Correlation functions and generalized Lyapunov exponents

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Correlation functions of one- and two-dimensional piecewise linear maps are analytically investigated. The asymptotic time behavior is shown to be given by the average inverse multiplier $\langle \mu_1^{-1}(\tau) \rangle$, for one-dimensional maps with absolutely continuous invariant measure. The decay rate γ coincides with the generalized Lyapunov exponent $\Lambda(q)$ at $q=2$, if the sign of the multiplier does not change during the time evolution, while, in general, it is larger than $\Lambda(2)$. The analysis of two-dimensional maps reveals the importance of the average second multiplier $\langle \mu_2(\tau) \rangle$ and of the average ratio $\langle \mu_2(\tau)/\mu_1(\tau) \rangle$ which, in some cases, can provide the leading long-time contribution.

I. INTRODUCTION

In the analysis of dissipative dynamical systems, most attention has been given to the evaluation of metric entropies, Lyapunov exponents, and fractal dimensions in order to characterize the structure of the attractors both from a dynamic and a static point of view. On the other hand, more standard statistical tools like correlation functions have only been discussed for simple systems, such as various one-dimensional piecewise-linear maps,[1] or in connection with particular phenomena like period doubling,[2] intermittency,[3] diffusion,[4] and "periodic chaos."[5] The decay of correlations in area-preserving maps has been investigated in Ref. 6.

In general, however, very little is known about relations between time correlation functions and the above-mentioned dynamical invariants, even for strictly hyperbolic systems.[7,8] In two and more dimensions, the difficulties arise essentially from the contribution of all Lyapunov numbers (multipliers) to the time decay. In nonhyperbolic systems there is, in addition, no clear separation between expanding and contracting directions and any Lyapunov exponent can assume both positive and negative values, when computed over long, but finite, times.[9] Finally, even for generic one-dimensional hyperbolic maps, no exact results are available. However, it is usually believed that, in "typical" chaotic situations, correlations decay exponentially as, for example, in the case of the Lorenz system.[10] Exceptions are given by the systems discussed in Refs. 3(b)–3(d) and 4, where sublinear diffusion occurs, leading to a power-law behavior.

In this work we evaluate correlation functions of the form

$$C_{AB}(\tau) \equiv \langle A(\mathbf{x}(t)) B(\mathbf{x}(t+\tau)) \rangle - \langle A \rangle \langle B \rangle \equiv \lim_{T\to\infty} \frac{1}{T} \int_0^T A(\mathbf{x}(t)) B(\mathbf{x}(t+\tau)) dt - \langle A \rangle \langle B \rangle , \quad (1.1)$$

where A and B are functions of the position $\mathbf{x}(t)$. Time t can be either continuous or discrete: in the latter case, the integral in Eq. (1.1) is replaced by a sum. The averages can be rewritten as

$$\langle A \rangle = \lim_{T\to\infty} \left[\frac{1}{T} \int_0^T A(\mathbf{x}(t)) dt \right] \equiv \int A(\mathbf{x}) \rho(\mathbf{x}) d^E x , \quad (1.2)$$

where an invariant probability measure $\rho(\mathbf{x})$ (ergodic with respect to the dynamical system) has been defined and E is the phase-space dimension.

In Sec. II, we compute correlation functions for some well-known one-dimensional maps with piecewise-constant invariant measure ρ, showing the connection with dynamical quantities like the generalized Lyapunov exponents $\Lambda_k(q)$ $(k=1,2,\ldots,E)$:[11,12] they characterize the fluctuations of the expansion rates along each invariant direction in phase space. More precisely, the correlation function is shown to coincide, up to multiplicative factors, with the average inverse multiplier $\langle \mu_1^{-1}(\tau) \rangle$, computed over a number τ of iterations. The decay rate γ is equal to $\Lambda_1(2)$ in all cases in which μ_1 has constant sign along the trajectory. In general, γ is larger than $\Lambda_1(2)$, indicating a faster decay. In Sec. III, we give an analytic solution of the 2D generalized baker transformation[13] in terms of symbol sequences. The resulting exact expansions for the two variables allow the evaluation of the correlation functions, performed in Sec. IV. Three different exponentially decaying terms have been identified. Besides the contribution already found in the 1D case, the average second multiplier $\langle \mu_2(\tau) \rangle$ and the average ratio $\langle \mu_2(\tau)/\mu_1(\tau) \rangle$ are present. For some dynamical systems, the new terms may yield the relevant contribution.

