Entropy function for multifractals

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Following an analogy to the formalism of statistical mechanics, an entropy function and a free energy are introduced for multifractals. These functions give a full description of the scaling behaviors of multifractals. The method of Halsey *et al.* [Phys. Rev. A **33**, 1141 (1986)] for characterizing multifractals can naturally be interpreted by the use of these functions. For the invariant set of a dynamical system, these functions are furthermore related to the measure-theoretic (Kolmogolov-Sinai) entropy, the topological entropy, and the Lyapunov exponent.

I. INTRODUCTION

In recent years it has been recognized that fractals are very often encountered in many systems. These are the sets which have noninteger Hausdorff dimensions. Some simple examples of fractals like the classical Cantor set, Sierpinski gasket, etc., have an exact self-similarity and can be constructed by applying a simple procedure iteratively. On the other hand, fractals which emerge from nontrivial physical processes do not have the simple scaling structures mentioned above in general and instead possess a spectrum of scaling indices. Sets of this kind are called multifractals.[1–7]

Kadanoff and co-workers[8–10] introduced a powerful method for characterizing multifractals based on a certain partition function. For dynamical systems, statistical mechanical formalism of chaos was considered by Kai and Tomita,[11] and Oono and Takahashi.[12] The example of Julia sets was discussed by Widom *et al.*[13] These works are relevant. Also we mention the highly mathematical work of Ruelle,[14] Bowen,[15] and Sinai.[16]

In this paper we formally follow the method of statistical mechanics to introduce an entropy function and its Legendre transform, the free energy, for multifractals. The method of Halsey *et al.*, is conveniently explained using these functions. Also these functions are related to important quantities like the measure-theoretic entropy (Kolmogolov-Sinai entropy), the topological entropy, and the Lyapunov exponent, in the case of dynamical systems. The work of Oono and Takahashi is related in this respect.

II. ENTROPY FUNCTION FOR FRACTALS

In this section we consider only topological aspects of fractals. The support of measure is considered here. The distribution of the measure on the support will be treated in Sec. III.

A. Entropy function

Let us consider a fractal set which has a systematic partitioning. More precisely, suppose the set is partitioned into $N(n)$ balls at the nth step. At the next step $(n+1)$, each of these balls is divided into some number of balls and we have $N(n+1)$ balls which constitute a partition at the $(n+1)$th step. In the special case where each ball is divided into the same number of new balls, say a, one simply has $N(n)=a^n$. Generally, $N(n)$ does not have to have the exact power behavior, but we expect that the limit $\ln a=\lim_{n\to\infty}\{[\ln N(n)]/n\}$ exists.

Now we are interested in the distribution of the diameter of balls l_i. As we will see later, however, it is natural to consider the distribution of the logarithm of l_i or the scaled variable

$$\varepsilon_i=-(\ln l_i)/n \ . \tag{2.1}$$

As n becomes large, l_i approaches zero, but ε_i takes a finite nonzero value. It can be understood that ε_i is a scaling index corresponding to l_i if (2.1) is written as $l_i=\exp(-n\varepsilon_i)$.

We write the number of balls whose scaling index lies between ε and $\varepsilon+d\varepsilon$ as $\Omega(\varepsilon)d\varepsilon$. Then we expect that $\Omega(\varepsilon)$ has the following scaling form as n becomes large,

$$\Omega(\varepsilon)=\exp[nS(\varepsilon)] \ , \tag{2.2}$$

where we call $S(\varepsilon)$ the entropy function (per step). We considered that (2.2) is the fundamental property of the fractal sets. Conversely, a fractal set is characterized by the *entropy function* $S(\varepsilon)$. The scaling form (2.2) corresponds to the fundamental property of thermodynamics that the entropy is an extensive quantity. In fact, this is a prerequisite for the existence of a thermodynamics.

