Calculations of laser-assisted electron–hydrogen-atom elastic scattering with consideration of higher-order terms of laser-modified wave functions

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(Received 5 June 1987; revised manuscript received 2 October 1987)

We study the problem of laser-assisted electron–hydrogen-atom elastic scattering in the general case when the electromagnetic field is elliptically polarized, and obtain the results for plane-polarized and circularly polarized electromagnetic fields as special cases. We extend the work of F. W. Byron and C. J. Joachain [J. Phys. B 17, L295 (1984)] by considering all higher-order terms of the laser-modified atomic wave function, which have increasingly important effects on the scattering amplitude for small momentum transfer when the number of photons emitted or absorbed increases. We set up recurrence relations for determining all higher-order coefficients in the spectral expansion of the laser-dressed wave function and also determine all higher-order terms in the energy shift and the time-dependent phase. Our general expression for the laser-assisted first-Born-approximation scattering amplitude for l-photon transfer is cast into a form which gives the probability for the case in which l^a photons are absorbed or emitted by the bound atomic electron and the remaining $(l-l^a)$ photons by the free projectile electron. We discuss the dependence of the scattering amplitude on the polarization of the electromagnetic field. Exchange effects in the presence of the laser are also taken into account in a simple though approximate manner. We further discuss the problem of replacing the above-mentioned first-Born-approximation scattering amplitude by some higher-order scattering amplitude in the presence of the time-dependent laser field.

I. INTRODUCTION

The purpose of this paper is to study the problem of elastic electron–hydrogen-atom scattering in the presence of a strong laser field when any number of photons may be transferred between the laser field and either the projectile or the bound electron or both. Earlier investigators[1–5] of the above problem have made an assumption in their analysis that the spatial part of the ground-state wave function (1s state) of the atom remains unchanged, which is not valid for a strong laser field. Recently Byron and Joachain[6] have considered a first-order laser modification of the ground state (1s state) which involves a suitable admixture of np states. It has been found that the contribution of the first-order correction term to the laser-assisted scattering amplitude dominates over that of the zeroth-order amplitude, involving the 1s-state component of the perturbed wave function, in the region of very small momentum transfer ($\mathbf{q}$). It appears that for one-photon transfer, the np-state (1s-state) component of the perturbed wave function mainly corresponds to the case when the photon is emitted or absorbed by the bound atomic (free projectile) electron. In this paper we intend to extend the work of Byron and Joachain[6] by considering all higher-order terms of the perturbed wave function which have for small $\mathbf{q}$ increasingly important effects on the laser-assisted scattering amplitude as the number of transferred photons increases.

We determine the perturbed atomic wave function in the general case when an elliptically polarized laser beam of angular frequency ω is present, using the method of Langhoff *et al.*[7] They[7] have studied a similar problem where the electromagnetic (EM) field interacting with the atom is plane polarized. We may note that plane polarization and circular polarization are special cases of the elliptical polarization considered in this paper. Langhoff *et al.*[7] have written the perturbed ground-state wave function in the form

$$\Psi(r,t)=\phi(r,t)\langle\phi\,|\,\phi\rangle^{-1/2}\exp\left[(i\hbar)^{-1}\left[E_0^{(0)}t+\int_{-\infty}^{t}\mathrm{Re}\Delta E(t')dt'\right]\right],$$

where in the exponential $E_0^{(0)}$ is the unperturbed ground-state energy, and the integral (to be defined later) gives "terms" signifying the energy shift and an oscillatory time-dependent "phase" (secular term). Introducing the perturbation expansion $\phi(r,t)=\sum_{n=0}^{\infty}\phi_n(r,t)$, Langhoff *et al.*[7] have set up a sequence of equations connecting various terms $\phi_n(r,t)$. Putting $\phi_n(\mathbf{r},t)=\sum_l\phi_{n,l}(\mathbf{r})\exp(il\omega t)$ in the present case of harmonic perturbation involving the EM field, we set up a sequence of equations involving spatially varying functions $\phi_{n,l}(r)$ where n and l may have arbitrary values. Langhoff *et al.*[7] have given the above set of equations when n has the specific values 1, 2, and 3 and the EM field is plane polarized. Finally, writing $\phi_{n,l}(\mathbf{r})$ in the form of a spectral expansion in terms of eigenfunctions of the unperturbed Hamiltonian, $\phi_{n,l}(r)=\sum_\kappa b_\kappa^{n,l}\phi_\kappa^{(0)}(\mathbf{r})$, we set up very useful recurrence relations for the expansion coefficients $b_\kappa^{n,l}$ which express the nth-order coefficient

$b_\kappa^{n,l}$ in terms of the lower-order ones $b_{\kappa'}^{n'l'}$ $(n'<n)$. Langhoff *et al.*[7] have given explicit expressions for $b_\kappa^{n,l}$, the energy shift, and the time dependent phase term up to third order in the perturbation expansion, whereas we have given here very general relations which help us to determine these quantities to all orders in the perturbation expansion, taking the EM field interacting with the atom to be elliptically polarized. We also write the normalized function $\Phi(\mathbf{r},t)=\phi(r,t)\langle\phi\mid\phi\rangle^{-1/2}$ in the form of the spectral expansion $\Phi(\mathbf{r},t)=\sum_n\sum_l\sum_\kappa B_\kappa^{n,l}\phi_\kappa^{(0)}e^{il\omega t}$ and express $B_\kappa^{n,l}$ in terms of $b_{\kappa'}^{n'l'}$.

Our general expression for the laser-assisted scattering amplitude when l photons are transferred between the particles and the EM field is cast into a form which gives the probability for the case when l_i^e and $l_f'^e$ (l_i^a and $l_f'^a$) photons are exchanged between the EM field and the free projectile electron (bound atomic electron) in the initial and the final state such that $l_i^e+l_f'^e+l_i^a+l_f'^a=l$. The expression for the laser-assisted scattering amplitude involves various field-free scattering amplitudes corresponding to different transitions like $1s\rightarrow ns$, $1s\rightarrow np$, etc. It is found that for the n-photon transfer process the part of the full analytic expression for the laser-assisted first-Born-approximation scattering amplitude which involves the nth-order perturbed wave function varies as q^{-1} or q^0 (i.e., constant) when $q\rightarrow 0$ according to whether n is odd or even, whereas the corresponding part involving the zeroth-order perturbed wave function varies as q^n. The above two limiting forms are shown to correspond mainly to the cases when all the n photons are emitted or absorbed by the atomic electron and the free electron, respectively. Using our general expression we obtain the result of Ref. 6 by considering up to first-order terms in the dipole interaction.

We also study the dependence of the laser-assisted amplitude on the directions of momentum transfer $\mathbf{q}$ and polarization vectors of the EM field in the two cases when the laser field is (i) plane polarized and (ii) circularly polarized.

Use of the lowest-order perturbation theory or the first Born approximation to calculate various field-free transition amplitudes occurring in the expression for the laser-assisted scattering amplitude is quite satisfactory for fast projectile electrons. This has recently been done by Francken and Joachain.[8] The lowest-order perturbation theory has also been used by Dubois *et al.*[9] to calculate the one-photon transition amplitude in electron-atom collision in the presence of a laser. Chowdhury and Bhakar[10] have suggested that for the low-frequency laser the above-mentioned Born-approximation result can be improved upon by replacing the field-free Born transition amplitudes by the corresponding Glauber amplitudes. These replacements are justified when certain conditions (connected with the time dependence of the interaction of the laser field with the colliding particles) to be discussed in Sec. III D are satisfied.

Byron and Joachain[11] have pointed out that there are some deficiencies of the Glauber approximation although it provides us with closed-form analytic expressions for the transition amplitudes. They[11] have replaced the field-free first-Born-approximation elastic scattering amplitude $f_{\rm el}^{\rm B1}$ occurring in the expression for the laser-modified scattering amplitude by the eikonal Born series (EBS) amplitude $f_{\rm el}^{\rm EBS}=f_{\rm el}^{\rm B1}+\bar{f}_{\rm el}^{\rm B2}+\bar{f}_{\rm el}^{\rm G3}$, where $\bar{f}_{\rm el}^{\rm B2}$ and $\bar{f}_{\rm el}^{\rm G3}$ are, respectively, the second Born term and the third-order term of the Glauber series for the direct amplitude. It may be pointed out that the small momentum transfer (q) region is generally of paramount interest since the effect of dressing of the atom (specially for transfer of odd number of photons) is large in this region because of the strong peaking of the $1s\rightarrow np$ transition amplitude. Some facts about numerical results and theoretical analysis of previous investigators[12,13] to be given in Sec. IV show that the result for the laser-assisted electron-atom scattering amplitude evaluated by Glauber approximation is likely to be quite close to the corresponding EBS result in the small-angle region for transfer of odd number of photons. We express different Glauber scattering amplitudes used here for numerical calculation in the form of a series expansion in the parameter η (to be defined later) when $q\rightarrow 0$.

To take into account the exchange effect we also determine the laser-assisted exchange scattering amplitude in the Glauber approximation using the procedure of Franco and Halpern.[14] For small-angle scattering which is of much interest in the present work, the first-order exchange effect is quite small and the field-free first-order exchange elastic amplitude is of order k^{-2} (k being the electron momentum) as compared with the corresponding direct part. We have extended the work of Prasad and Unnikrishnan[5] on laser-modified direct and exchange scattering by taking into account additional effects of all possible excited states like the np states. The final form of our result is also somewhat different from that of Prasad and Unnikrishnan.[5] Recently, Trombetta *et al.*[15] have improved upon the work of Ferrante *et al.*[4] on laser-assisted exchange scattering by considering exchange collisions through second order in the electron-electron interaction.