In Sec. V, we discuss possible extensions to more generic hyperbolic nonlinear systems. In particular, we show that the average inverse multiplier provides, in all cases, an upper bound to the decay rate.

II. ONE-DIMENSIONAL MAPS WITH PIECEWISE-CONSTANT INVARIANT MEASURE

In this section we show the relations between the decay of the correlation function $C(\tau)$ and dynamical quantities like generalized Lyapunov exponents for one-dimensional maps of the type $y_{n+1}=F(y_n)$, exhibiting invariant measures composed of a finite number of constant parts. We first consider the (asymmetric) Bernoulli shift

$$y_{n+1}=\begin{cases} y_n/p & \text{if } y_n \le p \\ (y_n-p)/q & \text{if } y_n > p \ , \end{cases} \tag{2.1}$$

and the tent map

$$y_{n+1}=\begin{cases} y_n/p & \text{if } y_n \le p \\ (1-y_n)/q & \text{if } y_n > p \ , \end{cases} \tag{2.2}$$

with $p+q=1$. The correlation function $C_{yy}(\tau)$ for these transformations has been studied in Ref. 1(a) by solving the eigenvalue equation for the Frobenius-Perron operator. We follow a different approach which allows discussing extensions to more generic systems. Equation (1.1) can be rewritten as

$$C_{yy}(\tau)=\int y_0 F^\tau(y_0)dy_0-\langle y \rangle^2 \ , \tag{2.3}$$

where the constancy of the measure $[\rho(y)=1]$ has been taken into account and $F^\tau(y_0)$ denotes the τth iterate of the initial condition y_0. The function $F^\tau(y)$ is composed of $N=2^\tau$ straight lines, the slope of which is the multiplier $\mu_1(\tau)=p^i q^{\tau-i}$, computed over τ steps (the index i indicates the number of times y_n was $\le p$). Therefore the integral (2.3) can be evaluated as a sum over N intervals of width $\Delta_j=[\mu_1(\tau)]^{-1}$:

$$\begin{aligned} C(\tau)&=\sum_{j=1}^{N}\int_{\Delta_j} y_0 F^\tau(y_0)dy_0-\langle y \rangle^2 \\ &=\sum_{j=1}^{N}\left[\bar{y}_j\int_{\Delta_j}F^\tau(\bar{y}_j+u)du \right.\\ &\qquad \left. +\int_{\Delta_j}uF^\tau(\bar{y}_j+u)du\right]-\langle y \rangle^2 \ , \end{aligned} \tag{2.4}$$

where the change of variables $y_0=\bar{y}_j+u$ has been made, with $\bar{y}_j$ denoting the abscissa of the midpoint of the jth interval. The first integral in Eq. (2.4) is just equal to $\langle y \rangle$ multiplied by the width Δ_j of the interval, since the distribution is uniform. Therefore, after performing the sum, this term yields $\langle y \rangle^2$. For the second integral, notice that $F^\tau(\bar{y}_j+u)=F^\tau(\bar{y}_j)+u\cdot\mu_1(\bar{y}_j;\tau)$, where $\mu_1(\bar{y}_j;\tau)$ is the multiplier computed over τ iterations, with initial condition $\bar{y}_j$. Insertion of this relation into the integral yields a linear and a quadratic term in u: the first one is zero and the second one gives, after the sum is evaluated,

$$C(\tau)=\tfrac{1}{12}\langle \mu_1(\tau)^{-1} \rangle \ , \tag{2.5}$$

in agreement with Ref. 1(a). There, however, the results were given explicitly as functions of the parameters, without making connections with dynamical quantities.

