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Analytic theory of the Saffman-Taylor fingers

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A complete analytic solution is presented for the selection of symmetric Saffman-Taylor fingers, in the small-surface-tension limit. In a previous paper [Combescot *et al.*, Phys. Rev. Lett. **56**, 2036 (1986)] we showed that the selection can be understood in terms of a nonlinear eigenvalue problem that describes the region where ordinary perturbation theory fails to converge. This "inner" problem is completely solved by systematically studying the asymptotic series of the (formal) solution of the problem. Two methods are proposed to extract the selection mechanism from this asymptotic series. Numerical results agree very well with other existing numerical results, and, somewhat surprisingly, with the results that we obtained by a WKB solution of the inner problem. Last, we complete our WKB treatment by determining explicitly the additive constant appearing in the WKB formula.

I. INTRODUCTION

Recently much progress has been done in the analytic understanding of the solvability mechanism for the Saffman-Taylor problem in the limit of small surface tension. It was shown simultaneously by Shraiman,[1] Hong and Langer,[2] and ourselves[3] that it is necessary to perform an asymptotic analysis beyond all orders in the small parameter in order to find the selection among the continuous family of zero surface tension solutions, discovered previously by Saffman and Taylor.[4] The first two papers proposed basically a linear approach to the problem; for example, in Ref. 2, a linear inhomogeneous integrodifferential equation for capillary corrections to the finger shape is derived and then solved by a WKB method which is here rather intricate because of the nonlocal features of the physics. On the contrary, in Ref. 3 (hereafter referred as I), we gave a fully nonlinear treatment of the problem, inspired by the work of Kruskal and Segur on the existence of needle crystals in geometrical models of solidification.[5] The two coupled equations for the interface shape, established by McLean and Saffman,[6] are directly extended into the complex domain without any linearization. They are then studied in the neighborhood of the singularities of the zero surface tension solutions where the regular perturbation expansion breaks down. In the Saffman-Taylor case, the positions of the singularities turn out to be very simply related to the relative width of the finger λ, the quantity of physical interest, through the combination $\alpha=(2\lambda-1)/(\lambda-1)^2$. We wrote in I a single ordinary differential equation which describes correctly the behavior of the solution in the singular region. After proper rescaling, the small parameter k (proportional to the surface tension, see below) disappears but it was found in I that the resulting equation still contains a parameter $a=|\alpha|^{3/2}/k$. The question of selection of λ is therefore converted into the one for a and the inner region is responsible for the whole selection mechanism. The eigenvalue nature of a becomes clear when one adds the right boundary condition at large distance to be satisfied by the inner solution. This is imposed by physical considerations. In order to get a smooth interface shape, only slightly differing from the zero surface tension shape, one simply requires that the inner solution matches at large distance the original Saffman-Taylor solution. This can be achieved only for discrete values of a, forming an infinite countable set a_n.

In I we were able to predict the behavior of the a_n's in the large-n limit but an explicit calculation of the lowest eigenvalues was missing. Since that time, various authors[7,8] have given numerical estimates of the lowest eigenvalues. To get the answer, they all integrated numerically the inner problem along particular paths in the complex plane, using a technique first discussed by Kruskal and Segur.[5] It is the main purpose of this paper to provide an essentially analytical solution of our eigenvalue problem. Beyond the mere quantitative interest of the calculation, we feel that our method clarifies and completes the Kruskal-Segur procedure. In particular, the appearance of exponentially small terms becomes conceptually more transparent and is now totally controlled, despite the nonlinearity of the inner problem. The paper is organized as follows: in Sec. II we give a detailed account of the content of paper I. At the end of the section, we show why the value $\lambda=\frac{1}{2}$ plays a special role in the solvability mechanism. Section III presents an analytic treatment of the inner problem. It is first recognized that all the information we need is contained in the asymptotic expansion around infinity of solutions of the inner problem. At large order in the expansion, the dominant growth rate of the coefficients of the series is in fact given by a linear recursion relation, as pointed out by Dashen *et al.*[9] in another context. On the basis of this simplifying observation, we propose two ways to extract from the asymptotic series the searched solvability condition. The former consists in solving a kind of renormalized linear problem, the latter in performing a Borel resummation of the divergent asymptotic series. At the end of this central section, we give our results for the three lowest values of a. In the last section we discuss the large-a limit where the inner region was shown in I to split into two subregions, defining a new inner problem. This led us to give a prediction for the large-n behavior of the a_n's, which depends on a constant issuing from the new inner problem. Here we calculate exactly this constant, following the lines developed in the previous section.

