Pendular Fabry-Pérot cavities as a paradigm for the dynamics of systems with delays

Ll. Bel, J-L. Boulanger, and N. Deruelle
Laboratoire de Physique Théorique, Institut Henri Poincaré, 11 rue Pierre et Marie Curie, 75005 Paris, France
(Received 6 April 1987)

We consider the dynamics of the suspended mirror of a pendular Fabry-Pérot cavity, taking into account the delays due to the finiteness of the velocity of light. We obtain an approximate equation for the asymptotic motion which allows a simple analysis of the behavior of the mirror and in particular of the instabilities that cavities with a large enough finesse or length can develop. The results, which are of relevance in cases of actual interest, can in principle be tested using presently available cavities. A pendular Fabry-Pérot cavity then appears as a perhaps ideal example of a delay system where theory, numerical, and laboratory experiments can be compared.

I. INTRODUCTION

Delay differential equations (also known as retarded, hereditary, difference differential equations, or equations with deviating arguments), that is, differential equations for a function x evaluated not only at time t but also at the retarded argument $(t-r)$ (the delay or time lag r being in general a function of t and x), occur in the description of countless physical phenomena, ranging from population dynamics[1] to acoustics,[2] optics[3,4] and classical field theories,[5,6] when the finite velocity of the propagation of the interaction must be accounted for.

The mathematical theory of such equations, that we shall henceforth assume to be purely retarded and with constant delays (see Refs. 7–9 for precise definitions), is poorly known, except when linear with constant coefficients, and exact solutions cannot in general be given in a closed form. Indeed, in contrast with ordinary differential equations, the solutions of delay equations, whatever their order of differentiation, depend in general on an initial function specified over some interval of length r, that is on an infinite set of initial conditions. For generic initial data, the solution is not C^{∞}, but is, however, of increasing smoothness: it is C^1 from $t=0$ onwards and C^n from $t=(n-1)r$ onwards.

In general, one then must resort to approximation schemes. One of them, initiated by Lagrange,[10] consists in transforming the retarded equation into an ordinary differential equation by replacing the function of the retarded argument by the first terms of a Taylor series, that is, by setting

$$x(t-r)=x(t)-r\dot{x}(t)$$

or

$$x(t-r)=x(t)-r\dot{x}(t)+r^2\ddot{x}(t)/2 ,$$

for example. Although often efficient, this procedure can be grossly incorrect, even for arbitrarily small delays, if the order of the resulting ordinary equation is higher than that of the original delay equation, unless an appropriate order reduction is performed.[11]

A more elaborate approach, associating a predictive, that is, ordinary, differential equation to the retarded system, has therefore been developed in order to tackle the problem more rigorously. It was initially conceived to solve the problem of motion of particles in classical field theories[12,13] and relied on the analytic dependence of the system on a tuning parameter (or coupling constant). It has been recently adapted to allow a description of the evolution from static to "laminar" and even "turbulent" regimes in processes involving delays.[14–17] The approach essentially exploits the seemingly general property of spontaneous prediction,[14–16] that is of rapid oblivion of the initial conditions. This allows one to attempt to approximate asymptotically the general solution of the retarded equation (which depends on an arbitrary initial function) by the general solution of an ordinary differential equation (which depends on a finite set of initial conditions). More precisely, the method consists in constructing a cascade of dominant reductions, that is a series of ordinary differential equations, each of which pertains to a different regime of the system, and of which the solutions asymptotically approach the exact solution in the regime considered. The order of differentiation of the dominant reduction is believed to increase as the behavior of the system evolves towards an increasing complexity.[17]

This concept of a cascade of dominant reductions as a road to chaos specific to retarded systems has been partially tested numerically[17] on a model of the hybrid device recently studied in nonlinear optics.[18,19] It, however, needs further support since, mathematically, the approach can be justified by plausibility arguments only.

We wish to argue here that Fabry-Pérot cavities (see Fig. 1), in which one of the semitransparent plates is replaced by a harmonically suspended mirror, might be a good paradigm for the dynamics of retarded systems: they could illustrate the different facets of the method and probe its validity, either by means of numerical simulations, comparing solutions of the exact (retarded) equation of motion with the solutions of the successive dominant reductions, or by means of laboratory experiments.

Indeed the parameter which characterizes the different regimes is not merely the time lag, that is the round-trip travel time τ of light in the cavity, which is usually very short. Rather, it is the ratio of the finesse F of the cavity (which also measures the number of round trips of light necessary to fill up the cavity) and the number of round trips of light during a period $2\pi/\Omega_0$ of oscillation of the mirror. In other words, for large $F\tau\Omega_0$, the delays must certainly be accounted for. However, it must be stressed that even when $F\tau\Omega_0$ is small, the delays might be of importance, even though further simplifications of the method are then possible (see below). Hence, currently available cavities, in particular those presently constructed as mock-ups of the future interferometric detectors of gravitational radiation, should be sensitive and flexible enough to be used to test the theory. The multistability due to the nonlinear radiation pressure force and the predicted instability caused by the time lag should be easily detectable.