To elucidate better the meaning of relation (2.5), we recall the notion of generalized Lyapunov exponents,[11,12] which provide a detailed description of the chaotic properties of dynamical systems. Let us first define the effective Lyapunov exponent

$$\lambda_1(x_0;\tau)=\frac{1}{\tau}\ln\prod_{i=0}^{\tau-1}|F'(x_i)|=\frac{1}{\tau}\ln|\mu_1(\tau)| \tag{2.6}$$

as the expansion rate over a finite number τ of iterates. The first generalized exponent $\Lambda_1(q)$ is then defined as

$$\Lambda_1(q)=\lim_{\tau\to\infty}\left[\frac{1}{\tau(1-q)}\ln\langle e^{\tau(1-q)\lambda_1(\tau)}\rangle\right] . \tag{2.7}$$

In the limit $q\to 1$, the usual definition of Lyapunov exponent is recovered. From Eqs. (2.5)–(2.7), in case of an everywhere positive (or negative) multiplier, we have

$$|C_{yy}(\tau)| \propto e^{-\tau\Lambda_1(2)} . \tag{2.8}$$

The reason for the occurrence of the argument $q=2$ is that $C_{yy}(\tau)$ is a *two*-time correlation function: this is reminiscent of the theory of generalized (Renyi) dimensions $D(q)$ which, for integer values of q, can be related to the estimate of q-point correlation functions.[14] In particular, the exponent $D(2)$ is called correlation dimension.[15]

In our case, the complete identification of the decay rate of $C_{yy}(\tau)$ with $\Lambda_1(2)$ only applies when the multiplier does not change sign during the time evolution [as, e.g., for the Bernoulli shift (2.1)], since the average in Eq. (2.7) involves the modulus of the multiplier, at variance with Eq. (2.5). Therefore, the decay rate

$$\gamma=-\limsup_{\tau\to\infty}\left\{\frac{1}{\tau}\ln|C_{yy}(\tau)|\right\} \tag{2.9}$$

is, in general, larger than $\Lambda_1(2)$. For the tent map (2.2), the average multiplier is 0, in the limit case $p=q=\frac{1}{2}$. Accordingly, Eq. (2.9) yields $\gamma=\infty$, indicating a δ-correlated process.

Although the exact result (2.5) has been derived for two special maps, it is not difficult to verify that it applies to any piecewise-linear map with constant invariant measure. The next degree of complexity is represented by maps with only piecewise-constant measure. As an example, we discuss the roof map[1(a)]

$$y_{n+1}=\begin{cases} c+y_n(1-c)/c & \text{if } y_n \le c \\ (y_n-1)/(c-1) & \text{if } y_n > c \end{cases} \quad (2.10)$$

From the solution given in Ref. 1(a), we already know that $C_{yy}(\tau)$ is the superposition of three distinct terms. On the basis of Eq. (2.5), we conjecture that for τ sufficiently large, the slowest contribution to $C_{yy}(\tau)$ is proportional to $\langle \mu_1^{-1}(\tau) \rangle$

$$\limsup_{\tau\to\infty} \left\{ \frac{\ln|C_{yy}(\tau)|}{\ln|\mu_1^{-1}(\tau)|} \right\} = 1 . \quad (2.11)$$

The evaluation of the average $\langle \mu_1^{-1}(\tau)\rangle$ for map (2.10) is not as trivial as in the previous examples, because the multipliers computed over a single iteration are correlated to each other. In fact, the interval $I_1 \equiv [0,c]$ is mapped onto $I_2 \equiv [c,1]$ and I_2 onto $I = I_1 \cup I_2 = [0,1]$. Therefore, since each trajectory must visit the interval I_2 after having visited I_1, the allowed multiplier values can be decomposed into products of uncorrelated terms: a two-step multiplier equal to $-1/c$, corresponding to the sequential visit $I_1 \mapsto I_2$, and a single-step multiplier $-1/(1-c)$, given by the sole visit of I_2 after I_2. This is sufficient to derive a recursion relation for the moment of order $1-q$ of the multiplier (in modulus), defined by

$$M(q,\tau) \equiv \langle e^{\tau(1-q)\lambda_1(\tau)} \rangle \sim e^{\tau(1-q)\Lambda_1(q)} . \quad (2.12)$$