II. DERIVATION OF THE INNER PROBLEM

Our starting point will be the McLean and Saffman equations for a half finger profile, established in Ref. 6. Before presenting them, we first recall the basic formulation of the Saffman-Taylor problem and some steps of the method used by McLean and Saffman to obtain their equations. This should help the reader to understand better the subsequent analysis.

We consider a Hele-Shaw channel, whose thickness b is much smaller than its width $2a$, along which a fluid of viscosity μ is being pushed by a nonmiscible second fluid of relatively negligible viscosity. Both fluids are incompressible. A single finger of the inviscid fluid is eventually formed and propagates at constant velocity U keeping a steady shape of width $2\lambda a$ (see Fig. 1). The velocity $\mathbf{u}$ of the viscous fluid (averaged across the vertical thickness b) obeys Darcy's law

$$\mathbf{u}=-\frac{b^2}{12\mu}\nabla p=\nabla\phi \ , \tag{1}$$

where p is the pressure and ϕ the velocity potential. Incompressibility implies that ϕ satisfies Laplace's equation

$$\nabla^2\phi=0 \ . \tag{2}$$

On the interface, the boundary conditions are

$$(\hat{\mathbf{n}}\cdot\nabla\phi)_{\rm int}=U\hat{\mathbf{n}}\cdot\hat{\mathbf{x}} \ , \tag{3}$$

where $\hat{\mathbf{n}}$ is the outward normal to the finger and $\hat{\mathbf{x}}$ the unit vector along its direction of propagation; and

$$\phi_{\rm int}=\phi_0+\frac{Tb^2}{12\mu}\kappa \ , \tag{4}$$

where ϕ_0 is the constant value of the potential inside the inviscid fluid, T is the surface tension between the two fluids, and κ the curvature of the interface. At the walls of the channel, free slip boundary conditions are assumed. Far behind the tip of the finger, the viscous fluid remains at rest ($\phi\to 0$), whereas, far ahead of the tip, it moves at velocity $V=\lambda U$.

As usual in two-dimensional problems involving harmonic functions, it is convenient to introduce a stream function ψ defined as the harmonic conjugate of ϕ. McLean and Saffman noted that this stream function becomes constant on the interface when one works in a frame of reference moving with the finger. If one defines $Z=X+iY$, $\hat{W}=\hat{\phi}+i\hat{\psi}$ as, respectively, the complex coordinate and the complex potential in this frame (for proper normalizations see Ref. 6), the transformation $Z\to\hat{W}$ maps conformally the flow region between the point $ABCDE$ in Fig. 1 into an infinite strip of unit width in the potential plane. A second conformal map

$$\hat{W}\to\sigma=s+it=e^{-(\hat{W}-\hat{\phi}_0)\pi} \tag{5}$$

maps finally the flow region in the upper half σ plane. In this representation, the interface AB goes into the real segment $0<s<1$, $s=0$ being the trailing part of the finger ($\hat{W}\to+\infty$) and $s=1$ corresponding to the finger tip ($\hat{W}=\hat{\phi}_0$). In Fig. 2 we have drawn the σ plane, indicating the positions of the various points of interest $ABCDE$.

Rather than solving directly for Z in terms of $\hat{W}$, it is simpler to study the quantity $d\hat{W}/dZ$ which by definition is the complex velocity $\hat{u}_x-i\hat{u}_y$ relative to the finger. At the interface this velocity has to be tangential to the profile and may be written as $\hat{q}e^{-i\hat{\theta}}$ where $\hat{\theta}$ is the angle between the tangent and the x direction. Note that $\hat{q}$, the modulus of the flow velocity at the interface, is equal to $d\hat{\phi}/d\hat{S}$ if $\hat{S}$ measures (dimensionless) arclength from the tip. Making use of the analytic properties of the function $\ln(d\hat{W}/dZ)$ in the upper half σ plane, one gets the first relation between $q=(1-\lambda)\hat{q}$ and $\theta=\hat{\theta}-\pi$

$$\ln q=\frac{s}{\pi}P\int_0^1\frac{\theta(s')}{s'(s'-s)}ds' \ . \tag{6}$$

On the other hand, Laplace's law for the pressure at the interface (4), is expressed in the moving frame. By differentiation with respect to arclength $\hat{S}$, this gives the second equation.