The dynamics of a pendular Fabry-Pérot cavity were initially investigated by Dorsel *et al.*[20] In Refs. 21 and 22 Deruelle and Tourrenc studied the effects of the time delay in the special configuration where the equilibrium position of the mirror corresponds to a maximum of the radiation force. In this particular case, using the method of Lagrange, they obtained an approximate ordinary differential system describing the motion, up to first order in terms of the delay and up to second order in terms of the mirror displacement from its equilibrium position. More recently, Aguirregabiria and Bel[23] derived from the linearized hereditary differential system the characteristic equation, which allowed them to study the bifurcation between stable and unstable solutions. They concluded that instabilities due to delays certainly cannot be ignored for values of the parameters pertinent to the future interferometric detectors of gravitational radiation. This completed some previous work by one of us.[24] However, these results are not sufficient to describe the asymptotic evolution of the system in detail.

In this paper we shall adapt the method used in Ref. 23, which is based on the series Liapounoff introduced in his so-called first method[25] (see Appendix A for an elementary introduction to this technique). We shall derive the dominant reduction of the equation of motion, for high finesse but arbitrary delays, and conclude by discussing in some detail the case of small delays.

II. DERIVATION OF THE APPROXIMATE DOMINANT REDUCTION

A. Outline of the method

We start from the exact hereditary differential equation describing the motion of the mirror[21] [see Eqs. (2)–(4) below]. We then expand it about a stationary solution in terms of the displacement. Following Liapounoff's first method (cf. Appendix A), we seek its general solution $X(t)$ under the form of a sum over the characteristic roots z_i of the linearized retarded equation. Hence we substitute

$$X(t)=k\exp(zt)+\bar{k}\exp(\bar{z}t)+hk^2\exp(2zt)+gk\bar{k}\exp(z+\bar{z})t+\bar{h}\,\bar{k}^2\exp(2\bar{z}t)+\cdots\,, \quad (1)$$

where z and its complex conjugate $\bar{z}$ are roots of the characteristic equation, and k is an arbitrary constant, into the hereditary differential equation. The parameters $h,g,\ldots$ are then formally determined by identifying successive powers of k and $\bar{k}$.

Now when either the finesse or the length of the cavity is large enough, the roots of the characteristic equation with the largest real part, z_1 and $\bar{z}_1$ say, have a positive real part (with $z_1\neq\bar{z}_1$), as shown in Ref. 23, so that the development (1) is dubious. However, there exists a suitable combination of $X(t)$ and its first and second derivatives which presumably does converge. Neglecting the terms which asymptotically tend to zero, we thus obtain the dominant reduction of the retarded system (expanded here up to the second order in terms of the displacement), that is the ordinary differential equation (of the second order in this case) whose general solution is (1), in which all the k but k_1 are set equal to zero. In some cases to be discussed below, the dominant reduction can be further simplified by an expansion in terms of the delay.

B. The hereditary differential system

Consider the Fabry-Pérot cavity of Fig. 1. The motion of the mirror is determined by the combined action of the mechanical restoring force and the radiation pressure of light. Its equation of motion has been obtained in Ref. 21. We shall write it in the form (see caption of Fig. 1 for notations):

$$\ddot{x}+\frac{\dot{x}}{Q}+x+x_0=(A\sin^2\theta)f\bar{f}\,, \quad (2)$$

where $x=4\pi[D(t)-D_m]/\lambda$ and $x_0=4\pi(D_m-D_0)/\lambda$, $D_m=\lambda(2m+1)/4$ being a resonance length of the cavity, and t the time in units of ω^{-1}. The constant A is $A=8\pi P/cM\lambda\omega^2$, c being the speed of light. f is equal, up to a phase, to $E/(P^{1/2}\sin\theta)$ where E is the electromagnetic field on the mirror at time t:

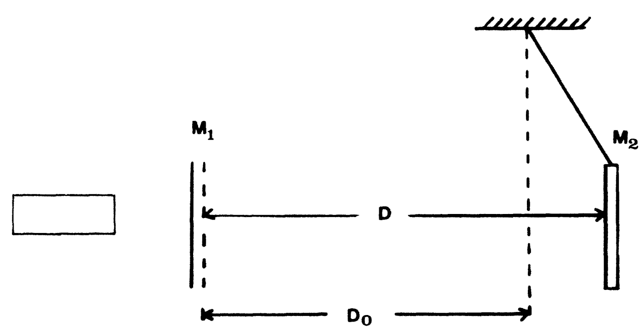

FIG. 1. The parameters of the cavity. P is the laser power, λ its wavelength. $R=\cos\theta$ is the reflectivity of the fixed mirror M_1. The mobile mirror M_2 (supposed to be without losses), of mass M, angular frequency ω, and quality factor Q is suspended to a wire anchored at D_0. The total length of the cavity is $D(t)$.

$$f(t)=1+\rho\exp[i(x_{(1)}-x_s)]f(t-r_{(1)}) \ . \tag{3a}$$

A recurrent application[23] yields

$$f=1+\sum_{n\geq 1}\rho^n\exp i\left[\sum_{1\leq j\leq n}(x_{(j)}-x_s)\right] , \tag{3b}$$

where x_s is the equilibrium position, $\rho=\cos\theta\exp ix_s$, and $x_{(j)}=x(t-r_j)$, r_j being the time needed by the light to make j round trips in the cavity ending at time t. In the static case, $x_{(j)}=$const, $f\bar{f}$ is proportional to the usual Airy function. In the following we shall in fact set $r_j=jr$ with $r=2D_0\omega/c$, which in all practical cases is an excellent approximation. The equilibrium position x_s is the stationary solution of Eqs. (2) and (3) and is related to the suspension point x_0 by

$$x_0+x_s=\frac{A\sin^2\theta}{(1-\rho)(1-\bar{\rho})} \tag{4}$$

(see Fig. 2).