$M(q,\tau)$ can be written as the sum of two distinct terms

$$M(q,\tau) = c^q M(q,\tau-2) + (1-c)^q M(q,\tau-1) , \quad (2.13)$$

where c is the probability of the subsequence I_1, I_2 which contributes with a multiplier $|\mu_1(2)| = 1/c$, raised to the power $1-q$, whereas $1-c$ refers to the subsequence I_2 [with multiplier $|\mu_1(1)| = 1/(1-c)$]. The linear equation (2.13) can be easily solved, yielding the growth rate $\Lambda_1(q)$ of $M(q,\tau)$ [see Eq. (2.12)]. The standard sigmoid shape is recovered with the two horizontal asymptotes $\Lambda_1(-\infty)$, $\Lambda_1(\infty)$ given by $\ln(1-c)$, $(\ln c)/2$ (their ordering depending on c).

Since the estimate of the decay rate γ [Eq. (2.9)] requires considering the multipliers *with* sign and the exponent $q=2$, we rewrite Eq. (2.13) as

$$\langle \mu_1^{-1}(\tau) \rangle = -c^2 \langle \mu_1^{-1}(\tau-2) \rangle - (1-c)^2 \mu_1^{-1}(\tau-1) \rangle . \quad (2.14)$$

The largest eigenvalue (in modulus) yields the growth rate $-\tilde{\gamma}$ of $\langle \mu_1^{-1}(\tau) \rangle$

$$\tilde{\gamma} = -\ln \left| \frac{(1-c)^2 + [(1-c)^4 - 4c^2]^{1/2}}{2} \right| , \quad (2.15)$$

which coincides with the slowest term of $C_{yy}(\tau)$ computed in Ref. 1(a). Hence, we see that the validity of Eq. (2.11) extends beyond the class of maps discussed above. Numerical simulations suggest that whenever the measure is absolutely continuous, the asymptotic time behavior of the correlation function coincides with that of the average inverse multiplier. As an example, we display, in Fig. 1 the logarithm of the modulus of the normalized correlation function $\bar{C}(\tau) = C_{yy}(\tau)/(\langle y^2 \rangle - \langle y \rangle^2)$ for a roof map with the maximum belonging to a period-4 cycle. The resulting probability distribution is composed of three constant parts. In the simulation, 6×10^8 iterations have been used for the correlation function and 10^8 for the multiplier. The convergence of $\bar{C}(\tau)$ to its asymptotic limit is slow for large τ, but a comparison with runs with lower statistics indicates the tendency to a very satisfactory agreement between $\ln|\bar{C}(\tau)|$ and $\ln|M(2,\tau)|$ (dashed and solid lines, respectively, in the figure) also for large values of τ.

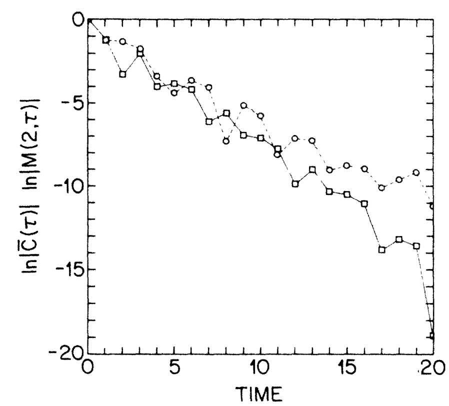

FIG. 1. Dashed curve: plot of the logarithm of the modulus of the normalized correlation function $\bar{C}(\tau) = C_{yy}(\tau)/(\langle y^2 \rangle - \langle y \rangle^2)$ for a modification of the roof map (2.10) vs time τ (number of iterations). Solid line: curve $\ln|M(2,\tau)|$ vs τ. In the simulation, 6×10^8 iterations have been used for the correlation function and 10^8 for the multiplier.