B. How to calculate the entropy function

We introduce a partition function by

$$Z(\beta)=\sum_{i=1}^{N} l_i^{\beta} = \sum_{i=1}^{N}\exp(-\beta n\varepsilon_i) \ . \tag{2.3}$$

Also the free energy (per step) is defined by [note that the free energy is usually $-(\ln Z)/\beta$]

$$F(\beta)=\frac{1}{n}\ln Z(\beta)=\frac{1}{n}\ln\left[\sum_{i=1}^{N}\exp(-\beta n\varepsilon_i)\right] . \tag{2.4}$$

We will show that the entropy function can be obtained from $F(\beta)$. The summation over the balls in (2.3) can be replaced by an integral over ε using the distribution $\Omega(\varepsilon)$ as

$$Z(\beta)=\int d\varepsilon\,\Omega(\varepsilon)\exp(-\beta n\varepsilon) = \int d\varepsilon\exp\{n[S(\varepsilon)-\beta\varepsilon]\} . \quad (2.5)$$

When n is large, the integral of (2.5) is dominated by the maximum of $S(\varepsilon)-\beta\varepsilon$. Denoting the value of ε which gives the minimum as $\langle\varepsilon\rangle$, we then have

$$\left.\frac{dS(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=\langle\varepsilon\rangle}=\beta . \quad (2.6)$$

Here note that $\langle\varepsilon\rangle$ depends on β. The free energy is obtained from (2.4) and (2.5) as

$$F(\beta)=S(\langle\varepsilon\rangle)-\beta\langle\varepsilon\rangle . \quad (2.7)$$

This is a Legendre transformation. From the entropy function $S(\varepsilon)$, one can determine β and $F(\beta)$ from (2.6) and (2.7). On the other hand, if $F(\beta)$ is known, $S(\varepsilon)$ and $\langle\varepsilon\rangle$ are obtained from $F(\beta)$. This is done by taking the derivative of (2.7) with respect to β. The use of (2.6) gives

$$\langle\varepsilon\rangle=-\frac{dF(\beta)}{d\beta} . \quad (2.8)$$

From (2.7) and (2.8), the entropy function is given by

$$S(\langle\varepsilon\rangle)=F(\beta)-\beta\frac{dF(\beta)}{d\beta} = -\beta^2\frac{d}{d\beta}[F(\beta)/\beta] . \quad (2.9)$$

Thus once the free energy is calculated as a function of β, $\langle\varepsilon\rangle$ is obtained from (2.8) for a given value of β, and $S(\varepsilon)$ at $\varepsilon=\langle\varepsilon\rangle$ is given by (2.9).

In order to understand the meaning of $\langle\varepsilon\rangle$, substitute (2.4) into (2.8) and one obtains

$$\langle\varepsilon\rangle=\sum_{i=1}^{N}\varepsilon_i\exp(-\beta n\varepsilon_i)/Z(\beta) . \quad (2.10)$$

Therefore $\langle\varepsilon\rangle$ is the average of ε_i with respect to a probability distribution which is proportional to $\exp(-\beta n\varepsilon_i)=l_i^{\beta}$.

The free energy also gives some other quantities of interest. The Hausdorff dimension of the set is given by $D_H=\beta_c$ where β_c is a zero of the free energy, i.e., $F(\beta_c)=0$ [see (2.3) and (2.4)]. For $\beta=\beta_c$, (2.7) gives $S(\langle\varepsilon\rangle_c)=\beta_c\langle\varepsilon\rangle_c$, where $\langle\varepsilon\rangle_c$ is the value of $\langle\varepsilon\rangle$ at $\beta=\beta_c$. The number of balls $\Omega(\langle\varepsilon\rangle_c)=\exp[nS(\langle\varepsilon\rangle_c)]$ is thus given by

$$\Omega(\langle\varepsilon\rangle_c)=\exp(\beta_c n\langle\varepsilon\rangle_c)]=\langle l\rangle_c^{-D_H} , \quad (2.11)$$

where $\langle l\rangle_c=\exp(-n\langle\varepsilon\rangle_c)$, and $\langle\varepsilon\rangle_c$ can be regarded as a representative scaling index for l_i.