$$kqs\frac{d}{ds}\left[qs\frac{d\theta}{ds}\right]+\cos\theta-q=0 \ , \tag{7}$$

where the dimensionless parameter k is defined as

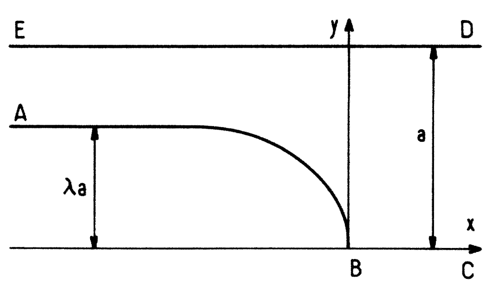


FIG. 1. The flow region in the physical z plane (half-finger profile).

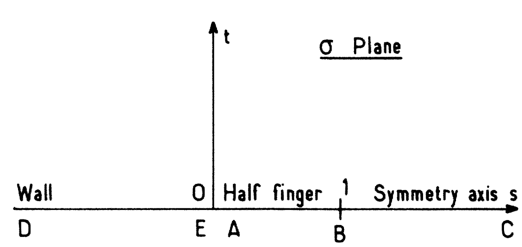


FIG. 2. The flow region in the σ plane.

$$k=\frac{Tb^2\pi^2}{12\mu Ua^2(1-\lambda)^2}\ . \tag{8}$$

These equations must be solved with the following boundary conditions:

$$\text{at the finger-tail,}\ \theta(0)=0\ \text{and}\ q(0)=1$$
$$\text{at the tip,}\ \theta(1)=-\frac{\pi}{2}\ \text{and}\ q(1)=0\ . \tag{9}$$

Once a solution of these equations is known, the corresponding value of λ is deduced via the relation

$$\ln(1-\lambda)=\frac{1}{\pi}\int_0^1\frac{\theta(s')}{s'}ds'\ . \tag{10}$$

Equations (6) and (7) together with (9) and (10) constitute the problem to which McLean and Saffman reduced the determination of finger shapes. In the $k=0$ case, the continuum of solutions discovered by Saffman and Taylor is recovered: $q_0(s)=[(1-s)/(1+s\alpha)]^{1/2}$, $\theta_0(s)=\cos^{-1}q_0(s)$ with $\alpha=(2\lambda-1)/(1-\lambda)^2$. Now we want to study the small k limit and show that this continuum is broken into a discrete countable family of solutions with λ decreasing toward $\frac{1}{2}$ as k goes to zero.

As explained in I, our strategy consists in identifying the regions in the σ-plane where the capillary term in (7) (or at least its analytic continuation) becomes dominant and regular perturbation expansion in k breaks down. After continuing the McLean and Saffman equations in the complex plane, one defines in this small region a rescaled problem (or inner problem) where the small parameter k has disappeared and which captures the singular effects of the capillary term. As a boundary condition on the solutions of this inner problem, one requires that they are asymptotic in the large distance limit to the zeroth-order solutions q_0, θ_0. It turns out that this can be achieved only for a discrete set of values of the parameter $a=|\alpha|^{3/2}/k$ and $\alpha>0$. The inner region contains therefore the whole selection mechanism.

In order to be more specific and to justify our statement, we will consider the whole finger and not a half finger as done by McLean and Saffman. The solution for the velocity in the lower part of the cell is obtained by symmetry with respect to the cell axis from the one in the upper part (more precisely we consider only such symmetric fingers). This means that the corresponding complex velocities are complex conjugates. In the complex σ plane, the lower part of the cell corresponds to the lower half plane and the solution in this half plane is complex conjugate of the one in the upper half plane. Since $\hat{q}e^{-i\hat{\theta}}$ is real in the interval $(1,\infty)$ of the real σ axis, the solution in the lower half plane is merely the analytic continuation through this interval of the one in the upper half plane. However, the σ representation is now rather inconvenient since the finger is represented by a cut [0,1] on the real axis. Although our analysis could be carried out in this representation, we introduce for convenience a new variable v by $\sigma=\cosh^{-2}(v/2)$. This is illustrated in Fig. 3 where the v plane has been drawn with the various lines of interest. The upper half finger corresponds to the interval $(-\infty,0]$ of the real v axis while $[0,\infty)$ corresponds to the lower half, the