III. GENERALIZED BAKER TRANSFORMATIONS

In this section we investigate how the results found for 1D maps are modified in higher-dimensional cases. In particular, we consider generalized baker transformations[13] of the following form:

$$x_{n+1}=\begin{cases} \alpha x_n & \text{if } y_n \le p \\ \beta x_n + 1 - \beta & \text{if } y_n > p \end{cases} , \quad (3.1)$$

where $\alpha+\beta<1$ and the equations for the y variable are the same as Eqs. (2.1) and (2.2). The asymptotic attractor can easily be recognized as a product of a continuum by a Cantor set. In fact, the action of the map can be described as follows. The unit square is cut horizontally at a height $y=p$ and the two parts are contracted in the x direction by a factor μ_2 which can either be equal to α or to β, depending on the value of y [see Eq. (2.1)], with probability p and q, respectively. The two resulting rectangles are then stretched along the y direction by a factor μ_1, which assumes the value $1/p$ in the lower part and $\pm 1/q$ in the upper one (according to the equation for y). The (Lyapunov) numbers μ_1 and μ_2 are the local (i.e., position-dependent) multipliers. The average Lyapunov exponents $\Lambda_1(1), \Lambda_2(1)$ are given by

$$\Lambda_1(1)=p\ln(1/p)+q\ln(1/q)\ ,$$
$$\Lambda_2(1)=p\ln\alpha+q\ln\beta\ . \tag{3.2}$$

Since the baker maps (2.1) and (3.1) are hyperbolic ($|\mu_1|>1$ everywhere), one can construct Markov partitions[16] and identify any point $\mathbf{x}\equiv(x,y)$ with the associated symbol sequence. The generating partition is obtained by cutting the square horizontally at $y=p$. Accordingly, it is possible to give the "bit" expansion for each variable. We illustrate the procedure for the case of the Bernoulli shift, discussing then the differences with the tent-baker map. It is convenient to write Eqs. (2.1) and (3.1) as

$$x_{n+1}=x_n\,\alpha\left[\frac{\beta}{\alpha}\right]^{a_n}+(1-\beta)a_n\ ,$$
$$y_n=y_{n+1}p\left[\frac{q}{p}\right]^{a_n}+pa_n\ , \tag{3.3}$$

where $S_n=\{a_n\}$ is the symbolic sequence whose elements a_n are given by

$$a_n=\frac{1+\mathrm{sgn}(y_n-p)}{2}\ , \tag{3.4}$$

and take the values 0 or 1, depending on whether y_n is smaller or larger than p. Iteration of Eq. (3.3) allows to write the expansions in the form

$$x_n=(1-\beta)\left[a_{n-1}+\sum_{i=1}^{\infty}a_{n-1-i}\alpha^i\prod_{j=1}^{i}\sigma^{a_{n-j}}\right]\ , \tag{3.5}$$

$$y_n=p\left[a_n+\sum_{i=1}^{\infty}a_{n+i}p^i\prod_{j=1}^{i}\rho^{a_{n-1+j}}\right]\ , \tag{3.6}$$

where the abbreviations $\sigma=\beta/\alpha$ and $\rho=q/p$ have been introduced. These expressions can be simplified by making use of symmetry properties of the Bernoulli-baker map. By exploiting the invariance of Eq. (3.1) under the transformation

$$x\to1-x\ ,$$
$$\alpha\leftrightarrow\beta\ , \tag{3.7}$$
$$a_n\to1-a_n\ ,$$

it is possible to transform Eq. (3.5) into

$$x_n=\frac{1-\beta}{\beta-\alpha}\left[-\alpha+(1-\alpha)\sum_{i=1}^{\infty}\alpha^i\prod_{j=1}^{i}\sigma^{a_{n-j}}\right]\ . \tag{3.8}$$

Similarly, for the y variable, the invariance of Eq. (3.6) under the transformations

$$y\to1-y\ ,$$
$$p\leftrightarrow q\ , \tag{3.9}$$
$$a_n\to1-a_n\ ,$$

leads to

$$y_n=\frac{p}{q-p}\left[-p+q\sum_{i=1}^{\infty}p^i\prod_{j=1}^{i}\rho^{a_{n-1+j}}\right]\ . \tag{3.10}$$

The use of expressions (3.8) and (3.10), in place of the corresponding ones [(3.5) and (3.6)], considerably simplifies the evaluation of correlation functions, because the only dependence on the symbols a_n is, now, in the exponent of the parameters σ and ρ.