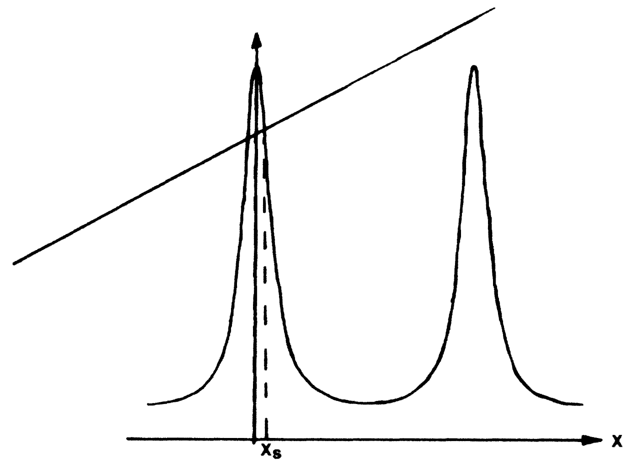

FIG. 2. The positions of equilibrium of the mirror. These lie at the intersection of the opposite of the mechanical restoring force (a straight line) and the radiation pressure force (an Airy function). When their number exceeds 1 the system is multistable (see Refs. 20–22), and the equilibrium position is chosen as shown in the figure.

C. The characteristic equation

In order to obtain the characteristic equation, we set

$$x(t)-x_s=X(t) , \tag{5}$$

and linearize f in X. Inserting the ansatz (1) for $X(t)$ into f and Eq. (2), we recover, at zeroth order in k, the equation (4) of the equilibrium configuration, and obtain at first order in k the characteristic equation

$$z^2+\frac{z}{Q}+1=\frac{A\sin^2\theta}{(1-\rho)(1-\bar{\rho})}\frac{i\Lambda(\rho-\bar{\rho})}{(1-\Lambda\rho)(1-\Lambda\bar{\rho})} , \tag{6}$$

where

$$\Lambda=\exp(-zr) \ . \tag{7}$$

This equation has already been analyzed under a different form in Ref. 23. In particular, the threshold at which the real part of z_1 and $\bar{z}_1$—the roots with the largest real part—becomes positive, has been determined. In the case when θ is small and x_s lies within a resonance peak (cf. Fig. 2), we set $y_s=2x_s\theta^{-2}$ so that $y_s=O(1)$. When $|z|r\theta^{-2}\ll 1$ the characteristic equation reduces to

$$z^2+\left[Q^{-1}-\frac{8ry_s}{\alpha\theta^2(1+y_s^2)^3}\right]z+\left[1+\frac{2y_s}{\alpha(1+y_s^2)^2}\right]=0 , \tag{8}$$

where $\alpha=\theta^4/8A$. The system develops a linear instability when the coefficient of z becomes negative (see below for application to cavities of special interest). The point to stress here is that the expansion parameter is not merely the round-trip travel time of light in the cavity, but $zr\theta^{-2}$, that is, as mentioned in the Introduction, $F\tau\Omega_0$ where Ω_0 is the effective angular frequency of the mirror, and the finesse F is $F=2\pi/\theta^2$.

If we call z_1 and $\bar{z}_1$ the solutions of (8), the linearized equation of motion can be written as

$$\ddot{y}-(z_1+\bar{z}_1)\dot{y}+z_1\bar{z}_1y=0 , \tag{9}$$

where $y=2(x-x_s)\theta^{-2}$.

D. The approximate dominant reduction

We now proceed to construct, up to the second order in the displacement, the dominant reduction of the hereditary equation of motion, that is the ordinary differential equation whose general solution is conjectured to approach the general solution of the retarded equation asymptotically. This equation reads (for $z_1\neq\bar{z}_1$)

$$\ddot{X}-(z_1+\bar{z}_1)\dot{X}+z_1\bar{z}_1X=a_{20}X^2+a_{11}X\dot{X}+a_{02}\dot{X}^2 , \tag{10}$$

where the coefficients a_{ij} are to be determined.