Another quantity of interest is the escaping rate exponent δ which is defined as

$$\sum_i l_i^D=\sum_i\exp(-n\varepsilon_i D)\sim\exp(-n\delta) \quad (2.12)$$

for a large n. Here D is the dimension of the space in which the fractal is embedded. From (2.4), it can be seen that δ is given by

$$\delta=-F(D) . \quad (2.13)$$

C. Example: Cantor set

The classical Cantor set is constructed by dividing an interval [0,1] as follows. First remove the middle one-third from [0,1]. Remove again the middle one-third from each of the two remaining intervals. This procedure is repeated infinitely many times. We see that at the nth step the partition of the Cantor set has 2^n intervals of equal length 3^{-n}. So ε is a constant, ln3. The free energy is simply calculated from (2.4) as

$$F(\beta)=\ln 2-\beta\ln 3 . \quad (2.14)$$

The entropy function, the Hausdorff dimension, and the escape rate exponent are easily calculated from (2.9), (2.13), and (2.14) as $S(\varepsilon)=\ln 2$ (for $\varepsilon=\ln 3$), $S(\varepsilon)=0$ (otherwise), $D_H=\ln 2/\ln 3$, and $\delta=\ln(\frac{3}{2})$.

III. FRACTALS WITH MEASURE

Let us generalize the entropy function introduced in Sec. II for a fractal with a probability measure. At the nth partition, the ith ball of the partition has a measure p_i. We introduce a scale index α_i by

$$p_i=l_i^{\alpha_i} \quad (3.1a)$$

or

$$\alpha_i=-\frac{1}{\varepsilon_i}\frac{1}{n}\ln p_i . \quad (3.1b)$$

We now have to consider the distribution of the two scale indices ε and α. We write the number of balls whose scale index lies between ε and $\varepsilon+d\varepsilon$, and α and $\alpha+d\alpha$ as $\Omega(\epsilon,\alpha)d\varepsilon d\alpha$. For multifractals, we expect that $\Omega(\varepsilon,\alpha)$ has the following scaling form for large n,

$$\Omega(\varepsilon,\alpha)=\exp[nQ(\varepsilon,\alpha)] , \quad (3.2)$$

where we call $Q(\varepsilon,\alpha)$ a generalized entropy function.

Following Refs. 8 and 9, we consider the generalized partition function

$$\Gamma(q,\beta)=\sum_i p_i^q l_i^{\beta} = \sum_i\exp[-n\varepsilon_i(\alpha_i q+\beta)] . \quad (3.3)$$

The generalized free energy is

$$G(q,\beta)=\frac{1}{n}\ln\Gamma(q,\beta) , \quad (3.4)$$

Obviously, we have the relations

$$Z(\beta)=\Gamma(q=0,\beta)$$

and (3.5)

$$F(\beta)=G(q=0,\beta) .$$

Using the generalized entropy $Q(\varepsilon,\alpha)$, (3.3) is written as

$$\Gamma(q,\beta)=\int d\varepsilon \int d\alpha \exp\{n[Q(\varepsilon,\alpha)-(\alpha q+\beta)\varepsilon]\} \ . \quad (3.6)$$

As in Sec. II, the maximum of the exponent dominates the integral and gives

$$G(q,\beta)=Q(\langle\varepsilon\rangle,\langle\alpha\rangle)-(\langle\alpha\rangle q+\beta)\langle\varepsilon\rangle \ , \quad (3.7)$$

where $\langle\varepsilon\rangle$ and $\langle\alpha\rangle$ give the maximum of $Q(\varepsilon,\alpha)-(\alpha q+\beta)\varepsilon$, so we have

$$\left.\frac{\partial Q(\varepsilon,\alpha)}{\partial\varepsilon}\right|_{\varepsilon=\langle\varepsilon\rangle,\ \alpha=\langle\alpha\rangle}=\langle\alpha\rangle q+\beta \quad (3.8)$$

and

$$\left.\frac{\partial Q(\varepsilon,\alpha)}{\partial\alpha}\right|_{\varepsilon=\langle\varepsilon\rangle,\ \alpha=\langle\alpha\rangle}=\langle\varepsilon\rangle q \ . \quad (3.9)$$

