Effect of cooperative atomic interactions on the natural linewidth of a single-mode laser

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The effect of cooperative atomic interactions on laser linewidth is investigated by suitably generalizing the Scully-Lamb theory of the laser. In particular, the single- and two-atom interactions with the field are included. It is shown that, under certain conditions, the cooperative interactions tend to reduce the natural linewidth.

I. INTRODUCTION

It is well known that ultimately the laser linewidth arises due to spontaneous emission. It is of considerable interest to find systems and effects that reduce the laser linewidth,[1,2] with potential applications in areas such as gravitational wave detection[3,4] and optical tests of metric theories of gravitation[5] in mind.

The linewidth of a single-mode laser can be evaluated in the Scully-Lamb model of the laser.[6] In this model a laser consists of a coupled system of a field and identical two-level atoms. This model is a single-atom model and it ignores the cooperative atomic interactions inside a laser. Recently the effect of cooperative atomic interactions on the photon statistics of a single-mode laser has been studied by many authors using a perturbative approach.[7–9] A nonperturbative treatment has been given in Ref. 10.

In this paper, we consider the effect of cooperative atomic interactions on the linewidth of a single-mode laser, by suitably generalizing the Scully-Lamb theory.[10] We show that the cooperative effects tend to decrease the natural linewidth of the laser at high intensities. In Sec. II we consider interaction of two two-level atoms with a single-mode quantized field and derive an expression for the field density matrix of the system. In Sec. III we derive an equation of motion for the field density matrix of the laser which includes the effects of cooperative atomic interactions. We then solve this equation to obtain an expression for the natural linewidth of the system. In Sec. IV we give numerical results and discuss our results.

II. INTERACTION OF TWO-LEVEL ATOMS WITH A SINGLE-MODE FIELD

We consider a system of two two-level atoms interacting with a single-mode quantized electromagnetic field. For simplicity, we assume the field to be at resonance with the atomic transitions. The Hamiltonian for the system, in the interaction picture, is therefore

$$H_I=\hbar\sum_{i=1}^{2} g_i(a\sigma_i^\dagger+a^\dagger\sigma_i)\ , \tag{1}$$

where a and $a^\dagger$ are the creation and annihilation operators for the field, $\sigma_i^\dagger,\sigma_i$ are the raising and lowering operators, and g_i is the atom-field coupling constant for the ith atom which contains the mode function $u(r_i)$, i.e., $g_i\equiv g(r_i)$.

The wave function $|\psi\rangle$ for the system can be written in terms of eigenstates $|\alpha,\beta,n\rangle=|\alpha,\beta\rangle\otimes|n\rangle$ where $|\alpha\rangle$ and $|\beta\rangle$ represent the states of atom 1 and atom 2, respectively, and $|n\rangle$ denotes the number state of the field. With $C_{\alpha\beta n}$ as the probability amplitude for the state $|\alpha,\beta,n\rangle$, $|\psi\rangle$ can be written as

$$\begin{aligned}|\psi(t)\rangle=C_{aan}(t)\,|a,a,n\rangle&+C_{ab\,n+1}(t)\,|a,b,n+1\rangle\\&+C_{ba\,n+1}(t)\,|b,a,n+1\rangle\\&+C_{bb\,n+2}(t)\,|b,b,n+2\rangle\ .\end{aligned} \tag{2}$$

The states $|a\rangle$ and $|b\rangle$ denote the upper and lower levels of the atoms, respectively.

The wave function $|\psi\rangle$ obeys the equation of motion

$$\frac{\partial}{\partial t}|\psi(t)\rangle=\frac{-i}{\hbar}H_I\,|\psi(t)\rangle\ . \tag{3}$$

With the help of Eqs. (1) and (2), Eq. (3) becomes

$$\dot{\tilde C}=-i\tilde M\tilde C\ , \tag{4}$$

where

$$\tilde C=\begin{bmatrix}C_{aan}\\C_{ab\,n+1}\\C_{ba\,n+1}\\C_{bb\,n+2}\end{bmatrix}\ , \tag{5a}$$

$$\tilde M=\begin{bmatrix}0&g_2\sqrt{n+1}&g_1\sqrt{n+1}&0\\g_2^*\sqrt{n+1}&0&0&g_1\sqrt{n+2}\\g_1^*\sqrt{n+1}&0&0&g_2\sqrt{n+2}\\0&g_1^*\sqrt{n+2}&g_2^*\sqrt{n+2}&0\end{bmatrix}\ . \tag{5b}$$

A solution of Eq. (4) is [with $C_1=C_{aan}$, $C_2=C_{ab\,n+1}$, $C_3=C_{ba\,n+1}$, and $C_4=C_{bb\,n+2}$ (Ref. 10)]

$$C_i(t+\tau)=\sum_{j=1}^{4}\sum_{k=1}^{4}\alpha_i^k e^{-i\lambda_k\tau}\alpha_j^{k*}C_j(t)\ , \tag{6}$$