To this end, we first expand f up to the second order in $x(t)-x_s=X(t)$ and replace $X(t)$ by the ansatz (1). The development of f up to the second order in terms of k and $\bar{k}$ then yields, for $t=0$,

$$(1-\rho)f=1+i\rho\left[\frac{k\Lambda}{1-\Lambda\rho}+\frac{\bar{k}\bar{\Lambda}}{1-\bar{\Lambda}\rho}+\frac{gk\bar{k}\Lambda\bar{\Lambda}}{1-\Lambda\bar{\Lambda}\rho}+\frac{hk^2\Lambda^2}{1-\Lambda^2\rho}+\frac{h\bar{k}^2\bar{\Lambda}^2}{1-\bar{\Lambda}^2\rho}\right]$$
$$-\frac{\rho}{2}\left[\frac{2k\bar{k}\Lambda\bar{\Lambda}(1-\rho^2\Lambda\bar{\Lambda})}{(1-\Lambda\rho)(1-\bar{\Lambda}\rho)(1-\Lambda\bar{\Lambda}\rho)}+\frac{k^2\Lambda^2(1+\Lambda\rho)}{(1-\Lambda\rho)(1-\Lambda^2\rho)}+\frac{\bar{k}^2\bar{\Lambda}^2(1+\bar{\Lambda}\rho)}{(1-\bar{\Lambda}\rho)(1-\bar{\Lambda}^2\rho)}\right] . \tag{11}$$

To get $\bar{f}$—see (3)—we transform ρ into $\bar{\rho}$ and i into $-i$ (leaving Λ and k unchanged, because x is real). Substituting (1) into (2) and using (11), we recover (4), (6) and (7) by identifying the terms constant and linear in k and $\bar{k}$. As for h and g, they are obtained by identifying the terms quadratic in k and $\bar{k}$. They read

$$h=\frac{Q(\Lambda)}{4z^2+2z/Q+1-P(\Lambda)},\quad \bar{h}=h(\bar{\Lambda},\bar{z}) \tag{12}$$

$$g=\frac{S}{(z+\bar{z})^2+(z+\bar{z})/Q+1-R}, \tag{13}$$

where z is a solution of the characteristic equation (6), (7) and where

$$P(\Lambda)=\frac{i(x_s+x_0)\Lambda^2(\rho-\bar{\rho})}{(1-\rho\Lambda^2)(1-\bar{\rho}\Lambda^2)}, \tag{14}$$

$$Q(\Lambda)=(x_s+x_0)\Lambda^2\left[\frac{\rho\bar{\rho}}{(1-\rho\Lambda)(1-\bar{\rho}\Lambda)}-\frac{\rho(1+\rho\Lambda)}{2(1-\rho\Lambda)(1-\rho\Lambda^2)}-\frac{\bar{\rho}(1+\bar{\rho}\Lambda)}{2(1-\bar{\rho}\Lambda)(1-\bar{\rho}\Lambda^2)}\right], \tag{15}$$

$$R=\frac{i(x_s+x_0)\Lambda\bar{\Lambda}(\rho-\bar{\rho})}{(1-\rho\Lambda\bar{\Lambda})(1-\bar{\rho}\Lambda\bar{\Lambda})}, \tag{16}$$

$$S=(x_s+x_0)\Lambda\bar{\Lambda}\left[\frac{\rho\bar{\rho}}{(1-\rho\Lambda)(1-\bar{\rho}\bar{\Lambda})}+\frac{\rho\bar{\rho}}{(1-\rho\bar{\Lambda})(1-\bar{\rho}\Lambda)}-\frac{\rho(1-\rho^2\Lambda\bar{\Lambda})}{(1-\rho\Lambda)(1-\rho\bar{\Lambda})(1-\rho\Lambda\bar{\Lambda})}-\frac{\bar{\rho}(1-\bar{\rho}^2\Lambda\bar{\Lambda})}{(1-\bar{\rho}\bar{\Lambda})(1-\bar{\rho}\Lambda)(1-\bar{\rho}\Lambda\bar{\Lambda})}\right]. \tag{17}$$

The general solution at the approximation considered is now known in terms of the characteristic roots of the linearized equation, provided that the series (1) converges. If the real part of z_1 and $\bar{z}_1$ becomes positive, the solution is not asymptotically stable and the development (1) is meaningless. Following the method outlined previously and in Appendix A, we look for the combination of X and its derivatives which does converge. Its limit is the dominant reduction, Eq. (10), whose coefficients a_{ij} are obtained by inserting the formal expressions (1) and (12)–(17) for X into Eq. (10), and then identifying the terms quadratic in k and $\bar{k}$

$$a_{20}(z-\bar{z})^2=hz(2z-\bar{z})+\bar{h}\bar{z}(2\bar{z}-z)-z\bar{z}g, \tag{18}$$

$$a_{11}(z-\bar{z})^2=z\bar{z}[-2h(2z-\bar{z})-2\bar{h}(2\bar{z}-z)+g(z+\bar{z})], \tag{19}$$

$$a_{02}(z-\bar{z})^2=z\bar{z}[h\bar{z}(2z-\bar{z})+\bar{h}z(2\bar{z}-z)-gz\bar{z}], \tag{20}$$

where h, $\bar{h}$, and g are given by Eqs. (12)–(17) and where $z=z_1$.

This completes the obtainment of the approximate dominant reduction for arbitrary delays. Note that when $z_1=\bar{z}_1$ the a_{ij} are not defined. Should that happen, the method would have to be slightly modified.