II. LASER-MODIFIED WAVE FUNCTIONS

A. Modification of projectile wave function by elliptically polarized laser

The electric field associated with the laser which in general can be taken to be elliptically polarized is given by

$$\mathcal{E}=-\frac{1}{c}\frac{\partial\mathbf{A}(t)}{\partial t} \tag{1}$$

$$=\hat{\mathbf{e}}_x\mathcal{E}_x\cos\omega t+\hat{\mathbf{e}}_y\mathcal{E}_y\sin\omega t\ , \tag{2}$$

where $\hat{\mathbf{e}}_x$ and $\hat{\mathbf{e}}_y$ are unit vectors along the x and y direction and $\mathbf{A}(t)$ is the vector potential.

The free-electron wave function in the presence of a laser field is of the form

$$\chi_k(\mathbf{r},t)=(2\pi)^{-3/2}\exp[i(\mathbf{k}\cdot\mathbf{r}-\mathbf{k}\cdot\boldsymbol{\alpha}-E_kt/\hbar+\gamma)]\ , \tag{3}$$

$$\gamma=-\frac{e^2}{2mc^2}\int^t A^2(t')dt'\ , \tag{4}$$

$$E_k = \hbar^2 k^2/2m \ , \tag{5}$$

$$\boldsymbol{\alpha} = \frac{e}{m\omega^2}\mathcal{E} \ , \tag{6}$$

$$\mathbf{k}\cdot\boldsymbol{\alpha} = k_x \alpha_{0x}\cos\omega t + k_y \alpha_{0y}\sin\omega t \tag{7}$$

$$= [\mathbf{k},\boldsymbol{\alpha}]\sin(\omega t + \delta_{\mathbf{k}} + \pi/2) \ , \tag{8}$$

where the notation $[\mathbf{k},\boldsymbol{\alpha}]$ stands for

$$[\mathbf{k},\boldsymbol{\alpha}] = [(k_x\alpha_{0x})^2 + (k_y\alpha_{0y})^2]^{1/2} \ , \tag{9}$$

and

$$\delta_{\mathbf{k}} = -\tan^{-1}\frac{k_y\alpha_{0y}}{k_x\alpha_{0x}} \ . \tag{10}$$

For plane-polarized light of the type $\mathcal{E}_x = 0$ and $\mathcal{E}_y \neq 0$, $[\mathbf{k},\boldsymbol{\alpha}]$ becomes $k_y\alpha_{0y}$ ($=\mathbf{k}\cdot\boldsymbol{\alpha}_0$, say) and $\delta_{\mathbf{k}} + \pi/2 = 0$.

For the circularly polarized light for which $\mathcal{E}_x = \mathcal{E}^c$ and $\mathcal{E}_y = \pm\mathcal{E}_x$ or equivalently $\alpha_{0x} = \alpha_0^c$ and $\alpha_{0y} = \pm\alpha_{0x}$ the term $[\mathbf{k},\boldsymbol{\alpha}]$ becomes $k\alpha_0^c$ and $\delta_{\mathbf{k}}$ has the value $\mp\tan^{-1}(k_y/k_x)$.

Using Eq. (8) we can write[16]

$$\chi_{k_{i,f}} = (2\pi)^{-3/2}\exp[i(\mathbf{k}_{i,f}\cdot\mathbf{r} - E_{k_{i,f}}t/\hbar + \gamma)]\sum_{l_{i,f}^e} J_{l_{i,f}^e}([\mathbf{k}_{i,f},\boldsymbol{\alpha}])\exp[-il_{i,f}^e(\omega t + \delta_{k_{i,f}} + \pi/2)] \ , \tag{11}$$

where the subscript i (f) refers to the initial (final) state of the electron and the superscript e of $l_{i,f}^e$, order of the Bessel function, implies that we consider the case of the electron.

B. Perturbation of the bound-state wave function by elliptically polarized laser

Perturbation of the bound-state wave function $\psi'(r,t)$ in the presence of a laser is obtained by using the following Hamiltonian[17] H_t for the target atom:

$$H_t = \frac{1}{2m}\left[\mathbf{p} + \frac{e}{c}\mathbf{A}(t)\right]^2 + U(r) \ . \tag{12}$$

For the low-frequency laser we ignore space dependence of $\mathbf{A}(t)$ and this leads to a dipole-type interaction[17] as shown below. Using the transformation[17]

$$\psi'(r,t) = \exp\left[\frac{-ie\,\mathbf{A}(t)}{\hbar c}\cdot\mathbf{r}\right]\psi(\mathbf{r},t) \ , \tag{13}$$

we obtain

$$i\frac{\partial\psi}{\partial t}(\mathbf{r},t) = (H^{(0)} + H^{(1)})\psi(\mathbf{r},t) \ , \tag{14}$$

where

$$H^{(0)} = \frac{1}{2m}p^2 + U(r) \tag{15}$$

and the dipole-type interaction

$$H^{(1)} = \mathbf{r}\cdot\mathcal{E}(t) \tag{16}$$

$$= h^{1,1}e^{i\omega t} + h^{1,-1}e^{-i\omega t} \ , \tag{17}$$

where

$$h^{1,\pm 1} = \tfrac{1}{2}(x\mathcal{E}_x \mp iy\mathcal{E}_y) \ . \tag{18}$$

In order to solve Eq. (14) we write (following the procedure of Langhoff *et al.*[7])

$$\psi(\mathbf{r},t) = a_0(t)\phi(\mathbf{r},t)\exp[(i\hbar)^{-1}E_0^{(0)}t] \ , \tag{19}$$

where $E_0^{(0)}$ is the energy of the unperturbed ground-state wave function $\phi_0^{(0)}$.

We write the following perturbation expansion of $\phi(\mathbf{r},t)$:

$$\phi(\mathbf{r},t) = \sum_{n=0}^{\infty}\phi_n(\mathbf{r},t) \ , \tag{20}$$

subject to the condition of intermediate normalization

$$\langle\phi_0|\phi_n\rangle = 0, \quad n \neq 0 \ . \tag{21}$$

It can be shown that for dipole interaction $\langle\phi_m|\phi_n\rangle = 0$ for $m+n$ odd. From the above equations Langhoff *et al.*[7] have shown that

$$\left[H^{(0)}(\mathbf{r}) - E_0^{(0)} - i\hbar\frac{\partial}{\partial t}\right]\phi(\mathbf{r},t) + [H^{(1)}(\mathbf{r},t) - \Delta E(t)]\phi(\mathbf{r},t) = 0 \ , \tag{22}$$

$$a_0(t) = \exp\left[(i\hbar)^{-1}\int_{-\infty}^{t}\Delta E(t')dt'\right] \ , \tag{23}$$

and

$$\langle\phi|\phi\rangle^{-1/2} = \exp\left[(\hbar)^{-1}\int_{-\infty}^{t}\mathrm{Im}\Delta E(t')dt'\right] \ , \tag{24}$$

where

$$\Delta E(t)=\langle \phi_0 | H^{(1)} | \phi \rangle = \sum_{n=1}^{\infty} E^{(n)}(t) \ . \quad (25)$$

The complex quantity $E^{(n)}(t)$ is given by

$$E^{(n)}(t)=\langle \phi_0 | H^{(1)} | \phi_{n-1} \rangle, \quad n>0 \ . \quad (26)$$

Using the above relations we can write

$$\psi(\mathbf{r},t)=\Phi(\mathbf{r},t)\exp \times \left[(i\hbar)^{-1} \left[E_0^{(0)} t + \int_{-\infty}^{t} \mathrm{Re}\Delta E(t')dt' \right] \right] , \quad (27)$$

where we have introduced the normalized function $\Phi(\mathbf{r},t)$ defined by[7]

$$\Phi(\mathbf{r},t)=\phi \langle \phi | \phi \rangle^{-1/2} \quad (28)$$

$$= \sum_{n=0}^{\infty} \Phi_n \ . \quad (29)$$

From Eqs. (22) and (20) we obtain the following sequence of equations for ϕ_n given by Langhoff *et al.*:[7]

$$\left[H^{(0)}(\mathbf{r}) - E_0^{(0)} - i\hbar \frac{\partial}{\partial t} \right] \phi_n(\mathbf{r},t) + [H^{(1)}(r,t) - E^{(1)}(t)]\phi_{n-1}(\mathbf{r},t) = \sum_{k\geq 2}^{n} E^{(k)}(t) \phi_{n-k}(\mathbf{r},t) \ . \quad (30)$$

For $n=1$, the right-hand side of Eq. (30) is taken to be zero. Considering the expansion

$$\phi_n(\mathbf{r},t)=\sum_l \phi_{n,l}(\mathbf{r}) e^{il\omega t} \ , \quad (31)$$

we have from (26)

$$E^{(n)}(t)=\sum_l (\epsilon_{+1}^{n,l} e^{i(l+1)\omega t} + \epsilon_{-1}^{n,l} e^{i(l-1)\omega t}) \ , \quad (32)$$

where

$$\epsilon_{\pm 1}^{n,l} = \langle \phi_0 | h^{1,\pm 1} | \phi_{n-1,l} \rangle \ . \quad (33)$$

In Eq. (33) n is even for dipole interaction.