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where t is some initial time, λ_k are the eigenvalues of matrix M, and α_i^k is the ith element of the corresponding eigenvector. The eigenvalues are determined to be

$$\lambda_k = \pm\frac{1}{\sqrt{2}}[(g_1^2+g_2^2)(2n+3)\pm\beta]^{1/2} , \quad (7)$$

where

$$\beta=[(2n+3)^2(g_1^2+g_2^2)^2 -4(n+1)(n+2)(g_1^2-g_2^2)^2]^{1/2} . \quad (8)$$

In Eq. (7), the upper sign inside the bracket is for $k=1,2$ and the lower sign is for $k=3,4$. The elements of the corresponding eigenvectors are

$$\alpha_1^k=\pm\frac{\lambda_k^2-(g_1^2+g_2^2)(n+2)}{[\beta^2\mp\beta(g_1^2+g_2^2)]^{1/2}} , \quad (9a)$$

$$\alpha_2^k=\pm\frac{[\lambda_k^2+(g_1^2-g_2^2)(n+2)]g_2\sqrt{n+1}}{\lambda_k[\beta^2\mp\beta(g_1^2+g_2^2)]^{1/2}} , \quad (9b)$$

$$\alpha_3^k=\pm\frac{[\lambda_k^2-(g_1^2-g_2^2)(n+2)]g_1\sqrt{n+1}}{\lambda_k[\beta^2\mp\beta(g_1^2+g_2^2)]^{1/2}} , \quad (9c)$$

$$\alpha_4^k=\pm\frac{2g_1g_2[(n+1)(n+2)]^{1/2}}{[\beta^2\mp\beta(g_1^2+g_2^2)]^{1/2}} . \quad (9d)$$

In the denominators, the upper sign is for $k=1,2$ and the lower sign is for $k=3,4$.

It therefore follows from Eq. (6) that, if at initial time t, the atoms are in the excited state $[C_2(t)=C_3(t)=C_4(t)=0]$, the probability amplitudes $C_i(t+\tau)$ at a later time are given by

$$C_i(t+\tau)=\kappa_i(\tau)C_1(t), \quad i=1,2,3,4 \quad (10)$$

where the coefficients κ_i are

$$\kappa_1(\tau)=\frac{[\beta-(g_1^2+g_2^2)]}{2\beta}\cos(\lambda_1\tau) +\frac{[\beta+(g_1^2+g_2^2)]}{2\beta}\cos(\lambda_3\tau) , \quad (11a)$$

$$\kappa_2(\tau)=-\frac{ig_2\sqrt{n+1}}{2\beta}\left[\frac{(4n+7)g_1^2-g_2^2+\beta}{\lambda_1}\sin(\lambda_1\tau) -\frac{(4n+7)g_1^2-g_2^2-\beta}{\lambda_3}\sin(\lambda_3\tau)\right] , \quad (11b)$$

$$\kappa_3(\tau)=-\frac{ig_1\sqrt{n+1}}{2\beta}\left[\frac{(4n+7)g_2^2-g_1^2+\beta}{\lambda_1}\sin(\lambda_1\tau) -\frac{(4n+7)g_2^2-g_1^2-\beta}{\lambda_3}\sin(\lambda_3\tau)\right] , \quad (11c)$$

$$\kappa_4(\tau)=\frac{2g_1g_2[(n+1)(n+2)]^{1/2}}{\beta}[\cos(\lambda_1\tau)-\cos(\lambda_3\tau)] . \quad (11d)$$

In physical situations the laser field is described by a mixture of states. The matrix elements for the field density matrix are given by

$$\rho_{nm}(t+\tau)=P_\psi[C_{aan}(t+\tau)C^*_{aam}(t+\tau) +C_{abn}(t+\tau)C^*_{abm}(t+\tau) +C_{ban}(t+\tau)C^*_{bam}(t+\tau) +C_{bbn}(t+\tau)C^*_{bbm}(t+\tau)] . \quad (12)$$

Here we have carried out the incoherent summations over the wave function amplitudes with P_ψ as the probability for the field to be in state $|\psi\rangle$. The probability amplitudes in Eq. (12) can be obtained by shifting the value of n appropriately in Eq. (10).

III. OFF-DIAGONAL ELEMENTS AND THE NATURAL LINEWIDTH

We want to investigate the effects of cooperative interactions, between two atoms, on the laser linewidth when the laser is oscillating at resonance. The model for this study is the same as that used in Ref. 10. In the present model, to describe cooperative effects in a laser, we restrict ourselves to a single-mode field and therefore we do not account for super-radiance. An equation of motion for the reduced density operator for the field ρ_{nm} can be obtained by taking the average coarse-grained time rate of change, due to the gain medium:

$$\dot\rho_{nm}=r\gamma\int_0^\infty d\tau e^{-\gamma\tau}\left[\sum_\alpha \rho_{an\,am}(t+\tau)-\rho_{n\,m}(t)\right] , \quad (13)$$

where r is the rate at which a pair of atoms is pumped in the excited state $|a\rangle$ and

$$p(\tau)\gamma e^{-\gamma\tau} \quad (14)$$

simulates the spontaneous emission where γ is the decay constant of the atomic levels. On substituting from Eqs. (10)–(12) into Eq. (13) and performing the integration over τ we get (with $\rho_n^k\equiv\rho_{n\,n+k}$)