Let us now give an expansion of the dominant reduction [Eqs. (10) and (18)–(20)] in terms of the delay, valid when $F\tau\Omega_0\ll 1$ and for small θ. A rather long but straightforward calculation yields

$$\ddot{y}+K\dot{y}+\Omega^2 y=0, \tag{21}$$

where y is the variable defined in (9) and where the "damping" function K and the "angular frequency" Ω are given by

$$K=\left[Q^{-1}-\frac{8ry_s}{\alpha\theta^2(1+y_s^2)^3}\right]-\frac{8r(1-5y_s^2)}{\alpha\theta^2(1+y_s^2)^4}y, \tag{22}$$

$$\Omega^2=\left[1+\frac{2y_s}{\alpha(1+y_s^2)^2}\right]+\frac{1-3y_s^2}{\alpha(1+y_s^2)^3}y. \tag{23}$$

This expansion of the dominant reduction can, in fact, be derived directly from the exact hereditary equations of motion (2)–(4); see Appendix B.

III. DISCUSSION

The preceding section has essentially consisted in replacing the exact equation for the displacement X of the mobile mirror of the Fabry-Pérot cavity represented in Fig. 1 by a much simpler equation, the "dominant reduction," of the form

$$\ddot{X}-\dot{X}[(z_1+\bar{z}_1)+a_{11}X+a_{02}\dot{X}]+X(z_1\bar{z}_1-a_{20}X)=0, \tag{24}$$

where z_1 and $\bar{z}_1$ are the roots with the largest real part of equation (6) and where the coefficients a_{ij} are defined by (18)–(20) and (12)–(17).

When the delay can be neglected, Eq. (24) reduces to

$$\ddot{X}+\dot{X}/Q+(z_1\bar{z}_1-a_{20}X)X=0, \tag{25}$$

and for very small displacements the motion is harmonic; its natural frequency ω becoming $\Omega_0=(z_1\bar{z}_1)^{1/2}\omega$ due to the radiation pressure force (see Fig. 2). When the term $a_{20}X$ becomes significant, the equation is elliptic and its solution is known.

When the delay can no longer be neglected and for very small displacements, Eq. (24) reduces to

$$X-(z_1+\bar{z}_1)X+z_1\bar{z}_1X=0\ , \tag{26}$$

and the motion is linearly unstable if the real part of z_1 is positive. In that case the nonlinear terms grow, and when they cannot be neglected any more, the sign of the coefficient of $\dot{X}$ in (24) may vary in time so that the nature of the attractor of the dynamical system is not obvious *a priori*. Preliminary numerical studies of the exact equation of motion indicate, however, that the system can approach a limit cycle.

Let us now consider cavities of actual interest, such as those planned to detect gravitational radiation and their present prototypes.[26] Typical values for the parameters (see Fig. 1) are

$$Q\approx 10^6,\quad \lambda\approx 0.6\ \mu\text{m},\quad \omega\approx 2\pi\ \text{rad s}^{-1} \tag{27}$$

and

$$P/M\approx 10\ \text{W kg}^{-1}\ , \tag{28}$$

a value relevant for the future detectors ($P\approx 1$ kW, $M\approx 100$ kg) as well as their prototypes ($P\approx 1$ W, $M\approx 100$ g). These values yield $A\approx 3.5\times 10^{-2}$. On the other hand, the cavities are very different as regards the delays. Since the reflectivity of the mobile mirror can have various values (ranging from $\theta\approx 10^{-1}$ to $\theta\approx 10^{-1.5}$) and since the length D of the gravitational wave detectors will be of the order of 1 km when their present prototypes are 1 m long, we shall leave θ and D as free parameters.

Suppose first that Eq. (24) can be expanded up to the first order in terms of the delay. We showed in the preceding section that it then reduces to Eqs. (21)–(23). With the values chosen for the parameters it reads

$$\ddot{y}+\dot{y}[Q^{-1}-(1.2\times 10^{-8})D\theta^{-6}(1-2y)]+y[1+0.14\theta^{-4}(1-0.5y)]=0\ . \tag{29}$$

To be specific we chose in (29) $y_s=1$ (see Fig. 2). The case $y_s=0$ (equilibrium position at a maximum of the radiation pressure force) has been considered in Refs. 21–22. It is clear that for all the values of D (1 m $<D<1$ km) and θ ($0.03<\theta<0.1$) considered, and for small displacements ($y\ll 1$), Eq. (29) can be further simplified to

$$\ddot{y}-(1.2\times 10^{-8})D\theta^{-6}\dot{y}+0.14\theta^{-4}y=0\ . \tag{30}$$

In all cases then [at least when Eq. (29) can be used—see below], the system is linearly unstable. Its behavior is depicted in Fig. 3 for four different typical cases.

Now, as shown previously, the dominant reduction (24) can be expanded in terms of the delay only if $|z|r\theta^{-2}\ll 1$, where z is the characteristic root of (30) and where $r=2D\omega/c$. One easily sees that this condition is satisfied by the present prototypes ($D\approx 1$ m) of the detectors but will be violated by the future detectors ($D\approx 1$ km) if the reflectivity of the mobile mirror becomes significantly smaller than 10^{-1}. Indeed for $D=1$ km and $\theta=10^{-1.5}$, for example, $|z|r\theta^{-2}\approx 5\times 10^2$. The description of such cavities therefore cannot be based on the simple analysis of Appendix B which is valid only for small delays. In that case then we have to use the full-fledged method which yields the "dominant reduction" (24).