Thus $G(q,\beta)$ is obtained from $Q(\varepsilon,\alpha)$ using (3.7), (3.8), and (3.9). On the other hand, once $G(q,\beta)$ is calculated, $\langle\varepsilon\rangle$, $\langle\alpha\rangle$, and $Q(\langle\varepsilon\rangle,\langle\alpha\rangle)$ are given by

$$\langle\varepsilon\rangle=-\frac{\partial}{\partial\beta}G(q,\beta) \ , \quad (3.10)$$

$$\langle\alpha\rangle\langle\varepsilon\rangle=-\frac{\partial}{\partial q}G(q,\beta) \ , \quad (3.11)$$

and

$$Q(\langle\varepsilon\rangle,\langle\alpha\rangle)=G(q,\beta)-q\frac{\partial G(q,\beta)}{\partial q}-\beta\frac{\partial G(q,\beta)}{\partial\beta} \ . \quad (3.12)$$

Since $\langle\varepsilon\rangle$ and $\langle\alpha\rangle$ are functions of q and β, different regions with scaling indices ε and α are explored by changing the values of the parameters q and β. Thus $Q(\langle\varepsilon\rangle,\langle\alpha\rangle)$ is implicitly a function of q and β. Also note that from (3.10) and (3.11), $\langle\varepsilon\rangle$ and $\langle\alpha\rangle$ are related by a Maxwell relation

$$\frac{\partial}{\partial q}\langle\varepsilon\rangle=\frac{\partial}{\partial\beta}(\langle\alpha\rangle\langle\varepsilon\rangle) \ . \quad (3.13)$$

The entropy $S(\varepsilon)$ for the scaling index ε defined in Sec. II is related to the generalized entropy as

$$\exp[nS(\varepsilon)]=\int d\alpha \exp[nQ(\varepsilon,\alpha)] \ . \quad (3.14)$$

Since the maximum of $Q(\varepsilon,\alpha)$ with respect to α occurs when $q=0$ [see (3.9)], $S(\varepsilon)$ is given by (3.12) with $q=0$. This also can be seen more easily from (3.5).

We can also consider an analogous entropy for α which should be given by

$$\exp[nS'(\alpha)]=\int d\varepsilon \exp[nQ(\varepsilon,\alpha)] \ . \quad (3.15)$$

Since the maximum of $Q(\varepsilon,\alpha)$ with respect to ε occurs when $\alpha q+\beta=0$ [see (3.8)], (3.7) and (3.15) give

$$S'(\alpha)=G(q,\beta) \ , \quad (3.16)$$

where q and β satisfy a relation [see (3.12)]

$$q\frac{\partial G(q,\beta)}{\partial q}+\beta\frac{\partial G(q,\beta)}{\partial\beta}=0 \ , \quad (3.17)$$

and α is given by

$$\alpha=-\beta/q \ . \quad (3.18)$$

A. Critical point

Recall that in Sec. II the critical value β_c is the Hausdorff dimension of the set and the corresponding value of $\langle\varepsilon\rangle_c$ gives the representative scale index. In the present case the critical value of β depends on q,

$$G(q,\beta_c(q))=0 \ , \quad (3.19)$$

and $\beta_c(q)$ can be regarded as a set of generalized dimensions. The scaling index $\langle\varepsilon\rangle_c$ which corresponds to $\beta_c(q)$ could be considered as being representative for a particular value of q. From (3.7), (3.8), and (3.19), we see that $Q(\langle\varepsilon\rangle,\langle\alpha\rangle)$ at the critical point satisfies a relation

$$Q(\langle\varepsilon\rangle_c,\langle\alpha\rangle_c)=\left.\frac{\partial Q(\varepsilon,\langle\alpha\rangle_c)}{\partial\varepsilon}\right|_{\varepsilon=\langle\varepsilon\rangle_c}\langle\varepsilon\rangle_c \ . \quad (3.20)$$

By solving this differential equation, we have

$$Q(\langle\varepsilon\rangle_c,\langle\alpha\rangle_c)=\langle\varepsilon\rangle_c f(\langle\alpha\rangle_c) \ , \quad (3.21)$$

where $f(\langle\alpha\rangle_c)$ is given by

$$f(\langle\alpha\rangle_c)=\left.\frac{\partial Q(\varepsilon,\langle\alpha\rangle_c)}{\partial\varepsilon}\right|_{\varepsilon=\langle\varepsilon\rangle_c} \ . \quad (3.22)$$