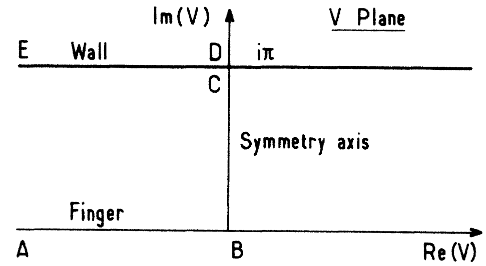


FIG. 3. The flow region in the v plane. Symmetry properties become obvious.

physical region being mapped into the strip $0\le \mathrm{Im}v\le\pi$. The symmetry with respect to the cell axis is now reflected in a symmetry with respect to the imaginary v axis, the complex velocities $\hat{q}e^{-i\hat{\theta}}$ on the left side and the right side being complex conjugate.

Since we want to continue the McLean and Saffman equations on the positive real v axis, we must still use $\theta=\hat{\theta}-\pi$ in order that $\theta(v)$ is continuous when one goes from the upper to the lower half finger. Since $\theta(v)+\pi/2$ is nothing else but the angle between the channel axis and the outward normal $\hat{n}$ to the finger, it is clearly an odd function of v on the real axis. However, on the lower half finger the angle of the velocity with the x axis is no longer $\hat{\theta}$ but $\hat{\theta}-\pi$. Since $\hat{q}e^{-i\hat{\theta}}$ is the complex velocity, this implies that $q(v)$ is odd and not even as is the modulus of the velocity.

Therefore both quantities $\theta(v)+\pi/2$ and $q(v)$ are odd functions of v. We can check that the continuation of the McLean and Saffman equations is compatible with this parity. In terms of the v variable Eqs. (6) and (7) read

$$\ln q(v)=-\frac{1}{\pi\cosh^2(v/2)}\times \mathrm{P}\int_{-\infty}^{0}\frac{\tanh(v'/2)\theta(v')}{\tanh^2(v'/2)-\tanh^2(v/2)}dv'\ , \tag{11}$$

$$kq\coth\left[\frac{v}{2}\right]\frac{d}{dv}\left[q\coth\left[\frac{v}{2}\right]\frac{d\theta}{dv}\right]-q+\cos\theta=0\ . \tag{12}$$

Equation (11) is simply extended in the upper half complex v plane:

$$\ln q(v)-i\theta(v)=-\frac{1}{\cosh^2(v/2)}\times\int_{-\infty}^{0}\frac{\tanh(v'/2)\theta(v')}{\tanh^2(v'/2)-\tanh^2(v/2)}=I(v)\ . \tag{13}$$

Since $I(v)$ is even with respect to v, we obtain by letting v go through the upper half plane to the real positive axis

$$\ln q(-v)+i\theta(-v)=\ln q(v)-i\theta(v)\ . \tag{14}$$

This leads to $\ln q(-v)=\ln q(v)+i\pi$ if we use the fact that $\theta(v)+\pi/2$ is odd. This shows that Eq. (13) respects the odd parity of $q(v)$ and $\theta(v)+\pi/2$. The same is obviously true for Eq. (12). The existence of a symmetric finger satisfying the McLean and Saffman equation is therefore equivalent to the existence of an odd solution $q(v)$ and $\theta(v)+\pi/2$ to Eqs. (12) and (13). Another way to say it is that q and $(\theta+\pi/2)$ have to be purely imaginary on the imaginary v axis.