APPENDIX A: DOMINANT REDUCTION OF ORDINARY NONLINEAR DIFFERENTIAL EQUATIONS

The general solution of the homogeneous linear second-order differential equation

$$\ddot{x}+a\dot{x}+bx=0 \tag{A1}$$

is

$$x=k_1\exp(z_1t)+k_2\exp(z_2t)\ , \tag{A2}$$

where k_1 and k_2 are arbitrary constants and where z_1 and z_2 (with $z_1\neq z_1$) are the roots of the characteristic equation

$$z^2+az+b=0\ . \tag{A3}$$

If both z_1 and z_2 have negative real parts, the motion is asymptotically stable, that is, x tends to zero as t tends to infinity.

Let us now consider the nonlinear differential equation

$$\ddot{x}+a\dot{x}+bx=f(x) \tag{A4}$$

with $f(0)=\dot{f}(0)=0$. Under some technical conditions on the function $f(x)$, Liapounoff[25] proved that if the roots z_1 and z_2 of the characteristic equation of the linearized equation, that is to say (A1), have both negative real parts, then the motion is asymptotically stable.

To prove this result, Liapounoff (in the so-called Liapounoff first method) considers the infinite series

$$x=k_1\exp(z_1t)+k_2\exp(z_2t)+c_{20}k_1^2\exp(2z_1t)+2c_{11}k_1k_2\exp[(z_1+z_2)t]+c_{02}k_2^2\exp(2z_2t)+\cdots\ . \tag{A5}$$

θ \ D	1 m	1 km
10^{-1}	37 (upper) / 170 (lower)	37 (upper) / 0.17 (lower)
$10^{-1.5}$	370 (upper) / 0.17 (lower)	(crossed out)

FIG. 3. Onset of the instability. For different values of the length of the cavity and of the reflectivity of the mobile mirror, we give here the angular pulsation (upper corner) and the e-folding time (lower corner) in units of ω^{-1}.

Once it has been shown that the coefficients c_{ij} can be determined algorithmically in order that (A5) formally satisfy Eq. (A4) for arbitrary constants k_1 and k_2, then (A5) is the general solution of Eq. (A4) (at least when $z_1 \neq z_2$) and Liapounoff's result is intuitively obvious. To prove it rigorously he had to show that the series (A5) converges for finite values of t greater than some value t_0 and that it converges to zero when t tends to infinity.

Here we shall follow the same line of argument to introduce the concept of dominant reduction. In order that the discussion be clearer we shall restrict ourselves to the case of ordinary nonlinear equations of the type (A4). The extension of the framework to the delay systems considered in the main body of this paper is straightforward.

Let us therefore consider again the linear equation (A1), but this time when the real part of one of the roots z_1 and z_2 is positive. To be specific we shall consider the case when z_1 and z_2 are real and, respectively, positive and negative. The solution (A2) will then not be asymptotically stable unless k_1 is zero. However, if we eliminate k_1 between expression (A2) and its derivative with respect to t we obtain

$$\dot{x} - z_1 x = k_2(z_2 - z_1)\exp(z_2 t) , \tag{A6}$$

and since $\exp(z_2 t)$ tends to zero, the left-hand side of this expression is asymptotically stable. Considering its limit, we obtain the differential equation

$$\dot{x} - z_1 x = 0 . \tag{A7}$$

This equation is a reduction of Eq. (A1), that is, a differential equation of lower order whose solutions are also solutions of Eq. (A1). Using the terminology introduced in Ref. 17 it is actually its dominant reduction, i.e., a differential equation of lower order whose solutions asymptotically approach the solutions of Eq. (A1) in the case considered.

Consider now again the nonlinear equation (A4) and the infinite series (A5). If z_1 has a positive real part, this series is generically meaningless, since an infinite subset of its terms diverges exponentially as t tends to infinity. Let us, however, formally determine the coefficients c_{ij} as before, thus formally defining the general solution of Eq. (A4). Let us then, still in a formal way, eliminate k_1 from expression (A5) and the series obtained by taking its derivative with respect to t. This elimination leads to an expression of the following form:

$$\dot{x} - z_1 x - S(x) = E(t) , \tag{A8}$$

where $E(t)$ is an infinite sum of exponentially decreasing functions of t and where

$$S(x) = a_{20}x^2 + a_{11}x\dot{x} + a_{02}\dot{x}^2 + \cdots , \tag{A9}$$

the a_{ij} being some coefficients determined algorithmically.

To keep with Liapounoff's standards of rigor one should show that the series $S(x)$ and $E(t)$ so defined converge in some domain and that $E(t)$ tends to zero as t goes to infinity. To our knowledge these crucial results have not yet been proven, and we shall here assume them.