From Eqs. (30) and (31) we obtain now the recurrence relations connecting various $\phi_{n,l}$ in the general case of elliptically polarized electric field instead of the plane polarized one considered by Langhoff *et al.*,[7]

$$(H^{(0)} - E_0^{(0)} \pm l\omega)\phi_{n,\pm l} + (h^{1,\mp 1}\phi_{n-1,\pm(l+1)} + h^{1,\pm 1}\phi_{n-1,\pm(l-1)}) = \sum_s \sum_q (\epsilon_{\pm 1}^{s,\pm(l-1-q)} + \epsilon_{\mp 1}^{s,\pm(l+1-q)})\phi_{n-s,\pm q} \ . \quad (34)$$

C. Spectral representation, secular terms, and energy shift of perturbed wave function

Let us expand $\phi_{n,l}$ in terms of eigenfunctions $\phi_\kappa^{(0)}(r)$ of the unperturbed Hamiltonian

$$\phi_{n,l} = b_\kappa^{n,l} \phi_\kappa^{(0)}(\mathbf{r}) \ , \quad (35)$$

$$\phi_{0,l} = b_\kappa^{0,l} \phi_\kappa^{(0)}(\mathbf{r}) \ , \quad (36)$$

where

$$b_\kappa^{0,l} = \delta_{l0}\delta_{\kappa 0} \ . \quad (37)$$

Summation over repeated subscript κ is in the above equations and henceforth is implied.

It will be very useful to set up recurrence relations for various expansion coefficients $b_\kappa^{n,l}$ not previously expressed. Putting $\omega_{\kappa 0} = E_\kappa^{(0)} - E_0^{(0)}$ (subscript 0 refers to the ground state) and using above relations we have

$$b_\kappa^{n,\pm l} = \frac{1}{\omega_{\kappa 0} \pm l\omega} \left[-\sum_{\kappa'} (h_{\kappa\kappa'}^{1,\mp 1} b_{\kappa'}^{n-1,\pm(l+1)} + h_{\kappa\kappa'}^{1,\pm 1} b_{\kappa'}^{n-1,\pm(l-1)}) + \sum_{s=2,4,\ldots} \sum_q (\epsilon_{\pm 1}^{s,\pm(l-1-q)} + \epsilon_{\mp 1}^{s,\pm(l+1-q)}) b_\kappa^{n-s,\pm q} \right] , \quad (38)$$

where

$$\epsilon_{\pm 1}^{s,t}=\sum_j h_{0j}^{1,\pm 1} b_j^{s-1,t} , \tag{39}$$

$$h_{\kappa\kappa'}^{1,\pm 1}=\langle \phi_\kappa^{(0)} | h^{1,\pm 1} | \phi_{\kappa'}^{(0)} \rangle ,$$
$$h_{oj}^{1,\pm 1}=h_{\kappa j}^{1,\pm 1}\delta_{\kappa j} . \tag{40}$$

In Eq. (39), s is even due to the nature of dipole interaction. The above recurrence relation expresses nth-order expansion coefficient $b_\kappa^{n,\pm l}$ in terms of lower-order coefficients $b_{\kappa'}^{n',\pm l'}$ $(n'<n)$. For the plane-polarized laser defined by $\mathcal{E}_x \neq 0$, $\mathcal{E}_y=0$ in Eq. (18), $h_{\kappa\kappa'}^{1,-1}=h_{\kappa\kappa'}^{1,+1}$ and $\epsilon_{+1}^{s,t}=\epsilon_{-1}^{s,t}$ in Eqs. (38) and (39).

We now consider normalized function $\Phi(\mathbf{r},t)$ related to $\phi(\mathbf{r},t)$ by Eq. (28). From Eqs. (28), (29), and (20) it can be shown that for dipole-type interaction,

$$\Phi_n=\sum_s \beta_{2s}\phi_{n-2s} , \tag{41}$$

where

$$\beta_{2s}=-\tfrac{1}{2}\tilde{\beta}_{2s}+(-\tfrac{1}{2})(-\tfrac{3}{2})\frac{1}{2!}\sum_{s_1}\tilde{\beta}_{2s_1}\tilde{\beta}_{2s-2s_1} + (-\tfrac{1}{2})(-\tfrac{3}{2})(-\tfrac{5}{2}) \times\frac{1}{3!}\sum_{s_1}\sum_{s_2}\tilde{\beta}_{2s_1}\tilde{\beta}_{2s_2}\tilde{\beta}_{2s-2s_1-2s_2}+\cdots . \tag{42}$$

In Eq. (42) the terms $\tilde{\beta}_{2s}$ are given by

$$\tilde{\beta}_{2s}=\sum_{m=1}^{2s-1}\langle \phi_m | \phi_{2s-m}\rangle \tag{43}$$

$$=\sum_p \tilde{\beta}^{2s,p} e^{ip\omega t} , \tag{44}$$

where

$$\tilde{\beta}^{2s,\pm p}=\sum_{t=1}^{2s-1}\sum_q (b_\kappa^{t,\pm q})^* b_\kappa^{2s-t,\pm(p+q)} \tag{45}$$

$$=(\tilde{\beta}^{2s,\mp p})^* , \tag{46}$$

where we also use the notation $\beta_{2s}=\sum_p \beta^{2s,p}e^{ip\omega t}$.

In analogy with the relations (31) and (35) we put

$$\Phi_n=\sum_l \Phi_{n,l}e^{il\omega t} , \tag{47}$$

and write the spectral expansion

$$\Phi_{n,l}=B_\kappa^{n,l}\phi_\kappa^{(0)}(r) . \tag{48}$$

In view of the relations (31), (35), and (41)–(48), the expansion coefficient in Eq. (48) is given by

$$B_\kappa^{n,l}=\sum_s\sum_p \beta^{2s,p} b_\kappa^{n-2s,l-p} . \tag{49}$$

Explicit values of $b_\kappa^{n,l}$, $\beta^{2s,p}$, and $B_\kappa^{n,l}$ for specific values of n and l are given in the Appendix.

The secular term of the wave function is determined with the help of Eqs. (27), (25), and (32) we have

$$\mathrm{Re}\int_{-\infty}^{t}\sum_n E^{(n)}dt=\sum_{l=2,4,\ldots l}\frac{\sin l\omega t}{l\omega}\sum_{n=l,l+2,\ldots}(\epsilon_{+1}^{n,l-1}+\epsilon_{-1}^{n,-(l-1)}+\epsilon_{-1}^{n+2,l+1}+\epsilon_{+1}^{n+2,-(l+1)}) + t\sum_{n=1,2,\ldots}(\epsilon_{-1}^{2n,1}+\epsilon_{+1}^{2n,-1}) . \tag{50}$$

It is obvious that $\sum_n(\epsilon_{-1}^{2n,1}+\epsilon_{+1}^{2n,-1})$ is the energy shift.

III. LASER-ASSISTED SCATTERING AMPLITUDE

A. Direct scattering amplitude

The Hamiltonian of the system is

$$H=H_t+H_f+V_d , \tag{51}$$

where H_t has already been defined by Eq. (12). H_f is the Hamiltonian of free electron in the presence of laser field and V_d is the electron-atom interaction in the direct (initial) arrangement channel. S-matrix element for the elastic scattering amplitude in Born approximation is given by

$$S_{\mathrm{el}}^B=-\frac{i}{\hbar}\int_{-\infty}^{+\infty}dt\langle \chi_{k_f}(\mathbf{r},t)\Psi_f(\mathbf{r},t) | V_d | \chi_{k_i}(\mathbf{r},t)\Psi_i(\mathbf{r},t)\rangle . \tag{52}$$

Using Eq. (11) and the summation theorem[16] for Bessel functions we can show that the product of free-electron wave functions occurring in Eq. (52) has the following form:

$$\chi^*_{k_f}\chi_{k_i}\Bigg/\left[\frac{1}{(2\pi)^3}\exp[i\mathbf{q}\cdot\mathbf{r}+(E_{k_f}-E_{k_i})t/\hbar]\right]$$

$$=\sum_{l_i^e}\sum_{\substack{l^e \\ l^e=l_i^e+l_f'^e}} J_{l_i^e}([\mathbf{k}_i,\boldsymbol{\alpha}])J_{(-l_f'^e-l_i^e)+l_i^e}([\mathbf{k}_f,\boldsymbol{\alpha}])\exp[il_i^e(-\delta_{\mathbf{k}_i}+\delta_{\mathbf{k}_f})-i(l_f'^e+l_i^e)(\omega t+\delta_{\mathbf{k}_f}+\pi/2)] \quad (53)$$

$$=\sum_{l^e} J_{l^e}([\mathbf{q},\boldsymbol{\alpha}])\exp[-il^e(\omega t+\delta_{\mathbf{q}}+\pi/2)] \ , \quad (54)$$

where $[\mathbf{q},\boldsymbol{\alpha}]=[(q_x\alpha_{0x})^2+(q_y\alpha_{0y})^2]^{1/2}$ and $\delta_q=-\tan^{-1}(q_y\alpha_{0y}/q_x\alpha_{0x})$. The above result can also be obtained by using the Bessel function expansion[16] of $\exp[-i\mathbf{q}\cdot\boldsymbol{\alpha}]$ which is a factor of $\chi^*_{k_f}\chi_{k_i}$.