By substituting (3.22) into (3.8) and (3.9), we obtain

$$f(\langle\alpha\rangle_c)=\langle\alpha\rangle_c q+\beta_c(q) \quad (3.23)$$

and

$$\left.\frac{df(\alpha)}{d\alpha}\right|_{\alpha=\langle\alpha\rangle_c}=q \ , \quad (3.24)$$

respectively. And (3.23) and (3.24) give

$$\langle\alpha\rangle_c=-\frac{d\beta_c(q)}{dq} \ . \quad (3.25)$$

Thus once $\beta_c(q)$ is known by solving (3.19), $\langle\alpha\rangle_c$ and $f(\langle\alpha\rangle_c)$ are obtained from (3.25) and (3.23).

In terms of $f(\alpha)$, the number of balls $\Omega(\varepsilon,\alpha)$ is written, using (3.2) and (3.21), as

$$\Omega(\langle\varepsilon\rangle_c,\langle\alpha\rangle_c)=\exp[n\langle\varepsilon\rangle_c f(\langle\alpha\rangle_c)]$$
$$=\langle l\rangle_c^{-f(\langle\alpha\rangle_c)} \ , \quad (3.26)$$

where $\langle l\rangle_c=\exp(-n\langle\varepsilon\rangle_c)$ is a representative length. Here note that $\langle\varepsilon\rangle_c$ and $\langle\alpha\rangle_c$ are functions of q [see (3.10) and (3.25)], and then they are related to each other. If (3.26) is compared with (2.11), $f(\alpha)$ can be considered to be a set of generalized dimensions.

Renyi[20] introduced a set of dimensions D_q defined by

$$D_q=\lim_{l\to 0}\left[\frac{1}{q-1}\frac{\ln\chi(q)}{\ln l}\right] \ , \quad (3.27)$$

where

$$\chi(q)=\sum_i p_i^q=\Gamma(q,0) \ , \quad (3.28)$$

and the partition is made by balls with a uniform size l.

In Ref. 9 it was argued that $(q-1)D_q=-\beta_c(q)$, so one should have

$$\chi(q)=\Gamma(q,0)\sim l^{-\beta_c(q)} . \quad (3.29)$$

In the present approach, however, the distribution of l is taken into account, so the relation (3.29), in fact, can be regarded as determining a representative length l or a scale index $\varepsilon=-(\ln l)/n$. From (3.4) and (3.29), we have

$$\varepsilon=G(q,0)/\beta_c(q) \quad (3.30a)$$

and

$$l=\exp(-n\varepsilon)=\exp[-nG(q,0)/\beta_c(q)] . \quad (3.30b)$$

In general, $\beta_c(q)$ is not simply related to D_q, since the partition with balls of uniform diameter is used to define D_q. Thus $\beta_c(q)/(1-q)$ gives an independent set of generalized dimensions. As shown above, however, $f(\alpha)$ which is related to $\beta_c(q)$ by (3.23) is a more natural generalization of the Hausdorff dimension.

IV. SPECIAL CASES

In this section we examine two special cases where l_i or p_i is constant.

A. l_i constant

There is no distribution of $\varepsilon_i=-(\ln l_i)/n$. For the general case discussed previously, $\langle\varepsilon\rangle$ depends on the parameters q and β and is determined by (3.12). Moreover, if one chooses $\beta=\beta_c(q)$. $\langle l\rangle_c=\exp(-n\langle\varepsilon\rangle_c)$ is the representative length. In the present case of $l_i=$const. all the previous formulas are, of course, valid but $\langle\varepsilon\rangle$ [or $\langle l\rangle=\exp(-n\langle\varepsilon\rangle)$] does not depend on q and β and is simply given by a constant ε (or l). The free energy which is related to the distribution of ε is simply given by

$$F(\beta)=\ln a-\beta\varepsilon , \quad (4.1)$$

where $\ln a=-[\ln N(n)]/n$. However, we have a nontrivial distribution of the measure and the generalized free energy is written from (3.3) and (3.4) as

$$G(q,\beta)=-\beta\varepsilon+G(q,0) . \quad (4.2)$$

The generalized entropy is written from (3.12) and (4.2) as

$$Q(\varepsilon,\langle\alpha\rangle)=G(q,0)-q\frac{\partial G(q,0)}{\partial q}=-q^2\frac{\partial}{\partial q}[G(q,0)/q] , \quad (4.3)$$

where $\langle\alpha\rangle$ is given from (3.11) as

$$\langle\alpha\rangle=-\frac{1}{\varepsilon}\frac{\partial G(q,0)}{\partial q} . \quad (4.4)$$