The differential equation obtained by taking the limit of (A8)

$$\dot{x} - z_1 x - S(x) = 0 \tag{A10}$$

is then by construction the dominant reduction of Eq. (4). Indeed it is the differential equation whose general solution is expression (A5) in which one sets k_2 equal to zero [under the assumptions previously made about the convergence of $S(x)$ and $E(t)$].

Let us now present the simplest example for which the use of a series similar to (A5) can be thoroughly tested. Consider the following first-order nonlinear differential equation with real coefficients a and b

$$\dot{x} + ax = bx^2 , \tag{A11}$$

whose general solution is

$$x = \frac{k\exp(-at)}{1 + (kb/a)\exp(-at)} , \tag{A12}$$

where k is an arbitrary constant. Its linearized approximation is

$$\dot{x} + ax = 0 \tag{A13}$$

and the root of the characteristic equation is $z = -a$.

Consider now the formal series

$$x = k\exp(zt) + k^2 c_2 \exp(2zt) + \cdots , \tag{A14}$$

and impose the condition that it be a solution of (A11). One thus obtains algorithmically the formal general solution of (A11):

$$x = k\exp(zt)[1 + k(b/z)\exp(zt) + k^2(b/z)^2\exp(2zt) + \cdots] , \tag{A15}$$

which is nothing but the formal development of (A12) regardless of the fact that the expansion is permissible or not. The expansion converges only if

$$|k(b/z)\exp(zt)| \leq 1 . \tag{A16}$$

Therefore if z is negative there always exists a value of t_0 such that if $t \geq t_0$ then the series (A15) converges and we recover Liapounoff's result: if the characteristic root of the linearized equation (A13) is negative, the solution of the nonlinear equation (A11) is asymptotically stable and is given by the series (A15).

The dominant reduction of Eq. (A11) is obtained by formally eliminating the constant k between (A15) and its derivative. That it is meaningful for all values of its arguments follows from the fact that it is nothing but the differential equation (A11) we started from.

APPENDIX B: THE APPROXIMATE DOMINANT REDUCTION IN THE CASE OF A SMALL DELAY

The equation of motion of the suspended mirror of the Fabry-Pérot cavity of Fig. 1 [Eqs. (2)–(4)] can be written as (cf. Ref. 21 and see text for notations)

$$\ddot{x} + \dot{x}/Q + x + x_0 = \frac{A\sin^2\theta}{|1 - \cos\theta\exp[i\bar{x}(t)]|^2} , \tag{B1}$$

where the function $\bar{x}(t)$ is a solution of the delay equation

$$\exp[i\bar{x}(t)] = \frac{\exp[ix(t-r)]}{1+\cos\theta\{\exp[ix(t-r)]-\exp[i\bar{x}(t-r)]\}} , \quad \text{(B2)}$$

r being here the round-trip travel time of light in the cavity which satisfies

$$rc/\omega = 2D_m + \lambda[x(t)+x(t-r)]/4\pi . \quad \text{(B3)}$$

(In the following we shall in fact set $r=2D_0\omega/c$).

The position of equilibrium x_s of the mirror, that is the stationary solution of Eqs. (B1)–(B3), is such that

$$x_s + x_0 = \frac{A\sin^2\theta}{1+\cos^2\theta - 2\cos\theta\cos x_s} , \quad \text{(B4)}$$

see Fig. 2 and Eq. (4).

We now expand the equation of motion (B1)–(B3) about x_s, up to the second order in the displacement. Assuming θ to be small and setting

$$x - x_s = \theta^2 y/2 ,$$
$$x - x_s = \theta^2(y_r + iy_i)/2 , \quad \text{(B5)}$$
$$x_s = \theta^2 y_s/2 ,$$

we obtain the following approximate hereditary equation, valid for $y \ll 1$:

$$\ddot{y} + \dot{y}/Q + y = -2\frac{y_r y_s + y_i}{\alpha(1+y_s^2)^2} + \frac{y_r^2(3y_s^2-1)+y_i^2(3-y_s^2)+8y_r y_i y_s}{\alpha(1+y_s^2)^3} , \quad \text{(B6)}$$

where y_r and y_i are solutions of the following delay system:

$$\hat{y}_r = y_r(1-\theta^2/2) - \theta^2 y_s y_i/2 + \theta^2 y/2 + (\theta^2/2)\{\hat{y}_i[(y-y_r)(1-\theta^2/2)+y_i y_s\theta^2/2] - (1-\theta^2/2)y_r y_i - \theta^2 y_s(y_r^2-y_i^2)/4 + \theta^2 y_s y^2/4\} , \quad \text{(B7)}$$

$$\hat{y}_i = y_i(1-\theta^2/2) - \theta^2 y_s(y-y_r)/2 - (\theta^2/2)\{-\hat{y}_r^2/2 + y_i^2/2 + \hat{y}_r y + \theta^2 y_s y_r y_i/2 - \tfrac{1}{2}(1-\theta^2/2)(y_r^2-y_i^2) - \theta^2 y^2/4\} , \quad \text{(B8)}$$

where we have set $\hat{f} \equiv f(t+r)$.