$$\rho_n^k= A_n^k\rho_n^k+B_{n-1}^k\rho_{n-1}^k+C_{n+1}^k\rho_{n+1}^k+D_{n-2}^k\rho_{n-2}^k , \quad (15)$$

where

$$A_n^k=(N_A-2A^2N_Q)/N_Q(B+D)-C(n+\tfrac{1}{2}k) , \quad (16a)$$

$$B_n^k=N_B/N_Q , \quad (16b)$$

$$C_n^k=C[n(n+k)]^{1/2} , \quad (16c)$$

and

$$D_n^k=N_D[(n+1)(n+2)(n+k+1)(n+k+2)]^{1/2}N_Q . \quad (16d)$$

The expressions for N_A, N_B, N_D, and N_Q are rather lengthy and are given in the Appendix. The coefficients $A=2r(g_1^2+g_2^2)/\gamma^2$ and $B=8r(g_1^4+g_2^4)/\gamma^4$ are gain and self-saturation parameters, respectively, and the coefficient $D=16rg_1^2g_2^2/\gamma^4$ represents the cooperative atomic interactions in the laser. The cavity loss parameter $C=\omega/Q$ is included in the usual way.[6,11] Note that, in the limit $k=0$, the coefficients in Eqs. (15a)–(16d) become the diagonal coefficients as given in Ref. 10. Similarly, without cooperative interactions, i.e., with $D=0$, we obtain exactly the same coefficients as are given in Ref. 11.

Until now we considered a system in which atoms were being pumped in pairs. We now extend our system to a situation in which N atoms are being pumped at the rate γ. We keep only the single- and two-atom interactions with the field. It can be shown that the equation of motion for ρ_n^k remains the same except that the coefficients A, B, and D have the following generalized expressions:

$$A=\frac{2r}{\gamma^2}\sum_{i=1}^{N} g_i^2 , \tag{17a}$$

$$B=\frac{8r}{\gamma^4}\sum_{i=1}^{N} g_i^4 , \tag{17b}$$

$$D=\frac{16r}{\gamma^4}\sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j}}^{N} g_i^2 g_j^2 , \tag{17c}$$

where g_i is the coupling constant for the ith atom. It is worthwhile to mention that the cooperative parameter D is significant over the region where the coupling constants g_i and g_j overlap. It vanishes when the separation between two atoms is large due to the nature of the mode functions contained in the coupling constants. In the steady state an exact solution for the diagonal elements of the density matrix is given by[7]

$$\rho_n^0=\frac{(-1)^n\rho_0}{C_1^0\times C_2^0\times\cdots\times C_n^0}M(n) , \tag{18}$$

where

$$M(n)=\det\begin{vmatrix} A_0^0 & C_1^0 & 0 & \cdots & & \\ A_0^0 & A_1^0 & C_2^0 & & & \\ D_0^0 & B_1^0 & A_2^0 & & & \\ 0 & D_1^0 & B_2^0 & & & \\ \vdots & & & & & C_{n-1}^0 \\ & & & D_{n-3}^0 & B_{n-2}^0 & A_{n-1}^0 \end{vmatrix} . \tag{19}$$

The quantity ρ_0^0 is determined by the normalization condition. It follows from Eqs. (16) that

$$A_n^0+B_n^0+C_n^0+D_n^0=0 . \tag{20}$$

Using this condition and some properties of the determinant, we can express the solution as a product of continued fractions:

$$\rho_n^0=\rho_0^0\prod_{m=1}^{m} Q_m , \tag{21}$$

where

$$Q_m=\frac{1}{C_m^0}\left[\alpha_{m-1}^0+\cfrac{\beta_{m-1}^0}{\alpha_{m-2}^0+\cfrac{\beta_{m-2}^0}{\alpha_{m-3}^0+\cdots\cfrac{\beta_1^0}{\alpha_0^0}}}\right] \tag{22}$$

with

$$\alpha_m^0=B_m^0+D_m^0 , \tag{23a}$$

$$\beta_m^0=C_m^0D_{m-1}^0 . \tag{23b}$$

We can now calculate the off-diagonal elements $\rho_n^k(t)$ by making the following ansatz:[11]

$$\rho_n^k(t)=e^{-\mu(k)t}(\rho_n^0\rho_{n+k}^0)^{1/2} . \tag{24}$$

On substituting from Eq. (21) into Eq. (24), it follows from Eq. (15) that

$$\begin{aligned}\mu(k)=&-A_{\bar n}^k-(Q_{\bar n}Q_{\bar n+k})^{-1/2}B_{\bar n-1}^k\\ &-(Q_{\bar n-1}Q_{\bar n}Q_{\bar n+k-1}Q_{\bar n+k})^{-1/2}D_{\bar n-2}^k\\ &-(Q_{\bar n+k}Q_{\bar n+k+1})^{1/2}C_{\bar n+1}^k ,\end{aligned} \tag{25}$$

where we have replaced n by $\bar n$.

IV. RESULTS AND DISCUSSION

The laser linewidth W is given by $\mu(1)/2$.[11] Our results, for $D=0$, are exactly the same as those in Ref. 11. In Tables I and II we compare the linewidth W in the absence of cooperative effects ($D=0$) to the linewidth W in the presence of cooperative effects. It is evident that the cooperative atomic interactions tend to decrease the laser linewidth for higher intensities.

