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Level crossings of filtered dichotomous noise

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This paper shows that the level crossing rate of a process $y(t)$ which results from nonlinear filtering of non-Markov dichotomous noise $x(t)$ can be expressed as $\bar{N}_l=\beta[1-F^2(l)]p(l)$, where $p(y)$ is the density of $y(t)$, β is an inverse time parameter, and $F(y)$ is a nonlinearity in the system equation $\dot{y}+\beta F(y)=\beta x$. Four examples are given in which Monte Carlo methods are used to establish the veracity of the theoretical results. In a fifth example, the theoretical result is obtained by time averaging. The new result is compared with the crossing rate for Gaussian processes and it is found that $\bar{N}_l \sim B(l)p(l)$ in each case, where $B(y)$ is the second conditional moment in the *extended* Fokker-Planck equation for $p(y)$.

I. INTRODUCTION

In this paper we will be concerned primarily with level crossings of nonlinearly filtered dichotomous noise; specifically with level crossings of processes generated by the first-order nonlinear differential equation

$$\frac{dy(t)}{dt}+\beta F[y(t)]=\beta x(t) , \tag{1}$$

in which $F(\ )$ is a nonlinear function, $\beta>0$ is an inverse time parameter, and $x(t)$ is dichotomous noise (a binary process). $F(\ )$ will be assumed to be nonpathological and such that the output $y(t)$ is continuous and bounded over any finite time interval if its initial value is finite, and has a mean-square derivative. Aside from being a stationary random process, $x(t)$ will be permitted to be quite general. If $N(\Delta)$ denotes the number of traversals of $x(t)$ from one state to the other in the time interval $(0,\Delta)$, all that will be assumed is that

$$\lim_{\Delta\to 0} P\{N(\Delta)=0\}=1 , \tag{2}$$

which is quite unrestrictive. Under these conditions, we will show that the average number $\bar{N}_l$ of crossings of $y(t)$ per unit time with the level l is given by

$$\bar{N}_l=\beta[1-F^2(l)]p(l) , \tag{3}$$

where $p(y)$ is the marginal probability density function of $y(t)$. This same expression was previously shown to hold when $x(t)$ is the random telegraph signal[1] and so is now being generalized to all $x(t)$ with the property (2).

Filtered dichotomous noise processes have been the subject of a number of recent investigations;[1-10] but one of the most classic filtered binary processes is the output of the filter-limiter-filter system in which the limiter input is RC Gaussian noise $\eta(t)$.[11-13] In this case the dichotomous noise is $\text{sgn}[\eta(t)]$ and has, consequently, an average number of traversals per unit time which is infinite. However, as discussed by Rice [Ref. 14, Eq. (117) and following] an unpublished result of Slepian for the probability that $\eta(t)$ has no zero crossings in an interval of length Δ is given by

$$P\{N(\Delta)=0\}=\frac{2}{\pi}\sin^{-1}(e^{-\Delta/\tau_c}) , \tag{4}$$

where τ_c is the correlation time of the RC noise. Hence, (2) is satisfied by the dichotomous process $\text{sgn}[\eta(t)]$. Another wide class of binary process for which (2) is satisfied is that with intervals generated by an equilibrium renewal process, and a number of results are available for the output probability density in this case.[10,15]

We here derive the above expression for $\bar{N}_l$ by evaluating[1]

$$\bar{N}_l=\lim_{\Delta\to 0}\frac{1}{\Delta}P\{(y-l)(y_0-l)<0\} , \tag{5}$$

in which $y_0=y(0)$ and $y=y(\Delta)$. The equation of Kac[16] and Rice[17] for $\bar{N}_l$ is

$$\bar{N}_l=\int_{-\infty}^{\infty}|\dot{y}|\,p(l,\dot{y})d\dot{y} , \tag{6}$$

where $p(y,\dot{y})$ is the joint probability density function of $y(t)$ and its derivative. Although these two are equivalent, in certain cases $\bar{N}_l$ is more easily found from (5) since (6) requires the determination of $p(y,\dot{y})$ prior to its evaluation.