Consider the linear approximation of the equation of motion, that is, Eqs. (B6)–(B8) truncated at linear order. Setting $y=\exp(zt)$ we recover the characteristic equation (6) (for small θ), under a different form:

$$z^2 + z/Q + 1 = -\frac{\theta^4 y_s}{2\alpha(1+y_s^2)}\frac{1}{[\exp(zr)-(1-\theta^2/2)]^2+(y_s\theta^2/2)^2} . \quad \text{(B9)}$$

Let us now restrict ourselves to the case of a small delay and solve Eqs. (B7) and (B8) iteratively. When the delay is neglected (zeroth order) we have $\hat{y}_r = y_r = y$ and $\hat{y}_i = y_i = 0$. At first order we introduce f_r and f_i as

$$y_r = y + 2rf_r/\theta^2, \quad y_i = 2rf_i/\theta^2 , \quad \text{(B10)}$$

and expand all functions $\hat{f} = f(t+r)$ as $\hat{f} = f(t) + r\dot{f}(t)$, thus making a "Lagrange expansion." Inserting (B10) into (B7) and (B8) gives f_r and f_i straightforwardly, and hence y_r and y_i:

$$y_r = y - \frac{2r}{\theta^2(1+y_s^2)}\dot{y} + \frac{4ry_s}{\theta^2(1+y_s^2)^2}[1-\theta^2(1+y_s^2)/4]y\dot{y} , \quad \text{(B11)}$$

$$y_i = -\frac{2ry_s}{\theta^2(1+y_s^2)}\dot{y} - \frac{2r}{\theta^2(1+y_s)^2}[(1-y_s^2)-\theta^2(1+y_s^2)]y\dot{y} . \quad \text{(B12)}$$

Inserting the result into the equation of motion (B6), we finally obtain the dominant reduction for $F\tau\Omega_0 \ll 1$ and small θ, which is nothing but Eqs. (21)–(23).

[1]G. Volterra, *Leçons sur la Théorie Mathématique de la Lutte pour la Vie* (Gauthier-Villars, Paris, 1931).
[2]M. Kitano, T. Yabuzaki, and T. Ogawa, Phys. Rev. Lett. **50**, 713 (1983).
[3]K. Ikeda, Opt. Commun. **30**, 257 (1979).
[4]J. R. Ackerhalt, P. W. Milonni, and M. L. Shih, Phys. Rep. **128**, 205 (1985).
[5]J. Lapiedra and A. Molina, J. Math. Phys. **20**, 1308 (1979).
[6]Ll. Bel *et al.*, Gen. Relativ. Gravit. **13**, 963 (1981).
[7]R. Bellman and K. L. Cooke, *Differential-Difference Equations* (Academic, New York, 1963).
[8]L. E. El'sgol'ts and S. B. Norkin, *Introduction to the Theory and Application of Differential Equations with Deviating Arguments* (Academic, New York, 1973).
[9]R. D. Driver, *Ordinary and Delay Differential Equations* (Springer-Verlag, New York, 1973).
[10]L. Lagrange, *Nouvelle Méthode pour Résoudre les Equations Littérales par le Moyen des Series* (l'Académie Royale des Sciences et Belles-Lettres, Berlin, 1770), Vol. XXIV.
[11]Ll. Bel and H. Sirousse-Zia, Phys. Rev. D **32**, 3128 (1985).
[12]E. Kerner, J. Math. Phys. **6**, 1218 (1965).
[13]Ll. Bel and X. Fustero, Ann. Inst. Henri Poincaré **31**, 115 (1979).
[14]Ll. Bel, in *Relativistic Action at a Distance: Classical and*

Quantum Aspects, Vol. 162 of *Lecture Notes in Physics,* edited by J. Llosa (Springer-Verlag, Berlin, 1982).

[15]Ll. Bel, C. R. Acad. Sci. **294**, 463 (1982).

[16]Ll. Bel, in *Encuentros Relativistas 1982* (Universidad del Pais Vasco, Bilbao, 1982).

[17]J. M. Aguirregabiria and Ll. Bel, in *Frontiers of Non-Equilibrium Statistical Physics,* edited by G. T. Moore and M. O. Scully (Plenum, New York, 1986).

[18]K. Ikeda, H. Daido, and O. Akimoto, Phys. Rev. Lett. **45**, 709 (1980).

[19]H. M. Gibbs *et al.*, Phys. Rev. Lett. **46**, 474 (1981).

[20]A. Dorsel *et al.*, Phys. Rev. Lett. **51**, 1550 (1983).

[21]N. Deruelle and P. Tourrenc, in *Gravitation, Geometry and Relativity* (Springer-Verlag, Berlin, 1984).

[22]P. Tourrenc and N. Deruelle, Ann. Phys. (Paris) **10**, 241 (1985).

[23]J. M. Aguirregabiria and Ll. Bel, Phys. Rev. A **36**, 3768 (1987).

[24]Ll. Bel (unpublished).

[25]M. A. Liapounoff, *Problème Général de la Stabilité du Mouvement* (Princeton University Press, Princeton, NJ, 1949).

[26]R. Drever, in *Gravitational Radiation,* edited by N. Deruelle and T. Piran (North-Holland, Amsterdam, 1983).