Critical slowing down near a noise-induced transition point

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Transient behaviors of a nonlinear model system perturbed by external white noises are studied numerically. Results derived with both Stratonovich and Ito interpretations of stochastic calculi demonstrate that instability occurs at a transition point associated with a critical value of the noise parameter. It is found that the critical slowing down familiar in the equilibrium systems occurs near this nonequilibrium instability point. It is also found that for both calculi and over a broad range of initial conditions, the relaxation time diverges with an exponent which is in accord with the classical mean-field value.

I. INTRODUCTION

Considerable efforts have been made in the studies of nonlinear systems far from equilibrium.[1,2] Though these nonequilibrium systems are far more complicated than the equilibrium ones, similarities and analogies between them do exist. With the excitement of the tremendous breakthrough in the phase-transition and critical phenomena of the equilibrium statistical physics,[3,4] the nonequilibrium phase transitions have attracted considerable attentions.[5–9] Besides the familiar transition of the first and second order, new aspects of the noise-induced nonequilibrium phase transitions are also explored.[9] Systems demonstrating such interesting transition phenomena cover many fields in the natural sciences and beyond.[1,2,9]

The phase transition in nonequilibrium systems refers to the transition among different branches of stable steady states (SS). The critical behaviors appear as the relevant control parameter approaches such a unique value that the stable SS turns out to be unstable. As a system approaches the marginal stability, the relaxation processes become slower and slower. This is reminiscent of the well-known phenomena of critical slowing down in equilibrium systems.

Owing to the complexity and diversity of the systems, various types of slowing-down properties have been observed experimentally and concluded theoretically.[10–19] The estimated exponent which characterizes the divergence of the relaxation time varies considerably from system to system. The terminology of critical slowing down was widely used to describe all these time-dependent processes, some of which possess nondivergent relaxation times. There are also some theoretical reports rejecting the existence of critical slowing down.[20,21] The transient behaviors deserve more investigation in order to elucidate the nature of the slowing-down processes in nonequilibrium system.

It is well known in the equilibrium statistical mechanics that the van der Waals theory of fluid systems and the Landau-Ginsberg theory of magnetic systems predict critical phenomena, and that both theories are similar in the sense that they can be described by a simple-cubic equation in terms of the relevant order parameter.[3,4] The multistability which is required for a nonequilibrium transition is a result of the nonlinearity in the rate equation governing the time evolution of the system.[1,2] It is not surprising that one is tempted to adapt the cubic nonlinearity in investigating the critical slowing down and other critical properties in nonequilibrium systems.[22] The simple quadratic nonlinearity, which can be used to describe various physical, chemical, and biological systems,[1,2,23] should not be excluded from the list since the instability does occur very naturally in these simple systems. In this paper, we shall investigate a quadratic nonlinear system which was introduced by Eigen and Schuster[24] to study the self-organization at the macromolecular level. The rate equation and its slowing down in the deterministic sense are discussed in Sec. II. The stochastic treatment, by considering the random white noises, is presented in Secs. III and IV. Numerical results and discussions of the critical slowing down near the noise-induced instability are presented in Secs. V and VI.

II. MODEL SYSTEM AND DETERMINISTIC SLOWING DOWN

The model system describing the macromolecular self-replication under constraint can be expressed in the rate equation[24]

$$\frac{dx}{dt} = Wx - Wx^2/\Omega \; , \tag{1}$$

where x denotes the number of molecules which duplicates themselves precisely with a net replication rate W and Ω stands for the system size limited by a dilution process which is controlled externally. The reacting system is assumed to grow with $W > 0$ from an initial state of $x_0 < \Omega$. The deterministic equation (1) allows two SS solutions,

$$x_1^s = 0, \quad x_2^s = \Omega \; . \tag{2}$$

The stability analysis reveals that the first one is unstable and the second one is stable if $W > 0$. Marginal stability and instability of the SS with $x^s = \Omega$ will be encountered if W decreases through zero. In this model a negative W would mean that the replication process is slower than the degradation one.[24]