A second ingredient essential in the derivation is the general result of Mazo and Salz that[18]

$$E[\dot{y}(t)\,|\,y(t)]=0 , \tag{7}$$

whenever $y(t)$ has a mean-square derivative. When used in (1), this leads to $E[x(t)\,|\,y(t)]=F[y(t)]$ from which it follows that

$$P\{x(t)=\pm 1\,|\,y(t)=l\}=\frac{1\pm F(l)}{2} . \tag{8}$$

For this to be always positive, we must require $|F(y)|\le 1$ as the region of validity of (1). If $|F(y)|>1$, the implication, from the above, is that $y(t)$ does not have a mean-square derivative or else is not a stationary random process.

The next section gives a derivation of the general result. Four examples are presented in Sec. III and expression (3) verified in each case by Monte Carlo simulations. In a fifth example, (3) is deduced from a time-domain averaging. The final section discusses the results and compares them with the same results for Gaussian processes. This leads us to speculate as to a possible general form for the average number of level crossings of an arbitrary stationary, mean-square differentiable random process, and an interesting interpretation of the second conditional moment in the *extended* Fokker-Planck equation.

II. DERIVATION

The derivation closely parallels that for exponentially distributed intervals (the random telegraph signal) in Sec. 4 of Ref. 1 but there are important differences because of the more general nature of the dichotomous noise.

We first show that (5) can be written in the equivalent form

$$\overline{N}_l = \lim_{\Delta\to 0} \frac{1}{\Delta} P\{(y-l)(y_0-l)<0 \mid N=0\} , \tag{9}$$

in which $N=N(\Delta)$. To simplify writing the equations, the event $(y-l)(y_0-l)<0$ will be denoted by $\mathcal{A}$. Then, starting with (5), we have

$$\begin{aligned}\overline{N}_l &= \lim_{\Delta\to 0} \frac{P\{\mathcal{A}\}}{\Delta} \\ &= \lim_{\Delta\to 0} \frac{1}{\Delta}(P\{\mathcal{A} \mid N=0\}P\{N=0\}+P\{\mathcal{A} \mid N>0\}P\{N>0\}) \\ &= (\lim_{\Delta\to 0} P\{N=0\}) \lim_{\Delta\to 0} \frac{P\{\mathcal{A} \mid N=0\}}{\Delta} + (\lim_{\Delta\to 0} P\{N>0\}) \lim_{\Delta\to 0} \frac{P\{\mathcal{A} \mid N>0\}}{\Delta} .\end{aligned} \tag{10}$$

We now invoke (2), i.e., $P\{N=0\}\to 1$ as $\Delta\to 0$. This, of course, implies that $P\{N>0\}\to 0$ as $\Delta\to 0$. The desired result (9) would follow if the second limit in the second term on the right-hand side of (10) were finite. Indeed, this is the case because this term is no larger than $\overline{N}_l$ since $P\{\mathcal{A} \mid N>0\} \le P\{\mathcal{A}\}$.

We now proceed to evaluate (9) by expressing the probability in (9) as a double integral over the underlying joint probability density function with limits determined by the system equation (1). Equating the up-crossings and the down-crossings, to first order in Δ in the limits of integration, we have

$$\overline{N}_l = \lim_{\Delta\to 0} \frac{2}{\Delta} \int_{l-M}^{l} dy_0 \int_{l}^{l+M} dy\, p(y,y_0 \mid N=0) , \tag{11}$$

in which $M=\beta\Delta[1-F(l)]$. The probability density in the integrand can be further expressed as

$$\begin{aligned}p(y,y_0 \mid N=0) &= p(y \mid N=0,y_0,x_0=1)P\{x_0=1 \mid N=0,y_0\}p(y_0 \mid N=0) \\ &\quad + p(y \mid N=0,y_0,x_0=-1)P\{x_0=-1 \mid N=0,y_0\}p(y_0 \mid N=0) ,\end{aligned} \tag{12}$$