Note that $S'(\alpha)$, the entropy for α defined by (3.14), is the same as $Q(\varepsilon,\alpha)$ in this case. Also $f(\alpha)$ is given by $Q(\varepsilon,\alpha)/\varepsilon$ (we do not need to adjust β to the critical value, since $\langle\alpha\rangle$ does not depend on β). Thus we have

$$S'(\alpha)=\varepsilon f(\alpha)=Q(\varepsilon,\alpha) , \quad (4.5)$$

and (3.2) is written as

$$\Omega(\varepsilon,\alpha)=\exp[n\varepsilon f(\alpha)]=l^{-f(\alpha)} . \quad (4.6)$$

Also, we note that $\beta_c(q)$ is related to D_q as $\beta_c(q)/(1-q)=D_q$ in this case.

B. p_i Constant

Since there is no distribution of p_i in this case, the analysis is essentially reduced to that of Sec. II where a measure is not taken into account. The measure on each ball is given by

$$p=1/N(n)=1/a^n . \quad (4.7)$$

As seen from (2.3), (2.4), (3.3), and (3.4) the generalized free energy is related to the free energy as

$$G(q,\beta)=-q\ln a+F(\beta) . \quad (4.8)$$

From (3.10) and (4.3), the scaling index $\langle\varepsilon\rangle$ is written as

$$\langle\varepsilon\rangle=-\frac{\partial}{\partial\beta}G(q,\beta)=-\frac{\partial}{\partial\beta}F(\beta) . \quad (4.9)$$

Thus $\langle\varepsilon\rangle$ does not depend on q and is given by the same value as that of Sec. II [see (2.8)]. From (3.1) and (4.3), $\langle\alpha\rangle$ is related to $\langle\varepsilon\rangle$ by

$$\langle\alpha\rangle=(\ln a)/\langle\varepsilon\rangle . \quad (4.10)$$

This implies that the generalized entropy is nonzero only when (4.10) is satisfied, and it can be shown that

$$Q(\varepsilon,\alpha)=S(\varepsilon)=S'(\alpha)=\varepsilon f(\alpha) . \quad (4.11)$$

Thus the analysis in this case reduces to that of Sec. II with the free energy $F(\beta)$. This is simply because the distribution of α is essentially identical with that of ε due to the relation $\alpha\varepsilon=\ln a$.

At the critical point, $\beta=\beta_c(q)$ (4.8) gives

$$q\ln a=F(\beta) . \quad (4.12)$$

This equation was previously considered to identify q with the free energy.[17]

V. INVARIANT SET OF A DYNAMICAL SYSTEM

A fractal object often appears in a dynamical system as an invariant set with an invariant measure. An important quantity characterizing the chaotic behavior of the dynamical system is the Kolmogolov-Sinai entropy h.[18] If a series of partitions is appropriately chosen, h is given by

$$h=-\frac{1}{n}\sum_i p_i\ln p_i \quad (n\to\infty) . \quad (5.1)$$

Using (3.3), (3.4), and (3.12) it can be shown that the Kolmogolov-Sinai entropy and the entropy function $Q(q,\beta)$ are related[12] by

$$h=Q(q=1,\beta=0) . \quad (5.2)$$

Also the topological entropy[18] is given by $h_{\text{top}} = (1/n)\ln_i \sum 1$, so from (3.3) it is written as

$$h_{\text{top}} = Q(q=0, \beta=0) . \tag{5.3}$$

A. Escape problem

Let us consider a one-dimensional escape problem of a map $x_{n+1} = g(x_n) = a x_n(1-x_n)$ with $a > 4$. Most of the initial points on [0,1] go to $-\infty$. The remaining set has a Lebesgue measure zero and forms a (fractal) Cantor set. Also all the points in this set are unstable, and they are repellers.

