analysis of the DFM signal shows that to $O(10^{-12})$ and in the absence of other perturbations $s(f)$ in Eq. (5) is equal to the empirical $\sigma(f)$ measured by the heterodyne technique. For the remainder of this paper we will not distinguish between s and σ. Equation (4) permits a precise *in situ* measurement of the transformed phase shift ϕ_0 defined by Eq. (6). By resonanting the cavity with an optical frequency standard so that $f(N)$ is known, ϕ_0 is measured once σ, N, and ϕ_D are determined.[16] Such a technique cannot be used with conventional variables using Eq. (3) because L cannot be measured directly. Physically ϕ_0 is the phase shift at an effective plane of reflection defined by $\sigma(f)$ and may be an inherently more precise quantity than ϕ.

We now show that s and ϕ_0 are independent of frequency to first order near the coating center frequency f_c by expanding $\phi(f)$ in a power series about f_c and retaining terms up to third order. We define the ith derivative of $\phi(f)$ at f_c to be a_i. Expressing $s(f)$ and $\phi_0(f)$ in terms of the a_i gives

$$s(f)=\frac{c}{2L+(c/\pi)[a_1+\frac{1}{2}(f-f_c)^2a_3]}\ , \tag{7}$$

$$\phi_0(f)=(a_0-f_ca_1)-\frac{a_3}{2}f_c(f-f_c)^2\left[1+\frac{2}{3}\frac{(f-f_c)}{f_c}\right]\ , \tag{8}$$

where $a_2=0$ since $\phi(f)$ is odd about f_c. Thus, s and ϕ_0 are quadratic functions of frequency near f_c with zero slope at f_c. The large dispersion of multilayer dielectric mirrors (a_1 term) changes the effective optical length of the cavity, and is absorbed into s, but does not lead to a frequency-dependent mode spacing. Only the third derivative a_3 leads to a frequency dependence. For definiteness we assume that the multilayer consists of a large number of quarter wave elements of index n_H and n_L and that the first and last layers are of n_H. In this case a_0 and a_1 have been calculated analytically[18] and have the values $a_0=\pi$ and $a_1=\pi/[f_c(n_H-n_L)]$. Substituting into Eq. (6) we find

$$\phi_0(f_c)=\pi\left[1-\frac{1}{n_H-n_L}\right]\ . \tag{9}$$

The above theory agrees with our experimental results and shows how they may be extrapolated to longer and more accurate cavities. Using Eq. (7) one can show that the quadratic parameter $k=\sigma_0^2a_3/\pi$. We have numerically evaluated a_3 for nominal values of n_H and n_L and find $k=3.7\times10^{-2}$ Hz/(THz)^2, only 25% below experiment. This relation also shows how the parabola scales with cavity length: $k\sim1/L^2$. Thus, the mode spacing of a 10 m cavity varies by only 75 mHz over a ±25 THz range. Such a cavity would have a linewidth FWHM ~500 Hz and could compare frequencies with errors close to 10^{-12}, given the present DFM apparatus.

Another application of the theory is to show how precision in the 10^{-9} range may easily be reached without stable lasers or the DFM method. This is similar to previous work of Lichten[7] except that we use new variables and a single calibration constant, rather than numerical modeling. The variation of $\sigma(f)$ may be ignored at the 10^{-9} level and $\sigma(f)\sim\sigma_0$. σ_0 may then be determined using an ordinary laser frequency standard at frequency f_s and Eq. (4). The unknowns ϕ_0 and ϕ_D form a new calibration constant $C=-\sigma_0(\phi_0+\phi_D)/\pi$ and the desired σ_0 is given by $\sigma_0=(f_s-C)/M$. The cavity resonance frequencies $f(N)$ then obey the simple relation $f(N)=N\sigma_0+C$.

We have observed no aging of C (or of k and f_c) while the mirrors were continuously held in a vacuum for 420 days. An I_2-stabilized He-Ne laser[19] was used to measure $f(N)$ and σ was measured by DFM. On 2 January 1986, $C=226.3\pm0.5$ MHz while on 27 February 1987, $C=225.9\pm0.2$ MHz. This implies an aging rate $<2\times10^{-12}$/day.

In conclusion, we have shown how heterodyne measurements of cavity modes with stable lasers can yield a new form of interferometry which is independent of the systematic errors which limit the current virtual mirror technique. Interferometers can be empirically calibrated as "black box" resonators with an accuracy limited only by how precisely lasers can be locked to their resonances. The method is scalable to the 10^{-12} precision needed by the next generation of optical fundamental constant experiments.

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