The time evolution of the system is found by solving Eq. (1) and can be expressed as

$$x(t)=x_0\Omega[x_0+(\Omega-x_0)e^{-Wt}]^{-1} . \tag{3}$$

As $t\to\infty$, $x(t)\to X^s=\Omega$. In practice, numerical results show that for all initial states $x_0<\Omega$, $x(t)\sim\Omega$ as $Wt\gtrsim 15.0$. By defining the evolution time t_s as the time required for the system to relax towards the SS of $x^s=\Omega$, we have numerically that

$$Wt_s\simeq 15 . \tag{4}$$

This, together with Eq. (3), demonstrates that there exists a deterministic slowing down as the marginal stability of $W=0$ is approached and that this slowing-down phenomenon possesses a mean-field type of exponent $\gamma=1$, since we may rewrite Eq. (4) as

$$t_s=15(W-W_c)^{-1},\quad W_c=0 . \tag{5}$$

III. WHITE-NOISE FORMULATIONS

Stochastically, the parameters in Eq. (1) are subject to fluctuations since various types of noises may affect the replication and dilution processes. It was found that[19] the noise in Ω is most destructive to the system that the growing process might be subjected to the fluctuation catastrophe.[24] In this paper we consider that the net replication rate fluctuates randomly in time as

$$W_t=W+\sqrt{\mathcal{D}}\zeta_t , \tag{6}$$

where W and $\mathcal{D}$ are the mean value and the noise intensity of the replication rate, respectively. ζ_t is the generalized Gaussian process described by

$$\langle\zeta_t\rangle=0,\quad \langle\zeta_t,\zeta_{t'}\rangle=\delta(t-t') . \tag{7}$$

This white-noise realization is usually valid since the correlation time between random noises is negligible as compared with the macroscopic time scale of the reacting systems. The deterministic rate equation (1) is now written in a form of the stochastic differential equation,

$$\frac{dx}{dt}=W(1-x/\Omega)+\sqrt{\mathcal{D}}x(1-x/\Omega)\zeta_t , \tag{8}$$

where x is now a random variable. Due to the random nature of ζ_t, stochastic calculi must be chosen. We shall consider both the Stratonovich and Ito interpretations of the stochastic calculi.[9] The probability function $P(x,t)$ is then found to satisfy the following Fokker-Planck equation:

$$\begin{aligned}\partial_t P(x,t)=-\partial_x\{&[W+(2-\nu)(1-2x/\Omega)\mathcal{D}/2]\\ &\times x(1-x/\Omega)P(x,t)\}\\ &+(\mathcal{D}/2)\partial_{xx}[(1-x/\Omega)^2P(x,t)] ,\end{aligned} \tag{9}$$

where $\nu=1$ and $\nu=2$ stand for the Stratonovich and Ito interpretations, respectively.

An analytical solution of $P(x,t)$ is unlikely. The SS properties can be found with less effort by solving the stationary $P_s(x)$ in the form of the β functions.[25] Since our objectives are the transient behaviors of the system, approximation schemes are therefore needed to treat the problem.

IV. STOCHASTIC TRANSIENT BEHAVIORS

Being unable to find $P(x,t)$ in closed form, we turn to the moments of it,

$$\langle x^n\rangle=\int_0^\infty x^nP(x,t)dx . \tag{10}$$

From Eq. (9), we find that

$$\begin{aligned}\frac{d\langle x^n\rangle}{dt}=&\,nW\langle x^n(1-x/\Omega)\rangle\\ &+(2-\nu)n\langle x^n(1-x/\Omega)(1-2x/\Omega)\rangle\mathcal{D}/2\\ &+n(n-1)\langle x^n(1-x/\Omega)^2\rangle\mathcal{D}/2 ,\end{aligned} \tag{11}$$

where angular brackets represent the averages. Since most stochastic features are described by the first two moments, we truncated the infinite hierarchy of coupled differential equations (11) by using the moment expansion approximation[19,26]

$$\langle x^n\rangle\simeq\langle x\rangle^{n-2}[\langle x\rangle^2+n(n-1)\sigma/2] , \tag{12}$$

where $\sigma=\langle x^2\rangle-\langle x\rangle^2$ is the variance. This approximation involves expanding the third and higher moments in terms of the first two and so is expected to be valid if the relative fluctuation

$$\mathcal{R}(t)=\sqrt{\sigma(t)}/\langle x(t)\rangle \tag{13}$$

remains small. After some mathematics we finally find that the mean and the variance satisfy the following coupled equations in closed form:

$$\begin{aligned}\frac{d\xi}{d\tau}&=\mathcal{A}_{11}\xi+\mathcal{A}_{12}\eta ,\\ \frac{d\eta}{d\tau}&=\mathcal{A}_{21}\xi+\mathcal{A}_{22}\eta .\end{aligned} \tag{14}$$