It is easily seen from the expressions of the Saffman-Taylor solutions that the perturbation expansion in k becomes increasingly divergent near $s_0=-1/\alpha$. In the v plane, this singularity splits into a pair of singularities given by $v_0=i\pi\pm 2\,\arg\sinh(\sqrt{\alpha})$. There are other antecedents of s_0 which are obtained from v_0 by 2π translations along the imaginary axis but which are equivalent to v_0 due to the periodicity of $q(v)$ and $\theta(v)+\pi/2$. For $\alpha<0$ (or $\lambda<\frac{1}{2}$), the pair of singularities is purely imaginary, whereas for $\alpha>0$ ($\lambda>\frac{1}{2}$) it rotates by $\pi/2$ around its center $i\pi$ (see Fig. 4). Note that, in the case $\alpha>0$, the singularities in q are located in the physical plane on the walls, whereas, when $\alpha<0$, they are located on the axis of the channel. It should be kept in mind, however, that the flow is not singular in the physical domain: a singularity in q does not imply the existence of a singularity for the flow field, $qe^{-i\theta}$. It will turn out that the small-k limit corresponds to small values of $|\alpha|$ and we will be interested in studying Eqs. (12) and (13) in the neighborhood of $i\pi$. In this region $I(v)$ admits a regular expansion in powers of k, because it is entirely determined by the behavior of θ on the interface which will depart only slightly from θ_0 in the small-k limit. Hence we may write

$$I(v)=I_0(v)+kI_1(v)+O(k^2)\ . \tag{15}$$

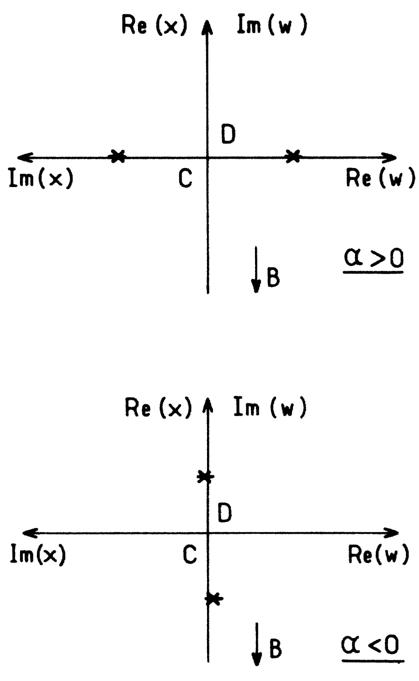


FIG. 4. An enlargement of the singular region. The singular points are denoted by an asterisk.

In the following we shall limit the expansion to zeroth order in k and use for q

$$q(v)=e^{iI_0(v)}e^{i\theta(v)}[1+O(k)]\ . \tag{16}$$

The factor $e^{iI_0(v)}$ may be simply calculated by going back to Eq. (12) at $k=0$:

$$q_0=\cos\theta_0=e^{i\theta_0}\frac{(1+e^{-2i\theta_0})}{2}\ , \tag{17}$$

which, by comparison with Eq. (16), gives at once

$$q(v)=e^{i\theta(v)}\frac{(1+e^{-2i\theta_0})}{2}[1+O(k)]\ , \tag{18}$$

Eq. (18) relates univocally the varying part of $q(v)$ and $\theta(v)$ near their singularities. Note, however, that Eq. (18) implies that the physical quantity $qe^{-i\theta}$ remains everywhere regular even at the singularities of q or $e^{i\theta}$. From the expression of q_0 and Eq. (17) we get for $v=i\pi+w$ with $|w|\ll 1$

$$q_0^{-2}\cong 4e^{-2i\theta_0}\cong\left[\frac{w^2}{4}-\alpha\right][1+O(w^2)]\ . \tag{19}$$

Equations (18) and (19) allow us to simplify the quantity $(q-\cos\theta)$ present in the differential equation as

$$q-\cos\theta=\frac{q}{4}\left[\frac{1}{q_0^2}-\frac{1}{q^2}\right][1+O(k,w^2)]\ . \tag{20}$$

By $O(k,w^2)$ we mean corrections which are at least either of order k or of order w^2 compared to the dominant terms. In the same spirit we may transform the differential operator near $i\pi$ as

$$kq\coth\left[\frac{v}{2}\right]\frac{d}{dv}\left[q\coth\left[\frac{v}{2}\right]\frac{d\theta}{dv}\right]=-iq\frac{w}{4}\frac{d}{dw}\left[w\frac{dq}{dw}\right][1+O(w^2)]\ . \tag{21}$$

