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Exact periodic solution in the semiclassical Jaynes-Cummings model without the rotating-wave approximation

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In the semiclassical version of the Jaynes-Cummings model without the rotating-wave approximation, an exact periodic solution has been obtained and studied for some specific values of the coupling constant and the detuning.

A two-state atom in a resonant cavity allowing only a single-field mode is described by the Jaynes-Cummings model.[1] The case of N two-state atoms was studied by Tavis and Cummings.[2] As is known under the rotating-wave approximation (RWA) the model is exactly solvable for the semiclassical as well as quantum-mechanical version. In the semiclassical case of the non-RWA Jaynes-Cummings model chaotic behavior has been discovered.[3,4] This discovery has shown that a two-level quantum system interacting with its own radiation field in a resonant cavity is a simple quantum system which can exhibit chaos. Various aspects of chaos in the Jaynes-Cummings model have been studied (see Refs. 5 and 6 for other references). However, the condition under which a solution shows chaotic or nonchaotic behavior is a problem that has not yet been solved. Also, up to now, no analytical solution has been found for the non-RWA version.

The purpose of this note is to present an example of an exact analytic solution in the semiclassical Jaynes-Cummings model. The solution is periodic and its Fourier transform is discrete.

In the Jaynes-Cummings model a single-mode field and a two-state atom couple to each other via the undamped Bloch-Maxwell equations

$$\dot{s}_1 = -s_2 \ , \tag{1a}$$

$$\dot{s}_2 = s_1 + s_3 E \ , \tag{1b}$$

$$\dot{s}_3 = -s_2 E \ , \tag{1c}$$

$$\ddot{E} + \mu^2 E = \alpha s_1 \ , \tag{2}$$

where the dimensionless parameter $\mu = \omega/\omega_0$, the coupling constant $\alpha = 8\pi N d^2 \omega_0 \hbar^{-1}$ and N is the number of two-level atoms. In (1) and (2) s_1, s_2, s_3 are components of Bloch's vector describing polarization and inversion. The electric field $E = 2d\tilde{E}/\hbar\omega_0$, d being the electric dipole moment of the atom, is dimensionless and equals the ratio of the Rabi frequency and the atomic transition frequency ω_0. In (1) and (2) the dot denotes the derivative with respect to the dimensionless time which is scaled with the atomic transition frequency ω_0. A different version of (2) in which $\ddot{s}_1$ appears instead of s_1 has also been studied[7] and chaos has been found.[4] The model of a two-level atom described by Eqs. (1) is valid under the assumption[8] that $E \ll 1$. It is well known that the system (1) and (2) possesses conservation laws for length of the Bloch vector and energy:

$$s_1^2 + s_2^2 + s_3^2 = 1 \ , \tag{3}$$

$$W = \alpha s_3 - \alpha s_1 E + \tfrac{1}{2}\mu^2 E^2 + \tfrac{1}{2}(\dot{E})^2 \ . \tag{4}$$

For (1) and (2) the following solution has been found:

$$E = E_0 \mathrm{cn}(\Omega t, k) \ , \tag{5}$$

$$\Omega^4 = \tfrac{1}{3}(\mu^2 - \tfrac{2}{27}) + \tfrac{1}{3}[\alpha^2 - 4(\mu^2 - \tfrac{1}{9})^3]^{1/2} \ , \tag{6}$$

and $k^2 = (\mu^2 - \frac{1}{3})/4\Omega^2 + \frac{1}{2}$, $E_0^2 = 4(\mu^2 - \frac{1}{3}) + 8\Omega^2$.

The inversion s_3 and the components of the dipole moment expressed in terms of E takes the form

$$s_3 = \alpha^{-1}\{-\tfrac{3}{2}(\mu^2 - \tfrac{1}{9})^2 + [\alpha^2 - 4(\mu^2 - \tfrac{1}{9})^3]^{1/2} + \tfrac{3}{4}(\mu^2 - \tfrac{1}{9})E^2 - \tfrac{3}{32}E^4\} \ , \tag{7}$$

$$s_1 = \alpha^{-1}[\tfrac{3}{2}(\mu^2 - \tfrac{1}{9})E - \tfrac{1}{8}E^3], \quad s_2 = -\dot{s}_1 \ . \tag{8}$$

The essential point of the solution is that it is valid only for

$$W = -2(\mu^2 - \tfrac{1}{9})(\mu^2 - \tfrac{5}{9}) + \tfrac{5}{3}[\alpha^2 - 4(\mu^2 - \tfrac{1}{9})^3]^{1/2} \ . \tag{9}$$

In this way the solution (5), (7), and (8) which may be expressed in terms of the elliptic Jacobian functions is valid only for specific values of W given by (9).