In the evaluation of Eq. (52) we use the following form of bound-state electron wave function:

$$\Psi_{i,f}(\mathbf{r},t)/\exp\left[(i\hbar)^{-1}\left[E_0^{(0)}t+\int_{-\infty}^{t}\mathrm{Re}\Delta E(t')dt'\right]\right]=\Phi_{i,f}(\mathbf{r},t)$$

$$=\sum_n\sum_{l_{i,f}^a}\Phi_{n,l_{i,f}^a}e^{il_{i,f}^a\omega t}=\sum_n\sum_{l_{i,f}^a}B_\kappa^{n,-l_{i,f}^a}\phi_\kappa^{(0)}(\mathbf{r})e^{-il_{i,f}^a\omega t} \ , \quad (55)$$

where values of $B_\kappa^{n,l}$ are further given in terms of $b_\kappa^{n,l}$ [depending upon the matrix element of electric dipole interaction as can be seen from Eqs. (37)–(40)] by the relation (49). We finally obtain the following form of the S_{el}^B matrix:

$$S_{\mathrm{el}}^B=(2\pi)^3 i\sum_{l=-\infty}^{\infty}\delta(E_{k_f}-E_{k_i}-l\omega)f_{\mathrm{el}}^{B,l} \ , \quad (56)$$

where the first-Born-approximation elastic-scattering amplitude $f_{\mathrm{el}}^{B,l}$ with transfer of l photons, is given by

$$f_{\mathrm{el}}^{B,l}=\sum_{n_f}\sum_{n_i}\sum_{l_i^e}\sum_{l_f'^e}\sum_{l_i^a}\sum_{l_f'^a}J_{l_i^e+l_f'^e}([\mathbf{q},\boldsymbol{\alpha}])\exp[-i(l_i^e+l_f'^e)(\delta_{\mathbf{q}}+\pi/2)](B_{\kappa_f}^{n_f,l_f'^a})^*B_{\kappa_i}^{n_i,-l_i^a}f_{\kappa_f\kappa_i}^B\delta_{l^e,l_i^e+l_f'^e}\delta_{l^a,l_i^a+l_f'^a}\delta_{l,l^a+l^e} \quad (57)$$

$$=\sum_{l^a}J_{l-l^a}([\mathbf{q},\boldsymbol{\alpha}])\exp[-i(l-l_a)(\delta_{\mathbf{q}}+\pi/2)]A_{l^a}^{\kappa_f\kappa_i}f_{\kappa_f\kappa_i}^B \quad \text{(say)} \ . \quad (58)$$

In Eq. (57) we take

$$l_f'^e=-l_f^e, \quad l_f'^a=-l_f^a \ . \quad (59)$$

In the above relation $f_{\kappa_f\kappa_i}^B$ is the field-free Born-approximation scattering amplitude for transition from quantum state κ_i to κ_f. In general κ_s in this paper stands collectively for the usual quantum numbers n_s,l_s,m_s.

The above derivation involving the relations (11) and (52)–(57) suggests that l_i^e and $l_f'^e$ photons are absorbed by the initial and final electron, respectively. Likewise l_i^a ($l_f'^a$) photons are absorbed by bound electrons in the initial (final) state.

For illustration the quantity $A_{l^a}^{\kappa_f\kappa_i}$ in Eq. (58) is given below for $l^a=0,1,2,3$ considering at least terms up to third order in electric dipole interaction. In Eq. (58) l^a (or $l-l^a$) may be thought of as the number of photons absorbed by the bound electron (or free electron)

$$A_0^{\kappa_f\kappa_i}=\delta_{\kappa_f 0}\delta_{\kappa_i 0}+\sum_{\substack{i \\ i=1,-1}}(B_{\kappa_f}^{1,i})B_{\kappa_i}^{1,i}+\{(B_{\kappa_f}^{2,0})^*\delta_{\kappa_i,0}+[\mathrm{c.t.}]\} \ , \quad (60)$$

$$A_{\pm1}^{\kappa_f\kappa_i}=(B_{\kappa_f}^{1,\pm1})^*\delta_{\kappa_i 0}+(B_{\kappa_f}^{3,\pm1})^*\delta_{\kappa_i 0}+(B_{\kappa_f}^{2,0})^*B_{\kappa_i}^{1,\mp1}+(B_{\kappa_f}^{2,\pm2})^*B_{\kappa_i}^{1,\pm1}+[\mathrm{c.t.}] \ , \quad (61)$$

$$A_{\pm2}^{\kappa_f\kappa_i}=(B_{\kappa_f}^{2,\pm2})^*\delta_{\kappa_i 0}+(B_{\kappa_f}^{1,\pm1})^*B_{\kappa_i}^{1,\mp1}+[\mathrm{c.t.}] \ , \quad (62)$$

$$A_{\pm3}^{\kappa_f\kappa_i}=(B_{\kappa_f}^{2,\pm2})^*B_{\kappa_i}^{1,\mp1}+(B_{\kappa_f}^{3,\pm3})^*\delta_{\kappa_i 0}+[\mathrm{c.t.}] \ . \quad (63)$$

In the above relations complementary terms [c.t.] imply that complex conjugation of previous terms with the changes $\kappa_f \leftrightarrow \kappa_i$ and $l^a_{i,f} \rightarrow -l^a_{i,f}$ are to be added.

B. Zeroth-order and first-order laser-modified amplitude for plane- and circularly polarized laser

Retaining terms up to first order in electric dipole interaction in Eq. (58) for l-photon transfer and using the explicit value of $B^{1,\pm 1}_{\kappa}$ given in the Appendix we have (omitting the superscript B)

$$f^l_{\rm el} = \exp[-il(\delta_{\rm q}+\pi/2)]\{J_l([\mathbf{q},\boldsymbol{\alpha}])f_{00} + J_{l-1}([\mathbf{q},\boldsymbol{\alpha}])\tilde{A}_{+1} + J_{l+1}([\mathbf{q},\boldsymbol{\alpha}])\tilde{A}_{-1}\}\ , \tag{64}$$

where

$$\tilde{A}_{\pm 1} = -\exp[\pm i(\delta_{\rm q}+\pi/2)]\sum_n\sum_m \left[\frac{(h^{1,\pm 1}_{npm,0})^* f_{npm,0}}{\omega_{n0}\pm\omega} + \frac{h^{1,\mp 1}_{npm,0} f_{0,npm}}{\omega_{n0}\mp\omega}\right] . \tag{65}$$

In Eq. (65) $h^{1,\pm 1}$ is already defined by Eq. (18). In the expression $f_{npm,0}$ suffix n is the principal quantum number, p stands for orbital angular momentum quantum number $l=1$ (appropriate for dipole-type interaction), $m\,(=+1,0,-1)$ for magnetic quantum number, and zero implies ground state. We define $\omega_{n0}=\omega_n-\omega_0$.

We now proceed to express $\tilde{A}_{\pm 1}$ in another compact form in the following. In view of the relations

$$\langle \mathbf{k}_f \mid V_d \mid \mathbf{k}_i \rangle = e^2\left\langle \mathbf{k}_f \left| \frac{1}{r_{12}} - \frac{1}{r_1} \right| \mathbf{k}_i \right\rangle = \frac{4\pi e^2}{q^2}(e^{i\mathbf{q}\cdot\mathbf{r}_1}-1) \tag{66}$$

and

$$e^{i\mathbf{q}\cdot\mathbf{r}} = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} i^l j_l(qr) Y^*_{lm}(\hat{\mathbf{q}}) Y_{lm}(\hat{\mathbf{r}})\ , \tag{67}$$

we can write

$$f_{nlm,0} = Y^*_{lm}(\hat{\mathbf{q}}) f_{nl,0}\ . \tag{68}$$

We can also write

$$\langle nlm \mid \mathbf{r}\cdot\mathbf{A} \mid 0 \rangle = \mid \mathbf{A} \mid Y^*_{lm}(\hat{\mathbf{A}}) M_{nl,0}\delta_{l,1}\ , \tag{69}$$

where $\mathbf{A}$ is any vector and $M_{nl,0}\delta_{l,1}$ or $M_{np,0}$ can be called the dipole matrix element.

Then using Eqs. (68) and (69) we can write

$$f_{0,nlm}\langle nlm \mid h^{1,\pm 1} \mid 0 \rangle = \frac{1}{2q}(\mathcal{E}_x q_x \mp i\mathcal{E}_y q_y) f_{0,nl} M_{nl,0} \tag{70}$$

$$= \frac{1}{2q}[\mathcal{E},\mathbf{q}]e^{\pm i\delta_{\rm q}} f_{0,nl} M_{nl,0}\ , \tag{71}$$

where $[\mathcal{E},\mathbf{q}] = m\omega^2/e[\boldsymbol{\alpha},\mathbf{q}]$ and $\delta_{\rm q}$ has already been defined after Eq. (54). The above relations enable us to express $\tilde{A}_{\pm 1}$ in the following more compact form in which the phase factor $e^{\pm i\delta_{\rm q}}$ does not appear:

$$\tilde{A}_{\pm 1} = \mp \frac{i}{2q}[\mathcal{E},\mathbf{q}]\sum_n \frac{M_{0,np} f_{np,0}}{\omega_{n0}\pm\omega} + \frac{M_{np,0} f_{0,np}}{\omega_{n0}\mp\omega}\ . \tag{72}$$

In the case of plane-polarized EM field obtained from Eq. (2) by putting $\mathcal{E}_x=0$, we have $[\mathcal{E},\mathbf{q}]=\mathcal{E}\cdot\mathbf{q}=\mid\mathcal{E}\mid\mid\mathbf{q}\mid\cos\lambda$ (λ is the angle between $\mathcal{E}$ and $\mathbf{q}$) and $\delta_q=-\pi/2$. In this case the above relation becomes the same as given by Byron and Joachain.[6] In the case of a circularly polarized EM field defined by

$$\mathcal{E}_x = \pm\mathcal{E}_y = \mathcal{E}^c = \frac{m\omega^2}{e}\alpha^c_0\ , \tag{73}$$

we have

$$[\mathbf{q},\mathcal{E}] = q\,\mathcal{E}^c = \frac{m\omega^2}{e} q\alpha^c_0 \tag{74}$$

and

$$\tan\delta^{(c)}_q = \mp q_y/q_x \tag{75}$$

when relations (74) and (75) are to be used for the circularly polarized EM field.