In the above, we introduce the reduced variables

$$\xi=\langle x\rangle/\Omega,\quad \eta=\sigma/\Omega^2,\quad \tau=Wt , \tag{15}$$

and

$$\begin{aligned}\mathcal{A}_{11}&=1+\alpha(2-\nu)/2-\xi[1+3\alpha(2-\nu)/2]+\alpha(2-\nu)\xi^2 ,\\ \mathcal{A}_{12}&=-1-3\alpha(2-\nu)/2+3\alpha(2-\nu)\xi ,\\ \mathcal{A}_{21}&=\alpha\xi(1-\xi)^2 ,\\ \mathcal{A}_{22}&=2+\alpha(3-\nu)-[8+6\alpha(5-2\nu)]\xi+6\alpha(7-3\nu)\xi^2 ,\end{aligned} \tag{16}$$

where $\alpha=\mathcal{D}/W$ is the relative noise intensity. The transient behaviors are now approximately described by $\xi(t)$ and $\eta(t)$, which can be solved numerically from Eq. (14).

The stochastic SS is defined here as the state with stationary ξ and η. In the framework of the present approximation, it means that the Gaussian $P(x,t)$ will have its peak location and width unchanged. We find that for both the Stratonovich and the Ito interpretations.

$$\langle x_1^s \rangle = 0, \quad \langle x_2^s \rangle = \Omega; \quad \mathcal{R}_1^s = \mathcal{R}_2^s = 0 \; . \tag{17}$$

These are similar to the deterministic SS and agree with the most probable value of x allowed by the SS probability function $P_s(x)$.

Numerical results show that the time evolution of $\langle x \rangle$ is essentially deterministic with relatively small $\sigma(t)$, which depends on the noise intensity and the initial conditions. By comparing the transient $\mathcal{R}(t)$ with those resulting from additive, linear, and quadratic multiplicative noises, it was found that[19] the fluctuation catastrophe is least possible for the present case. The deterministic-type SS, together with the insignificant $\mathcal{R}(t)$, allow us to probe the transient behaviors in detail since the approximation scheme remains valid even when the system encounters the instability.

V. NOISE-INDUCED SLOWING DOWN

Numerical results show that deviation from the deterministic behaviors remain insignificant unless the noise intensity $\mathcal{D}$ approaches a marginal value at which the surviving SS turns out to be unstable. There follows a new type of transition which is purely stochastic in nature.

The linear stability analysis result in that small deviation from which the SS will relax in time as $e^{\lambda t}$. It is found that for the surviving SS,

$$\begin{aligned} \lambda_1 &= -1 + \alpha(2-\nu)/2 \; , \\ \lambda_2 &= -6 + (11 - \nu - 2\nu^2)\alpha \; . \end{aligned} \tag{18}$$

Hence this SS is stable if $\max(\lambda) < 0$, or

$$0 < \alpha < 3\nu/(5-\nu^2) \; . \tag{19}$$

The SS system becomes marginally stable if

$$\begin{aligned} \alpha &= \alpha_c = 0.75 \quad (\nu = 1) \; , \\ \alpha &= \alpha_c = 6.0 \quad (\nu = 2) \; . \end{aligned} \tag{20}$$

In the above, the subscript c stands for the critical value of α. As $\alpha > \alpha_c$, the surviving SS becomes unstable, and we expect that a stochastic transition[27] from surviving to extinction will occur.

Numerical results demonstrate that the restoring of the SS from deviation or the transient towards the SS from a giving initial state x_0 will take a longer and longer time as $\alpha \to \alpha_c$. This slowing-down phenomenon is triggered by the noise parameter and is not predicted in deterministic theory.

In Fig. 1 we plot the time needed for a system to relax towards the stochastic SS, τ_s, versus the scaled noise factor α/α_c. It is surprising to find that τ_s diverges as $\alpha \to \alpha_c$ in a unique fashion which is almost independent of the stochastic calculi employed and the initial state with which the system starts.

By inspecting Fig. 1 we also find that the slowing-down phenomena occur only in a narrow region $\alpha \lesssim \alpha_c$. This resembles the critical slowing down in the equilibrium systems. Away from the *critical region,* the relaxation towards the SS from arbitrary x_0 is essentially deterministic, i.e., the relaxation time equals the deterministic value,

$$\tau_s^{(\mathrm{det})} \simeq 15.0 \; , \tag{21}$$

which is also independent of x_0.