A partition at the nth step of this Cantor set is given by a set of intervals with $\{x \mid 0 \le g^{(n)}(x) \le 1\}$, where $g^{(n)}(x)$ is an nth iterate of $g(x)$. There are 2^n intervals in the partition. Each interval contains a fixed point of $g^{(n)}(x)$, and for a large n, $dg^{(n)}(x)/dx$ is almost constant in an interval. Let us approximate $g^{(n)}(x)$ in an interval by a straight line with a slope $dg^{(n)}(x_i)/dx$, where x_i is a fixed point in the interval. In this approximation, the length of an interval is written as

$$l_i = [dg^{(n)}(x_i)/dx]^{-1} , \tag{5.4}$$

where x_i is a fixed point of $g^{(n)}(x)$. The partition function (3.3) is written as

$$\Gamma(q,\beta) = \sum_{\substack{j \\ \text{fix } n}} p_j^q [dg^{(n)}(x_j)/dx]^{-\beta} , \tag{5.5}$$

where the summation is over all the fixed point of $g^{(n)}$. The Lyapunov exponent γ is the average of $[\ln(dg^{(n)}/dx)]/n$ with respect to the invariant measure, so it can be shown that

$$\gamma = -\frac{\partial}{\partial\beta} G(q,\beta)\bigg|_{q=1,\ \beta=0} . \tag{5.6}$$

Moreover, the invariant measure in this approximation can be taken to be proportional to l_i, so we have

$$p_i = l_i \Big/ \sum_i l_i . \tag{5.7}$$

Then it can be shown that

$$G(q,\beta) = G(0, q+\beta) + q\delta , \tag{5.8}$$

where δ is the escape rate exponent defined by (2.12). So from (3.12), (5.2), (5.6), and (5.8) we have a relation

$$G(q=1, \beta=0) = h - \gamma + \delta , \tag{5.9}$$

which connects the Kolmogolov-Sinai entropy, the Lyapunov exponent, the escape rate exponent, and the generalized free energy.

VI. DISCUSSION

It has been emphasized by Halsey *et al.* that the function $f(\alpha)$ characterizes a distribution of a singular measure on a fractal. The function $f(\alpha)$ can be considered to be a generalization of the Hausdorff dimension [see (3.26)]. However, more information on the scaling behavior of multifractals is contained in the generalized entropy function $Q(\varepsilon,\alpha)$.

As schematically shown in Fig. 1(a), $Q(\varepsilon,\alpha)$ is nonzero in a domain of the ε-α plane. On line *AOB*, $Q(\varepsilon,\alpha)$ is maximum with respect to the variation of α and we have $S(\varepsilon) = Q(\varepsilon,\alpha)$, where $S(\varepsilon)$ is the entropy for the scaling index ε. The corresponding line in the q-β plane of Fig. 1(b) is $q=0$ [see (3.9)]. The entropy $S'(\alpha)$ for the scaling index α is given by $Q(\varepsilon,\alpha)$ on line *DOE* where $Q(\varepsilon,\alpha)$ is maximum with respect to the variation of ε. The corresponding line in the q-β plane is $q\alpha + \beta = 0$ [see (3.8)]. On line *CED*, we have $G(\varepsilon,\alpha) = 0$ and $Q(\varepsilon,\alpha)$ gives $f(\alpha)$ by

$$f(\alpha) = Q(\varepsilon,\alpha)/\varepsilon$$

or

$$f(\alpha) = \partial Q(\varepsilon,\alpha)/\partial\varepsilon$$

[see (3.21) and (3.22)]. This line corresponds to $\beta = \beta_c(q)$ in the q-β plane. The Hausdorff dimension is given by $\beta_c(0)$. Note that $\beta_c(q)$ could be formally defined beyond point C or D where the corresponding entropy is zero as $\beta_c(q) = -\alpha_{\max} q$ or $-\alpha_{\min} q$, where $\alpha_{\max}$ (or $\alpha_{\min}$) is the value α at point C (or D) in Fig. 1(b). However, $\beta_c(q)$ in these regions does not give any information on the scaling behavior of multifractals.