C. Explicit forms of some higher-order terms in laser-modified amplitudes

We may note that in Eq. (64) the field-free scattering amplitudes $f_{00}\rightarrow$constant and $f_{np,0}\rightarrow 1/q$ in the Born approximation when $q\rightarrow 0$. Keeping this in mind we find that in the case of single photon transfer (i.e., $l=1$) in the region of very small momentum transfer ($q\rightarrow 0$), the dominant part of the amplitude for the process corresponds to the photon absorption by bound electron (related to $J_{l-1}([\mathbf{q},\boldsymbol{\alpha}])\tilde{A}_1$ term) and is of order $1/q^2$ relative to that involving the free electron (the $J_{l=1}([\mathbf{q},\boldsymbol{\alpha}])f_{00}$ term). In another specific case of odd (or even) number of photons transferred, e.g., $l=3$ (or 2) we also expect that for sufficiently strong laser field the dominant contribution to the amplitude for this process in the region of very small q is given by the term in Eq. (58) characterized by $l^a=3$ (or 2) corresponding to the case when all the three (or two) photons are absorbed by bound electron. This can be seen from the following detailed expression for $A^{\kappa_f\kappa_i}_{l^a} f_{\kappa_f\kappa_i}$ for $l^a=\pm 2$ and ± 3 written in terms of matrix elements $h^{1,\pm 1}_{\kappa_f\kappa_i}$ with the help of Eqs. (62) and (63). This is further discussed in Sec. IV. We have

$$A_{l^a=\pm 2}^{\kappa_f \kappa_i} f_{\kappa_f \kappa_i} = \left[\frac{h^{1,\pm 1}_{n_2 l_2 m_2, nlm}}{\omega_{n_2 0} \pm 2\omega} \frac{h^{1,\pm 1}_{nlm,0}}{\omega_{n0} \pm \omega} - \frac{1}{2} \frac{h^{1,\pm 1}_{nlm,0} (h^{1,\mp 1}_{nlm,0})^*}{\omega_{n0}^2 - \omega^2} \delta_{n_2 l_2 m_0, 0} \right]^* \times f_{n_2 l_2 m_2, 0} + \left[\frac{h^{1,\pm 1}_{n_2 l_2 m_2, 0}}{\omega_{n_2 0} \pm \omega} \right]^* \frac{h^{1,\mp 1}_{n_1 l_1 m_1, 0}}{\omega_{n_1 0} \pm \omega} f_{n_2 l_2 m_2, n_1 l_1 m_1} + \text{c.t.} \tag{76}$$

and

$$A_{l^a=\pm 3}^{\kappa_f \kappa_i} f_{\kappa_f \kappa_i} = \left[\frac{1}{\omega_{n_3 0} \pm 3\omega} \left[-h^{1,\pm 1}_{n_3 l_3 m_3, n_2 l_1 m_2} \frac{h^{1,\pm 1}_{n_2 l_2 m_2, n_1 l_1 m_1}}{\omega_{n_2 0} \pm 2\omega} \frac{h^{1,\pm 1}_{n_1 l_1 m_1, 0}}{\omega_{n_1 0} \pm \omega} + \frac{h^{1,\pm 1}_{0,nlm} h^{1,\pm 1}_{nlm,0}}{\omega_{n0} \pm \omega} \frac{h^{1,\pm 1}_{n_3 l_3 m_3, 0}}{\omega_{n_3 0} \pm \omega} \right] + \frac{(h^{1,\mp 1}_{nlm,0})^* h^{1,\pm 1}_{nlm,0}}{\omega_{n0}^2 - \omega^2} \frac{h^{1,\pm 1}_{n_3 l_3 m_3, 0}}{\omega_{n_3 0} \pm \omega} \right]^* f_{n_3 l_3 m_3, 0} - \left[\frac{h^{1,\pm 1}_{n_3 l_3 m_3, nlm}}{\omega_{n_3 0} \pm 2\omega} \frac{h^{1,\pm 1}_{nlm,0}}{\omega_{n0} \pm \omega} \right]^* \frac{h^{1,\mp 1}_{n_1 l_1 m_1, 0}}{\omega_{n_1 0} \mp \omega} f_{n_3 l_3 m_3, n_1 l_1 m_1} + \text{c.t.} \ . \tag{77}$$

Keeping in mind the nature of dipole interaction [see Eqs. (16)–(18)] we can make some observations regarding the possible values of angular-momentum quantum numbers characterizing the transition amplitude in Eq. (58) for certain special cases corresponding to Eqs. (76) and (77). In relation (76) possible values of l_2 are 0 and 2 in the first term and zero only in the second term. The permissible value of both l_2 and l_1 is 1 in the case of the third term in Eq. (76). A study of the coefficient of $f_{n_3 l_3 m_3, 0}$ in Eq. (77) shows that l_3 is restricted to the values 1 and 3. Similarly in the term $f_{n_2 l_2 m_2, n_1 l_1 m_1}$, possible values of l_2 are 0 or 2 but that of l_1 is 1. The above analysis shows that when $l^a = 3$ (in general for odd values of l^a) and $l^a = 2$ (or even l^a), the term $A_{l^a}^{\kappa_f \kappa_i} f_{\kappa_f \kappa_i}$ involves among other quantities transition amplitudes $f_{np,1s}$ and $f_{ns,1s}$, respectively, whose asymptotic values for $q \to 0$ mainly determine the nature of the scattering amplitude for small q,

D. Glauber approximation or higher-order approximation of Laser-modified Born scattering amplitude

The complicated problem of determining higher-order corrections to the laser-assisted scattering amplitude in the presence of the time-dependent EM field has already been studied by Chowdhury and Bhakar[10] in the Glauber approximation. They have written the wave function of the projectile electron in the form

$$\tilde{\chi}_k(\mathbf{r},t) = \chi_k(\mathbf{r},t)\xi(\mathbf{r},t) \ , \tag{78}$$

where $\tilde{\chi}_k(\mathbf{r},t)$ satisfies the following equation for the projectile electron in the presence of an EM field $\mathbf{A}(t)$ (assumed to be almost spatially homogeneous in the region of electron atom collision) and the atomic potential V_d:

$$\left[\frac{1}{2m} \left[p + \frac{e\,\mathbf{A}(t)}{c} \right]^2 + V_d \right] \tilde{\chi}_k = i\hbar \frac{\partial \tilde{\chi}_k}{\partial t} \ . \tag{79}$$

The quantity $\chi_k(\mathbf{r},t)$ [already given by Eq. (3)] is the solution of Eq. (79) for $V_d = 0$. The quantity $\xi(\mathbf{r},t)$ then satisfies the following equation:

$$\frac{1}{2m} \left[-\nabla^2 - 2i \left[\mathbf{k} + \frac{e\,\mathbf{A}}{c} \right] \cdot \nabla \right] \xi + V_d \xi = i \frac{\partial \xi}{\partial t} \ . \tag{80}$$

Neglecting $\nabla^2 \xi$ in Eq. (80) and writing $\mathbf{r} = \mathbf{b} + \mathbf{z}$ ($\mathbf{z}$ taken along the direction of electron momentum) Chowdhury and Bhakar[10] have obtained the following solution for $\xi(\mathbf{r},t)$:

$$\xi(\mathbf{r},t) = \exp\left[-iF(t) \int_{-\infty}^{z} V_d(\mathbf{r}_z, \mathbf{b} + \mathbf{z}')dz' \right] \ . \tag{81}$$

In Eq. (80) if one assumes the smooth-phase approximation,[10] i.e., $\partial^2 F / \partial t^2 = 0$ and takes into account the usual value of the laser intensity and the short-range nature of the interaction, then one can write

$$F(t) \approx 1/v(t) \ , \tag{82}$$

where

$$v(t) = \frac{1}{m} \,|\, K_i(t) \,| \ . \tag{83}$$

$\mathbf{K}_i(t)$ is the shifted momentum in the EM field $\mathbf{A}(t)$ [see Eq. (1)] and is of the form

$$\mathbf{K}_i(t) = \mathbf{k}_i + \frac{e\,\mathbf{A}(t)}{c} \ . \tag{84}$$

The relations (82) and (83) also result if we assume $\partial \xi / \partial t \approx 0$ and $\nabla^2 \xi \approx 0$ in Eq. (80).

Using Eq. (81) we obtain the following expression for the direct Glauber amplitude $f^{G,d}_{\kappa_f \kappa_i}$ (or simply $f^{G}_{\kappa_f \kappa_i}$)

which can be used in place of $f_{\kappa_f\kappa_i}$ in Eqs. (64) and (65):

$$f^{G,d}_{\kappa_f\kappa_i}=\frac{iK_i}{2\pi}\int d^3r_2d^2b\,\phi^{(0)*}_{\kappa_f}(\mathbf{r}_2)\Gamma(\mathbf{r}_2,\mathbf{b},t_0)\phi^{(0)}_{\kappa_i}(\mathbf{r}_2)e^{i\mathbf{b}\cdot\mathbf{q}}\ , \tag{85}$$

where

$$\Gamma(\mathbf{r}_2,\mathbf{b},t_0)=\exp\left[-\frac{i}{v(t_0)}\int_{-\infty}^{\infty}V_d(\mathbf{r}_2,\mathbf{b}+\mathbf{z}')dz'\right]-1\ , \tag{86}$$

$$v(t_0)=\frac{1}{m}\,|\mathbf{K}_i(t_0)|\ , \tag{87}$$

where the time t_0, occurring as the argument of $\mathbf{A}(t_0)$ in the general case of elliptically polarized laser, is obtained by using the stationary-phase approximation of Chowdhury and Bhakar[18] who have considered the case of the plane-polarized EM field.