Usually, in experiments on chaotic systems, a single scalar time series w_n is available for a reconstruction of the attractor, through embedding techniques.[17] In the case of the baker map, neither x nor y can be used for this purpose: indeed the variable y is decoupled from x, while the evolution of the variable x cannot be inferred from the knowledge of its own past history alone. Therefore, it is convenient to consider the linear combination

$$w_n=x_n+y_n \tag{3.11}$$

as an appropriate embedding variable for this system.

IV. CORRELATION FUNCTIONS

In the case of the generalized baker transformation, the autocorrelation function $C_{ww}=\langle w_nw_{n+\tau}\rangle-\langle w\rangle^2$ can be evaluated by forming averages over the symbols a_n, using the expansions (3.8) and (3.10). Since the sequence S_n is a random process satisfying

$$\langle a_n\rangle=q\quad\text{and}\quad\langle a_ma_n\rangle=\begin{cases}q & \text{if } n=m\\ q^2 & \text{if } n\neq m\end{cases} \tag{4.1}$$

all the expectation values can be explicitly calculated. Noticing that the correlation $C_{xy}(\tau)=\langle x_0y_\tau\rangle-\langle x\rangle\langle y\rangle$ is identically zero, because the variable y does not depend on the variable x at all, one gets[18]

$$C_{xx}(\tau)=(\alpha p+\beta q)^\tau C_{xx}(0)\ ,$$
$$C_{yy}(\tau)=(p^2+q^2)^\tau C_{yy}(0)\ , \tag{4.2}$$
$$C_{yx}(\tau)=\kappa_{xx}(\alpha p+\beta q)^\tau+\kappa_{yy}(p^2+q^2)^\tau$$
$$+\kappa_{yx}(\alpha p^2+\beta q^2)^\tau\ ,$$

where the constants κ_{ij} and the average values $\langle x\rangle,\langle x^2\rangle,\langle y\rangle,\langle y^2\rangle$ are given in the Appendix. From Eq. (4.2), we see that not only the correlation function $C_{yy}(\tau)$ of the Bernoulli shift decays exponentially (as already seen in Sec. II), but all correlation functions of the two-dimensional map exhibit this behavior. The autocorrelation function for the "embedding" variable w_n is then a linear combination of Eqs. (4.2), which can be written in the following form:

$$C_{ww}(\tau)=\kappa_1\langle\mu_1^{-1}\rangle^\tau+\kappa_2\langle\mu_2\rangle^\tau+\kappa_3\langle\mu_2/\mu_1\rangle^\tau\ , \tag{4.3}$$

where we have introduced the average values of the one-step multipliers

$$\langle\mu_1^{-1}\rangle=p^2+q^2\ ,$$
$$\langle\mu_2\rangle=\alpha p+\beta q\ , \tag{4.4}$$
$$\langle\mu_2/\mu_1\rangle=\alpha p^2+\beta q^2\ .$$

Notice that successive values of μ_1,μ_2 are completely uncorrelated from each other, therefore $\langle\mu_k(1)\rangle^\tau$

$=\langle \mu_k(\tau)\rangle$. By comparing Eq. (4.3) with the 1D result (2.5), we see that two new terms contribute to the correlation function. In the case analyzed in this section (Bernoulli shift for the y variable), the dominating long-time term of C_{ww} is still the one which derives from the expanding multiplier. In fact, from the inequalities

$$\frac{\mu_2}{\mu_1} < \mu_2 < \frac{1}{\mu_1} < 1 \ , \tag{4.5}$$

between single-step quantities ($\mu_1, \mu_2 > 0$), it is clear that $\langle \mu_1 \rangle^{-\tau}$ is the slowest term.