We now have three functions $S(\varepsilon)$, $S'(\alpha)$, and $f(\alpha)$ for the scaling behavior of multifractals. In some special cases like those discussed in Secs. IV and V, these func-

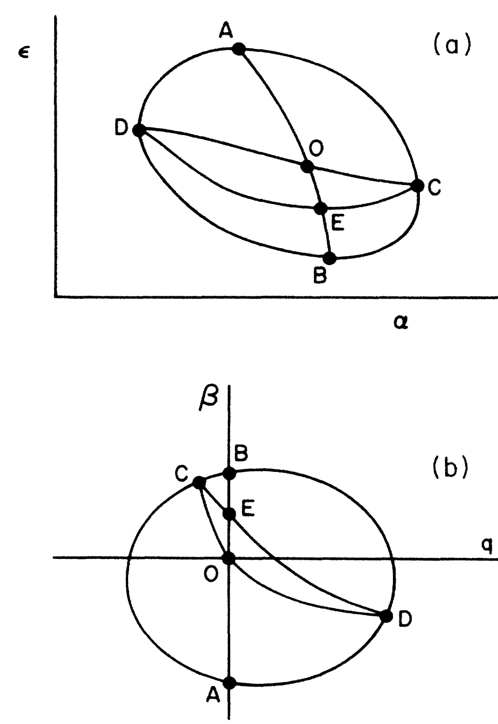

FIG. 1. (a) Schematic diagram of ε-α plane. The generalized entropy $Q(\varepsilon,\alpha)$ is nonzero only in region *ACBD*. (b) Schematic diagram of q-β plane. Points O, A, B, C, D, and E correspond to those in (a).

tions are not totally independent. However, for general multifractals, these functions are independent and give separate information on the scaling behavior. The support of the measure has a scaling structure, i.e., distribution of l_i and it is characterized by the entropy function $S(\varepsilon)$. Similarly, the distribution of the scaling index α for the measure is represented by $S'(\alpha)$. The function $f(\alpha)$ gives a set of generalized dimensions.

All of these functions are related to the generalized entropy $Q(\varepsilon,\alpha)$ which is most conveniently calculated from its Legendre transform $G(q,\beta)$. In a special case like $p_i=\text{const}$ (Sec. II B) or $p_i \propto l_i$ (Sec. V), it is possible to write $G(q,\beta)$ in terms of $F(\beta)$, since the distributions of ε and α are not independent. So an analogy to statistical mechanics is possible. However, we note that in general a partition with $p_i=\text{const}$ is not a natural choice so that we have to deal with the generalized free energy. Although one may wish to have a complete analogy to statistical mechanics, it is not possible in general and perhaps it is not necessary. We simply consider a "statistical mechanics" based on the generalized entropy $Q(\varepsilon,\alpha)$ and the generalized free energy $G(q,\beta)$.

In the ordinary statistical mechanics $\beta=1/k_BT$ is always positive since a state with a higher energy must have a smaller probability than a state with a lower energy. In the present analysis of fractals, the scaling index ε plays a role of the energy and we explore different values of ε by changing the parameter β. At $\beta=0$, all ε_i have the same weight and $\langle \varepsilon \rangle$ is given by the simple average of ε_i, $\langle \varepsilon \rangle_{\text{av}}$. If β is positive, we are focusing on the scaling index $\langle \varepsilon \rangle$ which is less than $\langle \varepsilon \rangle_{\text{av}}$. In the case of a negative β, we have $\langle \varepsilon \rangle$ which is larger than $\langle \varepsilon \rangle_{\text{av}}$. So there is nothing mysterious about a negative β here.

Note added. After submission of this manuscript, Professor D. Rand informed me of the work by Bohr and Rand,[19] in which they discuss a thermodynamic formalism of certain dynamical systems and introduce an entropy function for characteristic exponents. Their work is closely related to Ref. 12, and for some dynamical systems like the escape problem discussed in Sec. V their entropy function is related to the entropy function $S(\varepsilon)$ introduced in Sec. II of the present paper.

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