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APPENDIX

We have the following expressions for N_A, N_B, N_D, and N_Q:

$$\begin{aligned}N_A=&A_{10}A^{10}+A_{20}A^9(B+D)+A^8[A_{30}(B+D)^2+A_{31}(B^2-D^2)]+A^7[A_{40}(B+D)^3+A_{41}(B+D)(B^2-D^2)]\\ &+A^6[A_{50}(B+D)^4+A_{51}(B+D)^2(B^2-D^2)+A_{52}(B^2-D^2)^2]\\ &+A^5[A_{60}(B+D)^5+A_{61}(B+D)^3(B^2-D^2)+A_{62}(B+D)(B^2-D^2)^2]\\ &+A^4[A_{70}(B+D)^6+A_{71}(B+D)^4(B^2-D^2)+A_{72}(B+D)^2(B^2-D^2)^2+A_{73}(B^2-D^2)^3]\\ &+A[A_{80}(B+D)^7A_{81}(B+D)^5(B^2-D^2)+A_{82}(B+D^3)(B^2-D^2)^2+A_{83}(B+D)(B^2-D^2)^3] ,\end{aligned}$$

TABLE I. Linewidth as a function of mean photon number $\bar{n}$ and cooperation parameter D with $A=100\ \text{sec}^{-1}$ and $B=0.1\ \text{sec}^{-1}$.

$\bar{n}$ \ D	0.0	0.005	0.01	0.05	0.1
10	2.372 84	2.372 99	2.373 13	2.374 21	2.375 43
20	1.200 80	1.200 84	1.200 88	1.201 16	1.201 38
30	0.801 541	0.801 552	0.801 561	0.801 589	0.801 498
40	0.600 058	0.600 050	0.600 042	0.599 931	0.599 683
50	0.478 388	0.478 366	0.478 343	0.478 125	0.477 760
60	0.396 683	0.396 646	0.396 608	0.396 275	0.395 785
70	0.337 595	0.337 535	0.337 475	0.336 977	0.336 309
80	0.291 967	0.291 866	0.291 766	0.290 967	0.289 970
90	0.252 599	0.252 391	0.252 184	0.250 592	0.248 736

$$
\begin{aligned}
N_B = {} & B_1 A^9 + A^8(B_{20}B + B_{21}D) + A^7[B_{30}(B+D)^2 + (B+D)(B_{31}B + B_{32}D) + B_{33}(B^2-D^2)] \\
& + A^6[B_{40}(B+D)^3 + (B_{41}B + B_{42}D)(B+D)^2 + (B_{43}B + B_{44}D)(B^2-D^2)] \\
& + A^5[B_{50}(B+D)^4 + (B+D^3)(B_{51}+B_{52}) + (B+D)(B^2-D^2)(B_{53}B + B_{54}D) + B_{55}(B^2-D^2)^2] \\
& + A^4[B_{60}(B+D)^5 + (B+D)^4(B_{61}B + B_{62}D) + (B+D)^2(B^2-D^2)(B_{63}B + B_{64}D) + (B+D)(B^2-D^2)^2 B_{65} \\
& \quad + (B^2-D^2)^2(B_{66}B + B_{67}D)] \\
& \quad + A^3[(B+D)^5(B_{70}B + B_{71}D) + (B+D)^3(B^2-D^2)(B_{72}B + B_{73}D) \\
& \qquad + (B+D)^2(B^2-D^2)^2 B_{74} + (B+D)(B^2-D^2)^2(B_{75}B + B_{76}D + B_{77}(B^2-D^2)^3] \,,
\end{aligned}
$$

$$
\begin{aligned}
N_D = {} & A^3 D\{A^5 D_1 + A^4 D_2(B+D) + A^3[(B+D)^2 D_{30} + (B^2-D^2)D_{31}] + A^2[D_{40}(B+D)^3 + D_{41}(B+D)(B^2-D^2)] \\
& \quad + A[D_{50}(B+D)^4 + D_{51}(B+D)^2(B^2-D^2) + D_{52}(B^2-D^2)^2] \\
& \quad + D_{61}(B+D)^3(B^2-D^2) + D_{62}(B+D)(B^2-D^2)^2\} \,,
\end{aligned}
$$

$$
\begin{aligned}
N_Q = {} & A^8 Q_1 + A^7 Q_2(B+D) + A^6[Q_{30}(B+D)^2 + Q_{31}(B^2-D^2)] + A^5[Q_{40}(B+D)^3 + Q_{41}(B+D)(B^2-D^2)] \\
& + A^4[Q_{50}(B+D)^4 + Q_{51}(B+D)^2(B^2-D^2) + Q_{52}(B^2-D^2)^2] \\
& + A^3[Q_{60}(B+D)^5 + Q_{61}(B+D)^3(B^2-D^2) + Q_{62}(B+D)(B^2-D^2)^2] \\
& + A^2[Q_{70}(B+D)^6 + Q_{71}(B+D)^4(B^2-D^2) + Q_{72}(B+D)^2(B^2-D^2)^2 + Q_{73}(B^2-D^2)^3] \\
& + A[Q_{81}(B+D)^5(B^2-D^2) + Q_{82}(B+D)^3(B^2-D^2)^2 + Q_{83}(B+D)(B^2-D^2)^3] \\
& \quad + Q_{92}(B+D)^4(B^2-D^2)^2 + Q_{93}(B+D)^2(B^2-D^2)^3 + Q_{94}(B^2-D^2)^4 \,,
\end{aligned}
$$

where

TABLE II. Linewidth as a function of mean photon number $\bar{n}$ and cooperation parameter D with $A=100\ \text{sec}^{-1}$ and $B=1\ \text{sec}^{-1}$.