A similar derivation of $C_{ww}(\tau)$ can be performed by letting y evolve according to the tent map (2.2). Moreover, in both cases [(2.1) and (2.2)], the contracting multiplier may be allowed to assume negative values (α and β are taken with sign). However, the resulting lack of symmetry in the series expansions of x_n, y_n does not allow a simplified approach as in Sec. III and a more lengthy procedure has to be followed. The final result[18] confirms the presence of three distinct contributions to $C_{ww}(\tau)$, proportional to $\langle \mu_1^{-1}\rangle, \langle \mu_2\rangle, \langle \mu_2/\mu_1\rangle$, in this more general map also. As a consequence, either $|\langle \mu_2\rangle|$ or $|\langle \mu_2/\mu_1\rangle|$ can yield, in suitable ranges of parameter values, the leading contribution to the correlation function, rather than $|\langle \mu_1^{-1}\rangle|$ (see, for instance, the symmetric tent map where $\langle \mu_1^{-1}\rangle$ vanishes identically, for $\tau \geq 1$). Therefore, the sign of the multipliers has such an effect on correlation functions that it is not possible to infer the time behavior from a simple unique prescription.

V. CONCLUSIONS

In Sec. II we have shown that the leading behavior of the correlation function is determined by the average inverse multiplier, for one-dimensional piecewise-linear maps with absolutely continuous measure. This is clearly seen from our derivation: the amplitude of $C_{yy}(\tau)$ can be interpreted as the difference between the exact value of a suitable integral and its "approximate" estimate from the area below a histogram constructed on the elements of a Markov partition [see Eq. (2.4)]. Therefore, the correlation function is of the same order as the width of the elements used.

Moreover, our approach allows discussion of the limitations of this result and the possibility of future extensions to more generic maps. In fact, a crucial property of the map, exploited to derive Eq. (2.5), is the assumption that the invariant measure over any element of the Markov partition be transformed into the global invariant measure of the interval [0,1], in a finite number τ of iterations. This is the case of the maps [(2.1) and (2.2)], since any uniform distribution remains uniform under the action of a linear transformation. Of course, the same does not hold for nonlinear maps, where an additional characteristic time enters: namely, the relaxation time of a probability distribution towards the invariant measure. On the basis of these arguments, we expect $|\langle \mu_1^{-1}(\tau)\rangle|$ to provide an upper bound to the decay of the correlation function. Preliminary simulations confirm this conjecture showing, however, a finite difference between the two rates.

The analysis of simple two-dimensional maps revealed the existence of three terms in the correlation function. The general question of the possible occurrence of other combinations of the multipliers remains open.

The determination of the decay rate, even in simple dynamical systems, is a much more involved task than that of relating fractal dimensions and metric entropies to Lyapunov exponents. The preliminary results derived in the present paper suggest a promising approach to the problem.

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APPENDIX

Here we give the values of the constants appearing in Eq. (4.2). Since the sequence $\{a_n\}$ is δ correlated [see Eq. (4.1)], the evaluation of average values is straightforward; for example, one gets

$$\langle \sigma^{a_m}\sigma^{a_n}\rangle = \begin{cases} p+q\sigma^2 & \text{if } m=n \\ (p+q\sigma)^2 & \text{if } m\neq n \ . \end{cases}$$

For convenience, the following abbreviations are used:

$$r=p^2+q^2, \quad s=p\alpha+q\beta, \quad t=p^2\alpha+q^2\beta \ .$$

Notice that, in the text, these three quantities are recognized as average multipliers [see Eq. (4.4)]. Accordingly, we obtain

$$\langle x\rangle = \frac{q(1-\beta)}{1-s} \ ,$$

$$\langle x^2\rangle = q(1-\beta)^2\frac{1-p\alpha+q\beta}{(1-s)(1-p\alpha^2-q\beta^2)} \ ,$$

$$\langle y\rangle = \tfrac{1}{2} \ ,$$

$$\langle y^2\rangle = \tfrac{1}{3} \ .$$

The constants k_{ij} in Eq. (4.2) are given by

$$\kappa_{xx} = \kappa_0\frac{2s}{(1-s)(s-r)(\alpha+\beta)} \ ,$$

$$\kappa_{yy} = \kappa_0\frac{r}{(s-r)(t-r)} \ ,$$

$$\kappa_{yx} = \kappa_0\frac{(p-q)(\alpha-\beta)}{(1-s)(r-t)(\alpha+\beta)} \ ,$$

where $\kappa_0 = pq(1-\alpha)(1-\beta)/2$.

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