From the fact that E_0, Ω, and W are real and $|s_3| \leq 1$ one obtains appropriate conditions for μ and α under which the solution is valid. The detailed analysis shows that for $\mu^2 = \frac{1}{9}$, i.e., $\omega_0 = 3\omega$, there is no condition on the coupling constant α. This case corresponds to resonance because $s_3 = +1$ is reached and the amplitudes of E and s_3 attain maximal values. For $\mu = \frac{1}{3}$, Eq. (7) reduces to

$$s_3 = 1 - \frac{3}{32\alpha} E^4 , \tag{10}$$

and Ω, k, E_0 in (5) are given by

$$\Omega^4 = \tfrac{1}{81}(1+27\alpha) , \tag{11a}$$

$$k^2 = \frac{1}{2} \frac{-1+(1+27\alpha)^{1/2}}{(1+27\alpha)^{1/2}} , \tag{11b}$$

$$E_0^2 = \tfrac{8}{9}(-1+\sqrt{1+27\alpha}) . \tag{11c}$$

For $E = 0$, $s_3 = +1$ and for $E = E_0$, $s_3 = s_{3\,\min}$ given by

$$s_{3\,\min} = 1 - \frac{2}{27\alpha} (\sqrt{1+27\alpha} - 1)^2 . \tag{12}$$

From (4) for the value of W one obtains $W = 5\alpha/3$. Detailed study of (10) and (11) shows that for arbitrary α, one gets $s_{3\,\min} > -1$, i.e., the atom never reaches the ground state, and for $\alpha \rightarrow 0$, one gets $k \rightarrow 0$, $E_0 \rightarrow 0$, $s_{3\,\min} \rightarrow +1$. From the mathematical point of view Eqs. (10) and (5) with (11) are valid for arbitrary α. However, since for the two-level model of an atom $E \ll 1$ must hold, from (5) and (11c) we obtain $\alpha \ll 1$. For sufficiently small α Eqs. (10) and (5) with (11) describe oscillations of the inversion s_3 below and close to $+1$. The solution presented above demonstrates that strictly periodic exchange of energy between a single-mode field and a two-level atom may take place.

For the elliptic function, the Fourier spectrum is characterized by a series of cosine functions (see, for example, Ref. 9, p. 575) which shows that the spectrum of E is discrete. The highest value of the amplitude is for the fundamental frequency $\omega_f = \omega_0 \pi \Omega / 2K = 3\omega\pi\Omega/2K$, where $K = K(k)$ is the quarter period. Higher frequencies are given by $(2n+1)\omega_f$ where n is an integer. It has been found numerically that $\pi\Omega/2K > \frac{1}{3}$ and $\pi\Omega/2K \rightarrow \frac{1}{3}$ for $\alpha \rightarrow 0$. From (5), and (10) for $\mu = \frac{1}{3}$, it is clear that the Fourier transform of the inversion is also discrete.

Two questions seem to be essential for further understanding of solutions to Eqs. (1) and (2). First, does the resonant behavior appear for $\omega_0 = 3\omega$ only? Second, are other analytic solutions of Eqs. (1) and (2) possible? The main conclusion of this note is that the exact solution permits further studies of transition from nonchaotic to chaotic behavior. This solution seems to be an isolated solution because it is valid for some fixed values of μ and α and the corresponding value of W, and for other values of W chaotic behavior may appear. This question is now being investigated numerically. In particular, the Fourier transform (for example, for $W \neq 5\alpha/3$ when $\mu = \frac{1}{3}$, or $W = 5\alpha/3$ when $\mu \neq \frac{1}{3}$) should shed some light on what new frequencies in the chaotic region come into existence.

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