where $x_0=x(0)$. Now, the probability densities $p(y \mid N=0,y_0,x_0=\pm 1)$ must be δ functions since, starting at an initial state (x_0,y_0), the system will evolve in a deterministic way if the input remains fixed ($N=0$) according to the system equation (1). Hence, using (12) in (11) gives

$$\overline{N}_l = \lim_{\Delta\to 0} \frac{2}{\Delta} \int_{l-M}^{l} dy_0 P\{x_0=1 \mid N=0,y_0\}p(y_0 \mid N=0) \tag{13}$$

$$= 2\beta[1-F(l)]P\{x_0=1 \mid y_0=l\}p(l) \tag{14}$$

$$= \beta[1-F^2(l)]p(l) .$$

As $\Delta\to 0$, $N(\Delta)\to N(0)=0$ and since this is the certain event, it can be dropped in going from (13) to (14). Also, (8) was used to get the final result.

III. EXAMPLES

This section gives four examples of linear systems, $F(y)=y$, driven by non-Markov dichotomous noise in which explicit expressions for $\overline{N}_l$ are known. In the first three, the dichotomous noise has intervals generated by an equilibrium renewal process, and in the fourth it is limited RC Gaussian noise, the sgn$[\eta(t)]$ discussed in Sec. I. The theory was validated by Monte Carlo simulations in each example. Although Monte Carlo methods can rarely "prove" anything, it it interesting to see how close the actual and Monte Carlo results are. Since the agreement is so good, a table of values is given only for the first example.

The fifth example considers the nonlinear case $F(y)=k\,\mathrm{sgn}y$. Unfortunately, there are no known cases in which $p(y)$ or $\overline{N}_l$ is available when $F(y)$ is nonlinear. Instead of going through the Monte Carlo simulations, the time averages which they estimate can be done analytically and, in this way, the general result (3) deduced for dichotomous noise with an arbitrary equilibrium renewal interval density.

In the first four examples, $\beta=1$ for convenience, and the level crossing rates are related to their respective densities by $\overline{N}_l=(1-l^2)p(l)$; $|l| \le 1$. In the last example, $\overline{N}_l=\beta(1-k^2)p(l)$; $|l| < \infty$. The interval density, when appropriate, will be denoted by $f(t)$.

A. $F(y)=y$, gamma interval density

For the particular gamma interval density

$$f(t)=\frac{t}{4}e^{-t/2}, \quad t\geq 0 \tag{15}$$

the probability density $p(y)$ is a hypergeometric function [Ref. 10 (1986), Sec. 6]. Using the known result for it [Ref. 10 (1986), Eq. (54b)], we have

$$\overline{N}_l=\frac{|\Gamma(1-q)|^2}{\pi^{3/2}}(1-l^2)^{1/2}\ {}_2F_1(q,q^*;\tfrac{1}{2};l^2)\ , \tag{16}$$

in which ${}_2F_1$ denotes a hypergeometric function and $q=(1+i)/4$. Each of the intervals of the dichotomous noise was generated as the sum of two independent and identically distributed random variables with the exponential density $f(t)=\exp(-t)$ (variables with exponential densities are easily obtained from uniform variates by a logarithmic transformation).

10 000 intervals of the binary process were generated and filtered and the level crossings counted. The Monte Carlo results are compared in Table I with the exact result (16) and the agreement is to within 2% in all cases.

B. $F(y)=y$, McFadden interval density, $a=1$, $b=4$

In this case [Ref. 10 (1986), Sec. 7],

$$f(t)=3e^{-t}(1-e^{-t})^2, \quad t\geq 0 \tag{17}$$

and

$$\overline{N}_l=\tfrac{3}{44}(1-l^2)(7+l^2)\ . \tag{18}$$

Each of the intervals of the dichotomous noise was generated as the sum of three independent random variables with respective densities $\exp(-t)$, $2\exp(-2t)$, and $3\exp(-3t)$. Again, agreement between the Monte Carlo and exact results was excellent. For example, when $l=0.8$, the exact $\overline{N}_l=0.188$ and the Monte Carlo gave 0.191.