We have

$$\omega t_0=\cos^{-1}(l/[\boldsymbol{\alpha},\mathbf{q}])-(\delta_{\mathbf{q}}+\pi/2)\ . \tag{88}$$

The quantities on the right-hand side of Eq. (88) have already been defined immediately after Eq. (54). In the case of the plane-polarized EM field characterized by $\mathbf{A}(t)=\hat{\mathbf{e}}_y A_0\cos\omega t$ we obtain $\mathbf{A}(t_0)=\hat{\mathbf{e}}_y A_0 l/(\boldsymbol{\alpha}\cdot\mathbf{q})$. For very low-frequency laser $\mathbf{K}(t_0)$ can then be approximately replaced by $\mathbf{k}_i$.

We may point out that Byron and Joachain[6] have used the EBS amplitude $f^{\rm EBS}_{\kappa_f\kappa_i}$ instead of the Glauber amplitude $f^{G}_{\kappa_f\kappa_i}$ to calculate the field-free elastic ($1s\rightarrow1s$) amplitude occurring in Eq. (64). If we retain the $\nabla^2\xi$ term but neglect $\partial\xi/\partial t$ in Eq. (80) we obtain more accurate transition amplitudes which can be used in place of first-Born-approximation transition amplitudes appearing in Eq. (58) for the low-frequency laser.

In the following we give asymptotic values of f^G_{00} and $f^G_{npm,0}$ determined by Thomas and Gerjuoy[19] when $q\rightarrow0$ for low-frequency laser so that $K_i\approx k_i$.

$$f^G_{00}=f^G_{1s\rightarrow1s}$$

$$=-ik_i(2/a_0)^3\frac{1}{4}\left[\frac{\partial}{\partial\lambda}I_0(\lambda,q)\right]\Bigg|_{\lambda=2/a_0}\ , \tag{89}$$

where

$$I_0(\lambda,q\rightarrow0)=-4i\eta\Gamma(1+i\eta)\Gamma(1-i\eta)\lambda^{-2-2i\eta}(\lambda^2+q^2)^{-1+i\eta}\,{}_2F_1[1-i\eta,i\eta,1;\lambda^2(\lambda^2+q^2)]$$

$$=\left[\left[1+\eta^2\frac{\pi^2}{6}-i\eta^3c_1+\eta^4\frac{\pi^4}{120}-\tfrac{1}{2}i\eta^5c_2\right]-i\eta\left[1+\eta^2\frac{\pi^2}{6}+\eta^4\frac{\pi^4}{240}\right]\ln\left[\frac{q^2}{\lambda^2+q^2}\right]+\cdots\right]\ , \tag{90}$$

$$\eta=e^2m/k_i\ , \tag{91}$$

$$c_1=1+\frac{1}{2^2}(1+\tfrac{1}{2})+\frac{1}{3^2}(1+\tfrac{1}{2}+\tfrac{1}{3})+\frac{1}{4^2}(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4})+\cdots\ , \tag{92}$$

$$c_2/2=\frac{1}{2^2}(1+\tfrac{1}{2})+\frac{1}{3^2}\left[1+\frac{1}{2^2}\right](1+\tfrac{1}{2}+\tfrac{1}{3})+\frac{1}{4^2}\left[1+\frac{1}{2^2}+\frac{1}{3^2}\right](1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4})+\cdots\ , \tag{93}$$

and

$$f^G_{1s\rightarrow npm}=2ik_i(\tfrac{3}{2})^{1/2}\frac{1}{4!}\left[\frac{2}{a_0}\right]^4\frac{1}{n^3}\left[\frac{(n+1)!}{(n-2)!}\right]^{1/2}e^{-im\phi_{\mathbf{q}}}\sum_{j=0}^{n-2}\beta_j(n)(-1)^{j+1}\frac{\partial^{j+1}}{\partial\lambda^{j+1}}(\lambda,q)\Bigg|_{\lambda=(1/a_0)(1+1/n)}\ , \tag{94}$$

where $\phi_{\mathbf{q}}$ is the polar angle or $\hat{\mathbf{q}}$ in the plane normal to $\hat{\boldsymbol{\xi}}$ which is chosen perpendicular to $\hat{\mathbf{q}}$, and β_j is

$$\beta_j(n)=[(-n+2)_j/j!(4)_j](2/na_0)^j\ . \tag{95}$$

We have

$$I_1(\lambda,q\rightarrow0)=-4\eta\Gamma(1+i\eta)\Gamma(2-i\eta)\lambda^{-2i\eta-2}(\lambda^2+q^2)^{-1+i\eta}$$

$$\times\left[\left\{\frac{1-2i\eta}{1-i\eta}\left[\left[1+\eta^2\frac{\pi^2}{6}+\eta^4\frac{\pi^4}{120}\right]-\left[1+\frac{\lambda^2}{q^2}\right]^{-1}{}_2F_1[1-i\eta,i\eta,1;\lambda^2/(\lambda^2+q^2)]\right]\right.\right.$$

$$\left.\left.+(1+i\eta)\left[(1+i\eta)+(\eta^2+i\eta^3)\left[\frac{\pi^2}{6}-1\right]+(\eta^4+i\eta^5)\left[\frac{\pi^4}{120}-\frac{\pi^2}{6}+1\right]\right]\right\}q^{-1}+\cdots\right]\ . \tag{96}$$

We may note that in the evaluation of limiting values of $I_0(\lambda,q)$ and $I_1(\lambda,q)$ for $q\rightarrow0$ we retain terms at least up to $O(\eta^6)$ whereas Thomas and Gerjuoy[19] have shown terms of $O(\eta^2)$ and $O(\eta^1)$, respectively. Exact forms of I_0 and I_1 have been given in Ref. 19.

E. Exchange scattering amplitude

To take into account exchange effect in the scattering process we consider the following S-matrix for laser-assisted elastic-exchange scattering for the general case of elliptical polarization:

$$S_{\rm el}^{\rm ex} = -\frac{i}{\hbar}\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} dt\,\exp[i(E_{k_f}-E_{k_i})t - i\mathbf{q}\cdot\boldsymbol{\alpha}_0]X(t)\;, \tag{97}$$

where

$$X(t) = \int\int d\mathbf{r}_2 d\mathbf{r}_1 \exp\left[i\left[k_i+\frac{e\mathbf{A}}{c}\right]\cdot\mathbf{r}_{12}\right](r_{12}-z_{12})^{i\eta}\exp[i\mathbf{q}\cdot\mathbf{r}_2]\Phi(\mathbf{r}_2,t)F(r_2,r_{12},t)\;, \tag{98}$$

and

$$F(r_2,r_{12},t) = \Phi^*(r_2+r_{12},t)][\,|\mathbf{r}_2+\mathbf{r}_{12}| - (\mathbf{r}_2+\mathbf{r}_{12})\cdot\hat{\mathbf{Z}}]^{-i\eta}\;. \tag{99}$$

Using Franco and Halpern's[14] method of approximation we get

$$X(t) \simeq \frac{4\pi\Gamma(1+i\eta)}{|K_i|^{-2-2i\eta}}(-2i\mathbf{K}_i\cdot\hat{\mathbf{z}})^{i\eta}\int d\mathbf{r}_2 e^{i\mathbf{q}\cdot\mathbf{r}_2}\Phi_0(r_2,t)F(r_2,0,t)\;, \tag{100}$$

where $K_i(t)$ is given by Eq. (87). For the low-frequency laser $K_i(t)$ is to be replaced by k_i as before, and the quantity η is already defined by Eq. (91).

For the low-frequency laser field we use the stationary-phase approximation[18] to evaluate the time integral in Eq. (97). This is equivalent to replacing $X(t)$ by $X(t_0)$ where t_0 is given by Eq. (88). Unlike Prasad and Unnikrishnan[5] we do not expand the term $\exp[ie\,\mathbf{A}\cdot\mathbf{r}_{12}/c]$, part of the term

$$\exp\left[i\left[\mathbf{k}_i+\frac{e\mathbf{A}}{c}\right]\cdot\mathbf{r}_{12}\right]$$

occurring in Eq. (98), in terms of Bessel functions [as in Eq. (11)] before carrying out integration over the coordinate r_{12}. So the expression for the laser-assisted exchange-scattering amplitude $f_{\rm el}^{l,\rm ex}$ determined in this paper is somewhat different and simpler than that of Prasad and Unnikrishnan.[5] The above method of approximation shows that, for low-frequency laser-assisted elastic-exchange scattering, $f_{\rm el}^{l,\rm ex}$ is obtained from Eqs. (64) and (72) by replacing direct transition amplitudes f_{00} and $f_{np,0}$ by corresponding expressions for the field-free exchange amplitudes $f_{00}^{\rm ex}$ and $f_{np,0}^{\rm ex}$.