In Fig. 2 we plot with log-log scale the reduced excess time

$$R_t = (\tau_s - \tau_s^{(\mathrm{det})})/\tau_s^{(\mathrm{det})} \tag{22}$$

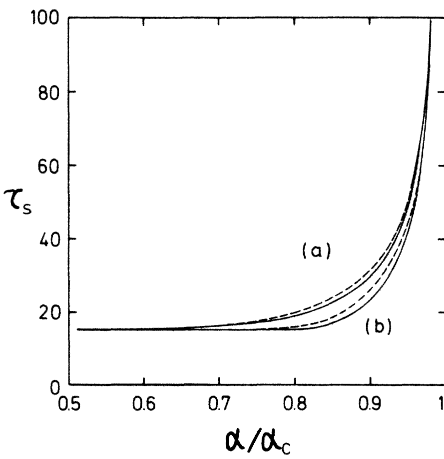

FIG. 1. Time $\tau_s = Wt_s$ required for a system to relax towards the surviving SS plotted against the reduced noise intensity α/α_c. Results derived from Stratonovich calculi (solid curves) are compared with those from Ito calculi (dashed curves). Broad range of initial conditions is included. (a) $\xi_0 = 0.1$ and (b) $\xi_0 = 0.9$.

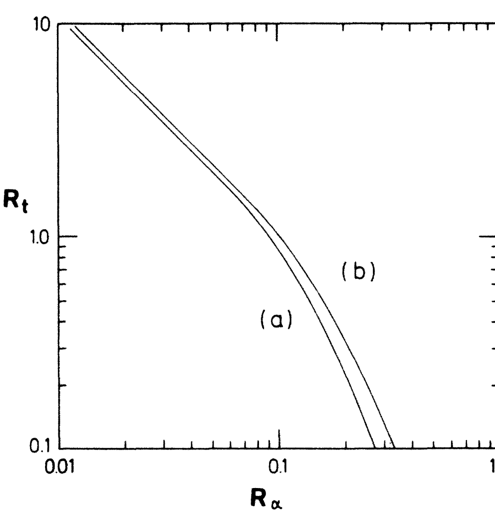

FIG. 2. Reduced relaxation time R_t plotted against the reduced noise parameter. Results of Stratonovich (a) and Ito (b) calculi are compared. For both curves, $\xi_0 = 0.1$ is assumed.

versus the reduced noise parameter

$$R_\alpha = (\alpha_c / \alpha) / \alpha_c \tag{23}$$

which measures the relative distance from the criticality. This enables us to derive the critical exponent γ for the slowing-down process. It is interesting to find that $\gamma \simeq 1.0$ for both stochastic calculi employed and over a wide range of initial conditions.

In Figs. 1 and 2 we intend to choose the scaled and reduced noise factors as an ordinate. These enable us to bring together all the curves derived from different calculi and x_0. These properties resemble the *universality* of the equilibrium critical phenomena.[4]

VI. DISCUSSION

We conclude that the transient behaviors near the nonequilibrium instability resemble the equilibrium critical slowing down. The critical exponent is found to be in accord with the mean-field value; this is a result of the fact that we do not consider the spatial inhomogeneity of the reacting systems.

The slowing-down phenomena demonstrated in our model system are induced by a mixed-order multiplicative noise. Mannella *et al.*[17,28] have investigated the mixing effects of the additive and linear multiplicative noises by using the electronic simulations and have found that the diverging properties of the slowing-down processes are rounded off. This resembles the round-off effects of the earth's gravity on the equilibrium fluid systems in the critical region.[29,30]

The critical slowing down of our model system possesses more features than in equilibrium systems. It describes not only the unique transient properties in restoring to the SS after perturbations but also the relaxation of the system towards the SS from an arbitrary initial state. The critical slowing-down behaviors are also found in the Kramers problems,[31] in which the activation process becomes very slow near the metastable-stable transition.[32,33]

The nonequilibrium systems are far more complicated than the equilibrium ones. Many more factors have to be taken into consideration, especially when external noises becomes significantly large. The noise-induced transition, which is essentially stochastic in nature, is a distinct type of phase transition. The transient behaviors and particularly the interesting slowing-down processes near the transition point surely deserve more efforts.

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