C. $F(y)=y$, McFadden interval density, $a=6$, $b=41$

In this case, the probability density turns out to have a bimodal shape [Ref. 10 (1986), Sec. 7, Fig. 4]. We have

$$f(t)=\frac{e^{-6t}(1-e^{-t})^{34}}{B(6,35)}, \quad t\geq 0 \tag{19}$$

$$\overline{N}_l=\frac{1}{\mu}\left[1-\sum_{k=0}^{5}\binom{46}{k}\{z^k(1-z)^{46-k}+z^{46-k}(1-z)^k\}\right], \tag{20}$$

where

$$z=\frac{1+l}{2},\ \mu=\sum_{k=0}^{34}\frac{1}{k+6}=1.995\,20\ldots\ . \tag{21}$$

Each of the intervals of the binary process was generated as the sum of 35 independent random variables with respective densities $6\exp(-6t),\ldots,40\exp(-40t)$. For $l=0.6$, the exact and Monte Carlo values of $\overline{N}_l$ were 0.461 and 0.465, respectively.

TABLE I. Comparison of exact and Monte Carlo results for $F(y)=y$ and gamma interval density.

l	$\overline{N}_l$ Monte Carlo	$\overline{N}_l$ Exact
0	0.227	0.231
0.2	0.225	0.229
0.4	0.217	0.221
0.6	0.202	0.206
0.8	0.173	0.176
0.9	0.143	0.145
0.95	0.117	0.117
0.99	0.0691	0.0678

D. $F(y)=y$, Filter-limiter-filter

In this example[11–13], $p(y)$ is known in closed form only when $\beta\tau_c=2$. The intervals of the dichotomous noise are no longer statistically independent and the statistics of the intervals are unknown. The result, using the result of Doyle *et al.* for $p(y)$,[11] is

$$\overline{N}_l=\frac{1}{\pi}(1-l^2)^{1/2}\ . \tag{22}$$

The simulations are more involved because the RC noise has to be generated. The same methods as employed in Ref. 13 were used and the exact and Monte Carlo results agreed to within 2% in all cases.

E. $F(y)=k\ \mathrm{sgn}y$, Equilibrium renewal interval density

This is a system for which an exact solution is not known. The system nonlinearity is

$$F(y)=k\ \mathrm{sgn}y, \quad 0<k<1$$

and the dichotomous noise is generated by an equilibrium renewal process with interval density $f(t)$. In this case the output $y(t)$ is a piecewise linear function that goes "away" from the origin with slopes $\pm\beta(1-k)$ and "towards" the origin with slopes $\pm\beta(1+k)$. Since the slope towards the origin is greater than that away, the effect is that the time trajectory looks like a random walk with a restoring force that tends to drive the particle back to the origin. For any given set of interval times, $y(t)$ can be exactly constructed from (1), and a typical trajectory obtained in this way is shown in Fig. 1.

Since neither $\overline{N}_l$ or $p(l)$ is known, they can both be estimated by Monte Carlo methods from a sample trajectory as time averages. As in the previous examples, $\overline{N}_l$ is obtained by counting the number of crossings with level $y=l$ in a time T. The density $p(l)$ is more difficult to estimate as the difference of the probability distributions $P\{y\leq l+\Delta\}$ and $P\{y\leq l\}$ divided by Δ for some small Δ, and each of the distributions as the percentage of time $y\leq l+\Delta$ or $y\leq l$, respectively.