IV. RESULTS AND DISCUSSION

We have performed numerical calculations for field strength $\mathcal{E}=0.02$ in atomic unit (a.u.) and angular frequency of the EM field $\omega=0.074$ a.u. for laser-assisted elastic electron scattering by hydrogen atoms at 100-eV electron energy using both Born and Glauber approximations. We have already mentioned in Sec. I that there are some inadequacies of the Glauber approximation. Byron and Joachain[11] have shown that the second-order term of the Glauber series for $1s\rightarrow 1s$ and $1s\rightarrow ns$ transition amplitudes unlike the corresponding second-Born-approximation term does not contain any real part which is connected with the polarizability of atom. The "imaginary part" of the second-order Glauber term is weakly divergent when $q\rightarrow 0$ due to the fact that the average excitation energy of the intermediate states is neglected in the derivation of this term. However, the above imaginary part closely agrees with the corresponding part of the second-Born-approximation amplitudes except when q is extremely small. Byron and Joachain[11] have used the eikonal Born series amplitude defined in Sec. I in place of the Glauber amplitude. We have already noted that the small-angle region is of great interest since the effect of laser modification of the atomic state (especially for odd number of photons transfer) is very large because the $1s\rightarrow np$ field-free transition amplitude is strongly peaked in the forward direction. In this connection we briefly mention some results given by previous investigators[12,13] for field-free transition amplitudes obtained by using the EBS method and the Glauber approximation. For field-free $1s$-np transitions the first-order Glauber amplitude term which is identical with the first Born term, is strongly peaked and gives the most dominant contribution and consequently the Glauber approximation is expected to give a satisfactory result in this case. It appears from the numerical result given by Byron and Latour[13] that there is a close agreement between the values of the differential scattering cross sections evaluated by the Glauber approximation and the EBS method in the case of $1s\rightarrow 2p$ transition and excitation of $n=2$ state of the atomic hydrogen when q is small. From Refs. 12 and 13 we also find that the values of differential scattering cross sections calculated by using the EBS amplitudes for $1s\rightarrow 1s$ and $1s\rightarrow 2s$ transitions agree more closely with the corresponding Glauber results than with the Born results. From above observations it is expected that the results for laser-assisted electron atom scattering cross sections obtained by Glauber approximation are likely to be quite close to the corresponding EBS results in the small-angle region for transfer of odd number of photons.

The laser-assisted differential cross sections (for direct scattering) evaluated by using the relation (64) which takes into account the effect of dressing of the ground state by the laser, are plotted in Figs. 1–3 for transfer of no photon, one photon, and two photons, respectively. The photons are taken to be plane polarized along **q**. Plotted results are given in both Born and Glauber approximations. In the above diagrams we also show the results considering unperturbed atomic wave functions which are related to the first term in Eq. (64). We may note that Byron and Joachain[6] in their numerical calculations based on a relation like Eq. (64) have replaced only the field-free elastic Born amplitude f_{00} by the eikonal Born series amplitude $f_{00}^{\rm EBS}$ (or $f_{\rm el}^{\rm EBS}$) but used the Born amplitudes $f_{np,0}^{B}$ for the $1s \rightarrow np$ transition amplitude $f_{np,0}$. Since for very small] momentum transfer the $1s \rightarrow np$ transition amplitude gives the dominant contribution to the differential scattering cross section for one photon transfer, we also use the Glauber amplitude $f_{np,0}^{G}$ (which is expected to be more accurate) for the $1s \rightarrow np$ transition term. We may note that the asymptotic forms of both $f_{np,0}^{B}$ and $f_{np,0}^{G}$ vary as q^{-1} as $q \rightarrow 0$, but they differ by a factor to be determined from the relations (94)–(96). Similarly, the asymptotic forms of f_{00}^{B} and f_{00}^{G}

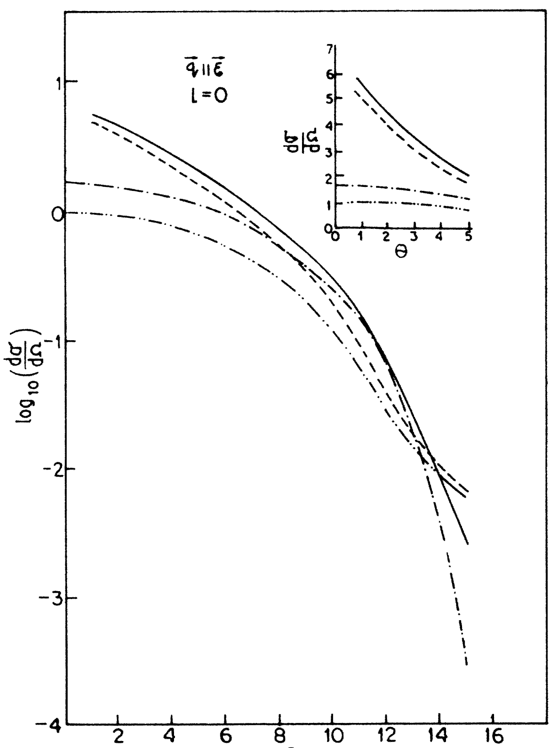

FIG. 1. The differential cross section for elastic electron-hydrogen scattering with the transfer of no photon ($l=0$) at an incident electron energy of 100 eV. The electric field is parallel to momentum transfer **q** and the field strength is 10^8 V cm^{-1} and angular frequency $\omega = 2$ eV/h: —··—, the first-Born-approximation differential cross section (neglecting dressing); —·—, the same including dressing of the target; — — —, Glauber differential cross section neglecting dressing of the target; ——, the same including dressing of the target. The angle θ is measured in degrees.

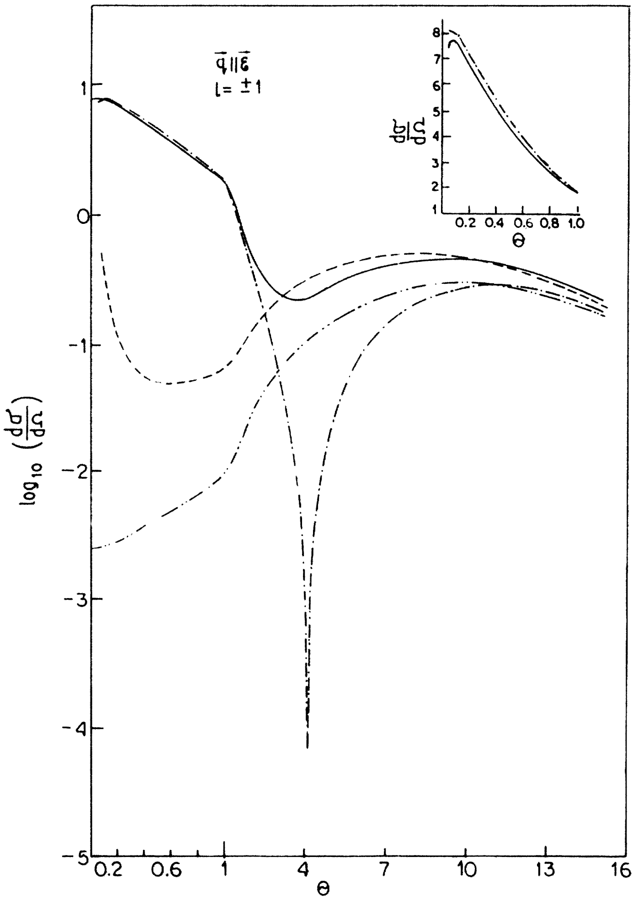

FIG. 2. The same as in Fig. 1 but with the one-photon transfer case.

can be obtained from Eqs. (89) and (93). The above asymptotic forms together with relation (64) determine the nature of the curves in the various diagrams. The great enhancement of the cross section for one-photon transfer ($l=1$) in Fig. 2 is due to the second term of Eq. (64) involving $f_{0,np}$ and $f_{np,0}$ which is multiplied by Bessel functions (depending upon q) of order zero. In view of the discussion given in the beginning of Sec. III C we may remark that in the above case the one-photon transfer process mostly involves the bound electron when $q \rightarrow 0$.

For two-photon transfer ($l=2$) the first term in Eq. (64) which corresponds to the case where both the photons are absorbed by the free projectile electron, varies in the Born approximation as q^2 as $q \rightarrow 0$. On the other hand, the second term in Eq. (64), which involves the product of the Bessel function of order 1 and the $1s \rightarrow np$ transition amplitude, becomes constant as $q \rightarrow 0$ and so dominates over the first term when $q \rightarrow 0$. In this case one of the two photons is absorbed by the free projectile electron and the other by the bound electron. The above asymptotic behavior as $q \rightarrow 0$ can be seen also from the nature of the curves in Fig. 2. Keeping in mind relations

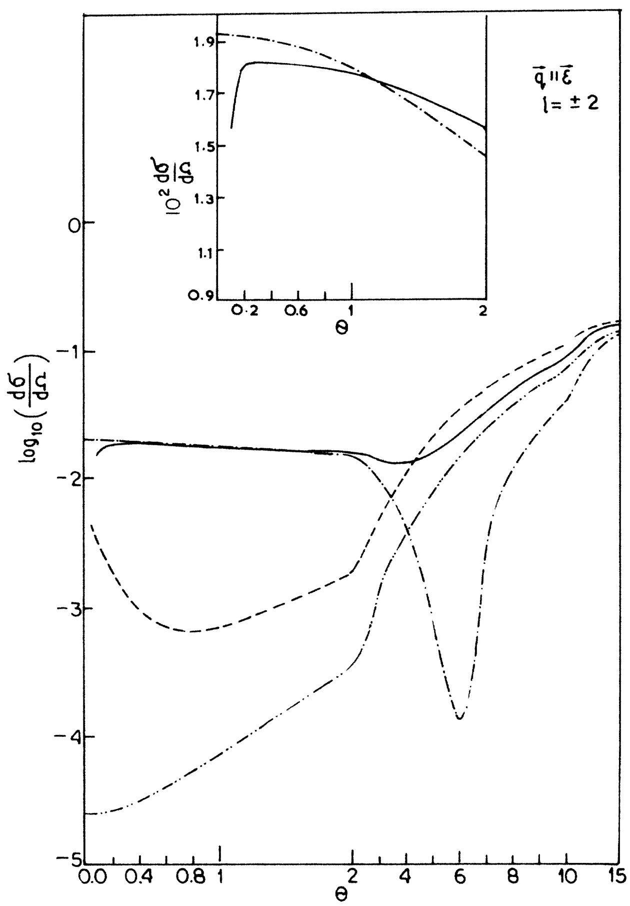

FIG. 3. The same as in Fig. 1 but with the two-photon transfer case.