$\bar{n}$ \ D	0.0	0.01	0.05	0.1	0.5	1.0
10	2.222 51	2.222 67	2.223 29	2.224 06	2.229 14	2.231 35
20	1.092 72	1.092 78	1.093 03	1.093 32	1.094 43	1.091 94
30	0.712 128	0.712 210	0.712 525	0.712 886	0.714 056	0.711 254
40	0.521 613	0.521 719	0.522 126	0.522 585	0.524 051	0.521 220
50	0.407 763	0.407 887	0.408 354	0.408 876	0.410 484	0.407 613
60	0.332 428	0.332 559	0.333 052	0.333 594	0.335 130	0.332 213
70	0.278 900	0.279 023	0.279 476	0.279 960	0.281 067	0.278 055
80	0.238 142	0.238 217	0.238 487	0.238 750	0.238 750	0.235 463
90	0.202 949	0.202 868	0.202 554	0.202 173	0.199 231	0.195 071

$$A_{10}=2(4)^8 \ ,$$

$$A_{20}=2(4)^7(7A_5+1) \ ,$$

$$A_{30}=\tfrac{1}{2}(4)^6(72A_5^2+16A_5+4k^2-A_1-A_2+2) \ ,$$

$$A_{31}=6(4)^6(A_3+A_4) \ ,$$

$$A_{40}=\tfrac{1}{2}(4)^5[(8A_5^2+4k^2-A_1-A_2)(10A_5+2)+10A_5(A_5^2-k^2)+18A_5-2k^2)] \ ,$$

$$A_{41}=\tfrac{1}{2}(4)^5[(A_3+A_4)(20A_5-16)+4k^2A_5] \ ,$$

$$A_{50}=2(4)^4\{8A_5^4+2A_5^3+(4-5k^2)A_5^2+(6k^2+A_1+A_2)A_5+[\tfrac{5}{2}k^4-\tfrac{7}{4}k^2+\tfrac{1}{4}(A_1+A_2)+\tfrac{1}{2}A_1A_2]\} \ ,$$

$$A_{52}=2(4)^4[3(A_3+A_4)^2-68A_3A_4] \ ,$$

$$A_{51}=(4)^4[(A_3+A_4)(-12A_5^2-16A_5+8k^2-2)+A_5^2(-4A_5+4+108k^2)+33(A_2A_3+A_1A_4)+(A_1A_3+A_2A_4)] \ ,$$

$$\begin{aligned}A_{60}=&\,32(6A^2+2k^2)(A_5^3-k^2A_5-A_5-2k^2)+64A_5(A_5^2-k^2)(6A_5^2+2k^2-6)+32A_1A_2(6A_5+2)\\&+192[(A_5^2+k^2)(A_5-2k-A_5^2-k^2)+k(A_3+A_4)(1-2A_5)-(A_5-1)A_1A_2+2kA_5]\\&+128A_5(3A_5^2-3A_5^4+8k^2A_5+3k^4+k^2) \ ,\end{aligned}$$

$$\begin{aligned}A_{61}=&\,32(6A_5^2+2k^2)(5k^2A_5+A_5-A_5^3)\\&+64(A_3+A_4)[2(A_5^3-k^2A_5-A_5-2k^2)+4A_5(A_5^2-k^2)-6A_5-2(A_1+A_2)+4k^2]\\&+128A_5(2k^2-A_3-A_4)(6A_5^2+2k^2-6)\\&-32[3(A_2A_3+A_1A_4)+(A_1A_3+A_2A_4)-4k^2(A_3+A_4)](6A_5+2)\\&+384[(A_3+A_4)(A_1+A_2-A_5+2k)+2k^2A_5(1-2A_5)+2A_5(A_5-1)(A_2A_3+A_1A_4)]\\&+256A_5[A_3+A_4+4k^2A_5+6(A_2A_3+A_1A_4)] \ ,\end{aligned}$$

$$A_{62}=32\{2(A_3+A_4)[2A_5(13k^2+1-A_5^2-A_3-A_4)+6(A_3+A_4)]+16A_3A_4(2-15A_5)-24(A_3-A_4)^2\} \ ,$$

$$\begin{aligned}A_{70}=&\,32A_5(A_5^2-k^2)(A_5^3-k^2A_5-A_5-2k^2)+4A_1A_2(6A_5^2+2k^2-6)\\&+32A_5[(A_5^2+k^2)(A_5-2k-A_5^2-k^2)+k(A_3+A_4)(1-2A_5)-(A_5-1)A_1A_2+2kA_5]\\&+4[A_5(3A_5-3A_5^3+8k^2)+k^2(3k^2+1)](A_1+A_2-4k^2) \ ,\end{aligned}$$