However, instead of going through the Monte Carlo simulations, analytical expressions can be written for the time averages and, from these, the desired result obtained. With the exception of the limits $\Delta\to 0$ and $T\to\infty$, these operations are identical to those performed

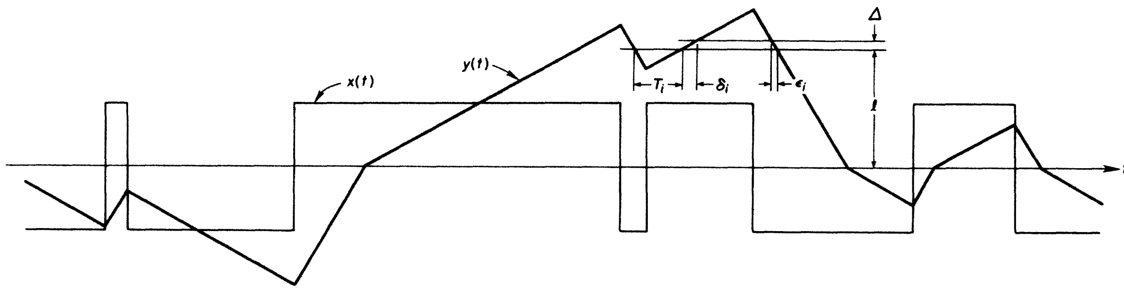

FIG. 1. Typical trajectory for $F(y)=k$ sgn$y$ and equilibrium renewal interval density.

in a Monte Carlo simulation. For each up-crossing with the level $y=l$, there is of course an ensuing down-crossing. Consider a pair of these and let T_i denote the time interval between the onset of the up-crossing and the down-crossing of the *previous* pair. Then, as shown in Fig. 1, the total time that $y(t)$ spends below the level $y=l$ for the *present* pair is T_i. The total time that the trajectory is below the level $y=l+\Delta$ for the *present* pair is then $T_i+\delta_i+\epsilon_i$ where δ_i is the time for the up-crossing to go between l and $l+\Delta$, and ϵ_i is the time for the down-crossing to go between $l+\Delta$ and l. Adding up all these times in a long interval of length T leads to

$$p(l)=\lim_{\Delta\to 0}\frac{1}{\Delta}[P\{y\le l+\Delta\}-P\{y\le l\}]$$

$$=\lim_{\substack{\Delta\to 0\\ T\to\infty}}\frac{1}{\Delta}\left[\frac{\sum_i (T_i+\delta_i+\epsilon_i)-\sum_i T_i}{T}\right].$$

Now, because of the different slopes in going through the up- and down-crossings, we have

$$\delta_i=\frac{\Delta}{\beta(1-k)};\ \epsilon_i=\frac{\Delta}{\beta(1+k)},$$

and, consequently,

$$p(l)=\lim_{\substack{\Delta\to 0\\ T\to\infty}}\frac{1}{T\Delta}\sum_i\left[\frac{\Delta}{\beta(1-k)}+\frac{\Delta}{\beta(1+k)}\right]$$

$$=\frac{\bar{N}_l}{\beta(1-k^2)},$$

and (3) is established for $F(y)=k$ sgn$y$.

This time averaging could, of course, be carried through for a general nonlinearity $F(y)$. However, because of the anomalies of processes like the dichotomous noise sgn$[\eta(t)]$, it may not be possible to carry it through in general.

IV. DISCUSSION

Although (3) holds for very general binary processes, the determination of $p(y)$ in the general case is still a difficult and unsolved problem. Exact results are available for the random telegraph signal for both linear and nonlinear $F(y)$. Some closed-form results are available for nonexponential intervals and computational methods have been investigated when there are no closed forms.[10,13]