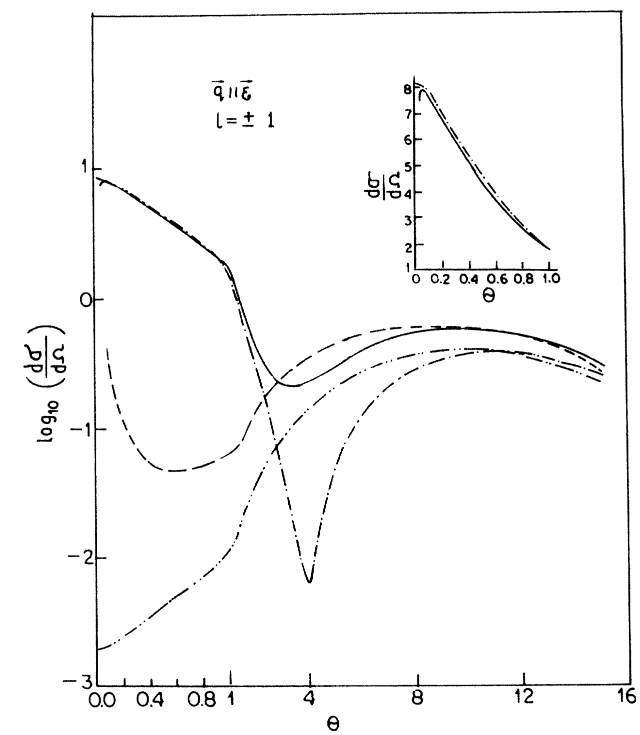

FIG. 4. The full differential cross section for elastic electron-hydrogen scattering with the transfer of one photon ($l=\pm 1$) considering both direct and exchange scattering amplitude.

(58) and (76) involving the $A_{l^a=2}$ term and the discussion at the end of Sec. III C we may note that when both the photons are absorbed by the atomic electron the scattering amplitude related to this term becomes constant for very small-angle scattering in the Born approximation. The above $A_{l^a=2}$ term, which is of second order in dipole interaction, gives a smaller contribution to the laser-assisted scattering cross section than the previously mentioned second term involving $\tilde{A}_{+1}$ in Eq. (64) which is of first order in dipole interaction. The $A_{l^a=2}$ term is not taken into account in our numerical calculation.

In view of the discussion given after Eqs. (76) and (77) we find that for three-photon transfer, the scattering amplitude related to the $A_{l^a=0}$ (or $A_{l^a=3}$) term in Eq. (58) corresponding to the case when all three photons are absorbed by the free (or bound) electron, varies as q^3 (or $1/q$) in the Born approximation when $q \to 0$.

It has already been mentioned that the exchange scattering amplitude is quite small compared to the direct scattering amplitude in the region of small momentum transfer where the effect of laser modification of the atomic wave function on the scattering amplitude is significant. In Fig. 4 we plot the differential cross section including the exchange effect for the scattering of unpolarized electrons by hydrogen atoms in the case of one-photon transfer. Comparing the curves of Figs. 2 and 4 we find that when the exchange effect is considered the differential scattering cross section at the Born minimum is considerably changed, but the corresponding change in the Glauber minimum is quite small.

The relations (64) and (72) show that for plane-polarized photon transfer ($l \neq 0$), the laser assisted direct scattering "amplitude" depends upon the angle between the momentum $\mathbf{q}$ and the electric field vector $\mathcal{E}$. The above amplitude is maximum when $\mathcal{E}$ is parallel to $\mathbf{q}$ and vanishes when $\mathbf{q}$ is perpendicular to $\mathcal{E}$. If we take the propagation vector of the laser along z axis then the scattering amplitude depends only on the magnitude of the perpendicular component $\mathbf{q}_\perp$ ($=\hat{\mathbf{e}}_x q_x + \hat{\mathbf{e}}_y q_y$) of the momentum transfer $\mathbf{q}$ and is independent of the direction of $\mathbf{q}_\perp$. Furthermore, this amplitude vanishes when $\mathbf{q}_\perp$ is zero.

APPENDIX

Explicit forms of some expansion coefficients $B_\kappa^{n,l}$, $b_\kappa^{n,l}$ and $\tilde{\beta}_{2s,p}$ associated with spectral representation of perturbed wave functions are given below:

$$B_\kappa^{0,l}=\delta_{l0}b_\kappa^{0,0}=\delta_{l0}\delta_{\kappa 0}\ , \tag{A1}$$

$$B_\kappa^{1,\pm1}=b_\kappa^{1,\pm1}=-\frac{1}{(\omega_{\kappa0}\pm\omega)}h_{\kappa0}^{1,\pm1}\ , \tag{A2}$$

$$b_\kappa^{2,\pm2}=-\frac{1}{\omega_{\kappa0\pm2\omega}}h_{\kappa\kappa'}^{1,\pm1}b_{\kappa'}^{1,\pm1}\ , \tag{A3}$$

$$b_\kappa^{2,0}=-\frac{1}{\omega_{\kappa0}}\sum_{\nu=1,-1}h_{\kappa\kappa'}^{1,-\nu}b_{\kappa'}^{1,\nu}\ , \tag{A4}$$

$$b_\kappa^{3,\pm3}=\frac{1}{\omega_{\kappa0}\pm3\omega}\left[-h_{\kappa\kappa'}^{1,\pm1}b_{\kappa'}^{2,\pm2}+\epsilon_{\pm1}^{2,\pm1}b_\kappa^{1,\pm1}\right]\ , \tag{A5}$$

$$b_\kappa^{3,\pm1}=\frac{1}{\omega_{\kappa0}\pm\omega}\left[-\sum_{\nu=1,-1}h_{\kappa\kappa'}^{1,\mp\nu}b_{\kappa'}^{2,\pm(1+\nu)}+\sum_{q=1,-1}\epsilon_{\pm1}^{2,\mp q}b_\kappa^{1,\pm q}+\epsilon_{\mp1}^{2,\pm1}b_\kappa^{1,\pm1}\right]\ , \tag{A6}$$

$$b_\kappa^{4,\pm1}=\frac{1}{\omega_{\kappa0}\pm l\omega}\left[-\sum_{\nu=1,-1}h_{\kappa\kappa'}^{1,\mp\nu}b_{\kappa'}^{3,\pm(l+\nu)}+\sum_q(\epsilon_{\pm1}^{2,\pm(l-1-q)}b_\kappa^{2,\pm q}+\epsilon_{\mp1}^{2,\pm(l+1-q)}b_\kappa^{2,\pm q})\right]\ , \tag{A7}$$

$$B_\kappa^{2,\pm l}=b_\kappa^{2,\pm l}+\beta^{2,\pm l}b_\kappa^{0,0},\quad l=2,0 \tag{A8}$$

$$B_\kappa^{3,\pm l}=b_\kappa^{3,\pm l}+\sum_p\beta^{2,\pm p}b_\kappa^{1,\pm(l-p)},\quad l=3,1 \tag{A9}$$

$$B_\kappa^{4,\pm l}=b_\kappa^{4,\pm l}+\sum_p\beta^{2,\pm p}b_\kappa^{2,\pm(l-p)}+\beta^{4,\pm l}b_\kappa^{0}\ , \tag{A10}$$

$$\tilde\beta^{2,\pm2}=-2\beta^{2,\pm2}=(b_\kappa^{1,\mp1})^*b_\kappa^{1,\pm1}\ , \tag{A11}$$

$$\tilde\beta^{2,0}=-2\beta^{2,0}=\sum_{q=1,-1}(l_\kappa^{1,-q})^*b_\kappa^{1,-q}\ , \tag{A12}$$

$$\tilde\beta^{4,\pm p}=\sum_q[(b_\kappa^{1,\mp q})^*b_\kappa^{3,\pm(p-q)}+(b_\kappa^{2,\mp q})^*b_\kappa^{2,\pm(p-q)}+(b_\kappa^{3,\mp q})^*b_\kappa^{1,\pm(p-q)}],\quad p=4,2,0 \tag{A13}$$

$$\beta^{4,p}=-\tfrac{1}{2}\tilde\beta^{4,p}+\tfrac{3}{8}\sum_{p_1}\tilde\beta^{2,p_1}\tilde\beta^{2,(p-p_1)}\ , \tag{A14}$$

$$\beta^{6,p}=-\tfrac{1}{2}\tilde\beta^{6,p}+\tfrac{3}{8}\sum_{p_1}(\tilde\beta^{2,p_1}\tilde\beta^{4,(p-p_1)}+\tilde\beta^{4,p_1}\tilde\beta^{2,(p-p_1)})-\tfrac{5}{16}\sum_{p_1}\sum_{p_2}\tilde\beta^{2,p_1}\tilde\beta^{2,p_2}\tilde\beta^{2,(p-p_1-p_2)}\ . \tag{A15}$$

The quantities like $\epsilon_{\pm1}^{s,t}$ in the above equations are given by Eq. (39).

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