$$\begin{aligned}A_{71}=&\,64A_5(2k^2-A_3-A_4)(A_5^3-k^2A_5-A_5-2k^2)+32A_5^2(A_5^2-k^2)(5k^2+1-A_5^2)\\&+2(A_3+A_4)[8A_1A_2-8(3A_5^2-3A_5^4+8k^2A_5+3k^2+3k^4+k^2)]\\&-4(6A_5^2+2k^2-6)[3(A_2A_3+A_1A_4)+(A_1A_3+A_2A_4)-4k^2(A_3+A_4)]\\&+64A_5[(A_3+A_4)(A_1+A_2-A_5+2k)+2k^2A_5(1-2A_5)+2(A_5-1)(A_2A_3+A_1A_4)]\\&+8(A_1+A_2-4k^2)[A_3+A_4+4k^2A_5+6(A_2A_3+A_1A_4)] \ ,\end{aligned}$$

$$\begin{aligned}A_{72}=&\,8A_5^2(2k^2-A_3-A_4)(5k^2+1-A_5^2)+8[(A_3+A_4)^2+4A_3A_4](6A_5^2+2k^2-6)\\&-16(A_3+A_4)[3(A_2A_3+A_1A_4)+(A_1A_3+A_2A_4)-4k^2(A_3+A_4)]\\&+2[A_3+A_4+4k^2A_5+6(A_2A_3+A_1A_4)]-128A_5[(4A_5A_3A_4)+(A_3-A_4)^2]-92A_3A_4(A_1+A_2-4k^2) \ ,\end{aligned}$$

$$A_{73}=32(A_3+A_4)[(A_3+A_4)^2+16A_3A_4] \ ,$$

$$\begin{aligned}A_{80}=&\,2A_1A_2(A_5^3-k^2A_5-A_5-2k^2)\\&+(A_1+A_2-4k^2)[(A_5^2+k^2)(A_5-2k-A_5^2-k^2)+k(A_3+A_4)(1-2A_5)-(A_5-1)A_1A_2+2kA_5] \ ,\end{aligned}$$

$$\begin{aligned}A_{81}=&\,4A_1A_2A_5(2k^2-A_3-A_4)\\&-2[3(A_2A_3+A_1A_4)+(A_1A_3+A_2A_4)-4k^2(A_3+A_4)](A_5^3-k^2A_5-A_5-2k^2)\\&+2(A_1+A_2-4k^2)^2[(A_3+A_4)(A_1+A_2-A_5+2k)+2k^2A_5(1-2A_5)+2(A_5-1)(A_2A_3+A_1A_4)]\\&-16(A_3+A_4)[4A_5A_3A_4+(A_3-A_4)^2]\end{aligned}$$

$$A_{82}=4[(A_3+A_4)^2+4A_3A_4](A_5^3-k^2A_5-A_5-2k^2)$$
$$-2[3(A_2A_3+A_1A_4)+(A_1A_3+A_2A_4)-4k^2(A_3+A_4)]A_5(5k^2+1-A_5^2)$$
$$-4(A_1+A_2-4k^2)[4A_5A_3A_4+(A_3-A_4)^2]$$
$$-8[(A_3+A_4)(A_1+A_2-A_5+2k)+2k^2A_5(1-2A_5)+2(A_5-1)(A_2A_3+A_1A_4)](A_3+A_4)\ ,$$

$$A_{83}=4[(A_3+A_4)^2+2A_3A_4](5k^2A_5-A_5^3+A_5)+4(A_3+A_4)[4A_3A_4A_5+(A_3-A_4)^2]\ ,$$

$$B_1=2(4^7)\ ,$$

$$B_{20}=4^7(3A_5+1)\ ,$$

$$B_{21}=4^7(A_5-1)\ ,$$

$$B_{30}=4^5(23A_5^2+5k^2)\ ,$$

$$B_{31}=4^5(16A_5+5)\ ,$$

$$B_{32}=-4^5(22A_5^2+18k^2+16A_5+5)\ ,$$

$$B_{33}=2(4^5)(A_3+A_4)\ ,$$

$$B_{40}=6(4^5)[A_5(A_5^2+k^2)]\ ,$$

$$B_{41}=4^5(6A_5^2+4k^2+3A_5)\ ,$$

$$B_{42}=4^5[2A_5^2(A_5+1)-3A_5(6k^2+1)-4k^2]\ ,$$

$$B_{43}=-2(4^5)(A_3+A_4)(A_5+2)-2k^2A_5\ ,$$

$$B_{44}=2(4^5)[3(A_3+A_4)A_5-4k^2A_5+2(A_3+A_4)]\ ,$$

$$B_{50}=4^3(3A_5^2+k^2)^2\ ,$$

$$B_{51}=4^3[4(A_3+A_4)(4k^2-3A_5^2)+8(2A_5+1)(A_5^2+k^2)+(A_5^2-k^2)$$
$$+4k^2A_5(3A_5+2)(16A_5^3+9A_5^2+7k^2+12k^2A_5^2+24k^2A_5)]\ ,$$