It is instructive to compare the work of Kac and Rice on level crossings of Gaussian processes to see what light, if any, it sheds on the present paper. Rice's expression for the average number of level crossings per unit time of a Gaussian process can be written as [Ref. 17, Eq. (3.3)-(13)]

$$\bar{N}_l=\bar{N}_0\frac{p(l)}{p(0)},\quad |l|<\infty \tag{23}$$

in which

$$p(l)=(2\pi\psi_0)^{-1/2}\exp(-l^2/2\psi_0),$$

$$\bar{N}_0=\pi^{-1}(-\ddot{\psi}(0+)/\psi_0)^{1/2},$$

$\psi(\tau)$ is the autocorrelation function of the process, and $\psi_0=\psi(0)$. Our expression (3) can be put into the form

$$\bar{N}_l=\bar{N}_0\frac{1-F^2(l)}{1-F^2(0)}\frac{p(l)}{p(0)},\quad |F(l)|\le 1 \tag{24}$$

where $\bar{N}_0=\beta[1-F^2(0)]p(0)$. Both of these can be obtained from

$$\bar{N}_l=\bar{N}_0\frac{B(l)p(l)}{B(0)p(0)} \tag{25}$$

by appropriate definitions of $B(l)$, $p(l)$, and $\bar{N}_0$.

If $B(y)$ is the second conditional moment in the *extended* Fokker-Planck equation for $p(y)$, then (23) and (24) each follows from (25). To see this in greater detail, it is necessary to examine the *extended* Fokker-Planck equation which is[19]

$$\frac{1}{2}\frac{d^2}{dy^2}[B(y)p(y)]-\frac{d}{dy}[A(y)p(y)]=0, \tag{26}$$

in which

$$A(y)=\lim_{\epsilon\to 0}\frac{1}{\epsilon^\nu}E[y(t+\epsilon)-y(t)\,|\,y(t)], \tag{27}$$

$$B(y)=\lim_{\epsilon\to 0}\frac{1}{\epsilon^\nu}E[\{y(t+\epsilon)-y(t)\}^2\,|\,y(t)]. \tag{28}$$

When $\nu=1$, this reduces to the classical Fokker-Planck equation; however, the classical equation degenerates to $0=0$ in some cases [because $A(y)$ and $B(y)$ are each zero] and it is then necessary to use $\nu>1$.[19]

For a Gaussian process (with $\nu=1$), $B(y)=\psi_0$ and (25) yields (23).

For the nonlinearly filtered dichotomous noise of the present paper, it is necessary to use $\nu=2$ (see Ref. 1). It is readily shown from the system equation (1) that $B(y)=\beta^2[1-F^2(y)]$. Now (25) yields (24).

Based upon these, it is tempting to speculate that (25) with the *extended* Fokker-Planck second conditional moment might hold for any stationary mean-square differentiable random process. If this were the case, one implication would be that (26) can *always* be written as[9]

$$\frac{1}{2}\frac{d}{dy}[B(y)p(y)]-A(y)p(y)=0 . \tag{29}$$

That the constant of integration on the right-hand side is indeed zero follows by integrating the left-hand side over the range of y, noting that $B(y)p(y)$ vanishes at the endpoint by (25) and that $E[A(y)]$ must vanish by (27). The solution to (29) is

$$B(y)p(y)=c\exp\left[\int^{y}\frac{2A(y')}{B(y')}dy'\right] , \tag{30}$$

where c is a constant. Also, if the $y(t)$ process were such that $y_{\min}\le y\le y_{\max}$, it is then necessary that $B(y)p(y)$ vanish at both endpoints since there cannot be any level crossings at the endpoints even if $p(y)$ were asymmetric.

Our general result (3) gives insight into level crossing rates and, in some cases, makes their calculation more tractable. In view of the fact that there has been extensive recent interest in the first-passage time problem for (1),[6,7] it is quite natural to ask if there is any connection between level crossing rates and first-passage times. On the surface, mean first-passage times appear to be much more difficult to determine and it would be highly desirable if some simpler way could be found to obtain them; i.e., specifically in some way through level crossing rates. This, we feel, is a fruitful area for future investigation.

The speculation $\bar{N}_l\sim B(l)p(l)$ stresses the utility of the *extended* Fokker-Planck equation and gives an interesting interpretation of the second conditional moment $B(y)$. Recently, the extended Fokker-Planck equation has also been shown to play a key role in extensions of Pearson's method from statistics for approximating distributions from moments.[9]

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