$$B_{52}=2(4^3)[2(A_3+A_4)(4A_5^2-k^2)+5(A_5^2-k^2)^2-(A_5^2-k^2)(28A_5^2+1)-4A_5(2A_5^2+3k^2)$$
$$+2(14A_5^2+18k^2-3k^2A_5^2)]\ ,$$

$$B_{53}=2(4^3)[4k^2A_5-(A_3+A_4)(16A_5+5)]\ ,$$

$$B_{54}=2(4^3)[4k^2A_5(10A_5-13)-(A_3+A_4)(26A_5^2+6k^2-24A_5-5)]\ ,$$

$$B_{55}=-2(4^3)(A_3^2+A_4^2+22A_3A_4)\ ,$$

$$B_{60}=32A_5(A_5^2+3k^2)(A_5^2-k^2)\ ,$$

$$B_{61}=(A_1+A_2)[64A_5+8(A_1+A_2)-32k^2+32A_5(2k^2-A_3-A_4)]$$
$$+(A_3+A_4)[-64A_5(A_5^2-k^2)]-32A_4(A_5^2-k^2)+64A_1A_2+256A_5k^r$$
$$+32(A_2A_3+A_1A_4)-64k(A_1-A_2)-32(A_1A_3+A_2A_4)]\ ,$$

$$B_{62}=(4)^3\{2A_5[\tfrac{1}{4}(A_1+A_2)(A_1+A_2-4k^2-1)-(A_2A_3+A_1A_4)+\tfrac{1}{2}k(A_1A_7-A_2A_6)$$
$$+\tfrac{3}{4}(A_2A_3+A_1A_4)+\tfrac{1}{4}(A_1A_3+A_2A_4)-k^2(A_3+A_4)]$$
$$+2A_5(A_1+A_2-4k^2)[\tfrac{1}{4}(A_3+A_4)-k^2-\tfrac{1}{8}(A_1+A_2)-\tfrac{1}{2}k^2-\tfrac{1}{8}]$$
$$-\tfrac{1}{2}A_1A_2(2A_5+1)-\tfrac{1}{2}A_5(2k^2-A_3-A_4)(A_1+A_2+4k^2)-\tfrac{1}{8}(A_2A_6+A_1A_7)(A_1+A_2-4k^2)\}\ ,$$

$$B_{63}=(4)^3[(A_3+A_4)(A_1+A_2-A_5+4k^2)-4kA_5(A_4-A_3)-\tfrac{3}{2}(A_2A_3+A_1A_4)-(A_1A_3+A_2A_4)-2k^2A_5]\ ,$$

$$B_{64}=(4)^3\{(A_3+A_4)[-2A_5(A_1+A_2-4k^2-1)+\tfrac{1}{2}(A_2A_6+A_1A_7)]$$
$$+\tfrac{1}{2}(A_1+A_2-4k^2)(A_4A_6+A_3A_7)-A_5(2k^2-A_3-A_4)(A_1+A_2+4k^2-1)+\tfrac{1}{2}(2A_5+1)$$
$$\times[3(A_2A_3+A_1A_4)+(A_1A_3+A_2A_4)-4k^2(A_3+A_4)]+4kA_5(A_4A_6-A_3A_7)\}\ ,$$

$$B_{65}=64A_5[3(A_3+A_4)^2-4k^2(A_3+A_4)-4A_3A_4]\ ,$$

$$B_{66}=64[3(A_3+A_4)^2+4A_3A_4]\ ,$$

$$B_{67}=-128\{(A_3+A_4)(A_4A_6+A_3A_7)+8(2A_5+1)[(A_3+A_4)^2+4A_3A_4]\}\ ,$$

$$B_{70}=(A_1+A_2+4k^2)[A_1A_2-3(A_2A_3+A_1A_4)-(A_1A_3+A_2A_4)+4k^2(A_3+A_4)+\tfrac{1}{16}(A_1+A_2-4k^2)]$$
$$-2A_1A_2+(A_1+A_2-4k^2)[4(A_2A_3+A_1A_4)-A_1A_2-2k(A_1-A_2)\ ,$$

$$B_{71}=(A_1+A_2+4k^2)[A_1A_2+3(A_2A_3+A_1A_4)+(A_1A_3+A_2A_4)$$
$$-4k^2(A_3+A_4)+\tfrac{1}{16}(A_1+A_2-4k^2-1)(A_1+A_2-4k^2)]$$
$$+(A_1+A_2-4k^2)[2k(A_1A_7-A_2A_6)-4(A_2A_3+A_1A_4)]-2A_1A_2[A_1+A_2+8k^2-1+2(A_3+A_4)]\ ,$$

$$B_{72}=-4(A_3+A_4)[A_1+A_2-2k(A_1-A_2)]+16[3(A_2A_3+A_1A_4)+(A_1A_3+A_2A_4)-4k^2(A_3+A_4)]$$
$$-8k(A_4-A_3)(A_1+A_2-4k^2)\ ,$$

$$B_{73}=-4(A_1+A_2-4k^2-1)(A_1+A_2)(A_3+A_4)$$
$$+[6(A_2A_3+A_1A_4)+2(A_1A_3+A_2A_4)-8k^2(A_3+A_4)](A_1+A_4+4k^2-1)$$
$$+\frac{k}{2}[(A_4A_6-A_3A_7)(A_1+A_2-4k^2)-(A_1A_7-A_2A_6)(A_3+A_4)]\ ,$$

$$B_{74}=2(A_1+A_2)[(A_3+A_4)^2-4A_3A_4]+8k^2[(A_3+A_4)^2+12A_3A_4]$$
$$-(A_3+A_4)[4(A_2A_3+A_1A_4)-(A_1A_3+A_2A_4)+4k^2(A_3+A_4)]\ ,$$

$$B_{75}=4(A_3-A_4)^2+32k(A_4^2-A_3^2)\ ,$$

$$B_{76}=4(A_3+A_4)^2(A_1+A_2)-8(4k^2+1)(A_3+A_4)^2$$
$$-32k(A_3+A_4)(A_4A_6-A_3A_7)-4(4k^2-1)(A_3+A_4)^2-16A_3A_4(A_1+A_2+4k^2-1)\ ,$$

$$B_{77}=-8(A_3+A_4)(A_3-A_4)^2\ ,$$

$$D_1=6(4^7)\ ,$$

$$D_2=(4^6)18A_5\ ,$$

$$D_{30}=2(4^5)(9A_5^2-k^2)\ ,$$

$$D_{31}=-4^6(A_3+A_4)\ ,$$

$$D_{40}=2(4^4)A_5(3A_5^2+k^2)\ ,$$

$$D_{41}=4^5A_5(A_3+A_4-6k^2)\ ,$$

$$D_{50}=+4^4(A_5^2-k^2)(2k^2-A_5^2)\ ,$$

$$D_{51}=4^4[(A_3+A_4)(2A_5^2-k^2)-3k^2A_5^2]\ ,$$

$$D_{52}=-2(4^3)[5(A_3A_4)^2+12A_3A_4]\ ,$$

$$D_{60}=0\ ,$$

$$D_{61}=(4^3)k^2A_5(A_3+A_4-k^2)\ ,$$

$$D_{62}=32A_5[(A_3+A_4)^2+4A_3A_4-4k^2(A_3+A_4)]\ ,$$

$$Q_1=4^8\ ,$$

$$Q_2=2(4^8)A_5\ ,$$

$$Q_{30}=2(4^6)(11A_5^2+k^2)\ ,$$

$$Q_{31}=4^7(A_1+A_4)\ ,$$

$$Q_{40}=4^6A_5(7A_5^2+k^2)\ ,$$

$$Q_{41}=2(4^6)A_5(A_3+A_4)+2k^2 ,$$

$$Q_{50}=4^4[(A_5^2-k^2)(A_5^2+7k^2)+16k^4] ,$$

$$Q_{51}=4^5[(A_3+A_4)(3A_5^2+10k^2)-k^2A_5^2] ,$$

$$Q_{52}=2(4^4)(3A_3^2+3A_4^2-130A_3A_4) ,$$

$$Q_{60}=4^4A_5(A_5^2-k^2)(A_5^2+3k^2) ,$$

$$Q_{61}=4^5A_5[(A_5^2+k^2-1)(\tfrac{1}{2}A_5^2+k^2)-k^2(2A_5^2-k^2)] ,$$

$$Q_{62}=2(4^4)A_5[k^2(A_5^2+2k^2-2)-24A_3A_4] ,$$

$$Q_{70}=64k^2(A_5^2-k^2)^2 ,$$

$$Q_{71}=64(A_5^2-k^2)[A_5^4-A_5^2+k^2(A_3+A_4)-2k^2A_5^2] ,$$

$$Q_{72}=64[k^2A_5^2(A_5^2-15k^2-1)-\tfrac{1}{2}(A_3+A_4)^2+(9A_5^2+11k^2)-14(A_5^2-k^2)A_3A_4 ,$$

$$Q_{73}=64A_3+A_4[(A_3+A_4)^2+28A_3A_4] ,$$

$$Q_{81}=-16k^2A_5(A_5^2-k^2)(3A_5^2-3k^2-1) ,$$

$$Q_{82}=-16k^2A_5[A_5^2(A_5^2-5k^2-1)+4(A_3+A_4)^2] ,$$

$$Q_{83}=32k^2A_5[(A_3+A_4)(k^2-1)+8A_3A_4] ,$$

$$Q_{92}=k^4(A_5^2-k^2+1)^2 ,$$

$$Q_{93}=-2k^4A_5^2(A_5^2-k^2+1) ,$$

$$Q_{94}=k^4A_5^4 ,$$

with

$$A_1=(2n+3)^2 ,$$

$$A_2=(2n+2k+3)^2 ,$$

$$A_3=(n+1)(n+2) ,$$

$$A_4=(n+k+1)(n+k+2) ,$$

$$A_5=(2n+k+3) ,$$

$$A_6=(4n+7) ,$$

$$A_7=(4n+4k+7) .$$

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