Finally we end up with

$$ikw\frac{d}{dw}\left[w\frac{dq}{dw}\right]=\left[q^{-2}+\alpha-\frac{w^2}{4}\right][1+O(k,w^2)]\ , \tag{22}$$

where the two small parameters are still present. By rescaling the function q and the variable w as $q=-iQ/|\alpha|^{1/2}$ and $w=2i|\alpha|^{1/2}x$ we get the equation given in I:

$$a^{-1}x\frac{d}{dx}\left[x\frac{d}{dx}Q\right]+Q^{-2}=\epsilon+x^2\ , \tag{23}$$

where $\epsilon=\mathrm{sign}(\alpha)$ and $a=(|\alpha|^{3/2})/k$. Note the rotation by $-\pi/2$ of the x axes of coordinates. This convention differs from the one chosen in I. The range of validity of the inner equation is determined by the condition that the neglected corrections are indeed small. This is the case as long as $|w^2|\ll 1$ or $|x|\ll 1/|\alpha|^{1/2}$, which corresponds to a large domain for the internal variable x. On the contrary, the perturbation expansion is valid as long as $x \gg a^{-1/3}$ [that is as long as $a^{-1}(xd/dx)(xd/dx)Q_0$ is small compared to Q_0^{-2}].

In the next section we will find that solutions of the Saffman-Taylor problem exist for values of the parameter a of order 1 or larger. This means that the inner region and the outer one have a large overlap and the matching procedure is then consistent. The usual behavior of Saffman-Taylor solutions when w is small may therefore serve as a boundary condition on Q in the large x limit:

$$Q \cong Q_0 \cong \frac{1}{x} \quad \text{for } |x| \to +\infty \quad \left[-\frac{\pi}{2} \le \arg x \le \frac{\pi}{2} \right] . \tag{24}$$

Equations (23) and (24) define our inner problem which depends on $(2\lambda-1)$ and k only via the sign of $(2\lambda-1)$ (as reflected by ϵ) and the parameter a. Solutions of this problem will be found only for discrete values of a and $\lambda > \frac{1}{2}$. Note that since we are seeking odd solutions analytic in the upper v plane their real parts have to vanish on the imaginary v axis, and another way of stating the boundary condition (24) is

$$Q \cong \frac{1}{x} \quad \text{for Im}x \to -\infty , \qquad \text{Im}Q = 0 \quad \text{for } x \text{ real} . \tag{25}$$

This is this last form, completely equivalent to the preceding one as we will see later on, which we used in I.

The next section is devoted to an exact calculation of the smallest values of a, corresponding to the lowest branches of solutions. As we have already mentioned in I, the results agree very well with known numerical calculations.[6,10] We recall that in the large-a limit we have shown in I that the inner region separates in turn into two smaller regions of width $a^{-2/7}$ around each singularity $x = \pm\epsilon^{1/2}$. Then by using a WKB approximation we predicted for the asymptotic behavior of the selected nth eigenvalue

$$a_n \cong 2(n + \tfrac{1}{2} - \delta)^2, \quad n \in N \tag{26}$$

where δ is a constant which depends only on a reduced inner problem, written around one isolated singularity. Surprisingly, in Sec. III, we will see that this relation seems to hold with a good precision, even for the lowest values of a. In Sec. IV, to be complete, δ is exactly calculated from direct analysis of the reduced inner problem.

Before turning to these sections, we will take a slightly different look at the problem and also try an approximate linear solution to our nonlinear eigenvalue problem, which serves as an introduction to our complete treatment of Sec. III. Up to now, we have considered that α and k go together to zero and we were led naturally to work at fixed $a = |\alpha|^{2/3}/k$. We will now work at fixed α and ask whether there is a solution which reduces, in the limit $k \to 0$, to the Saffman-Taylor solution. Since it does not cause any complications, we work with the original McLean Saffman equations with the variable s.

As we have seen, a perturbation expansion in k breaks down near the singularity of the Saffman-Taylor solution $q_0(s)$ located at $s_0 = -1/\alpha$. In the vicinity of s_0, the real solution is very different from $q_0(s)$ and the problem is finding out if there is a solution which behaves as $q_0(s)$ far away from s_0. We will show that, for $\alpha \neq 0$, this is not possible for all the directions in the complex plane.

As before, we extend the McLean and Saffman equations into the upper half s-plane. We have

$$\ln q(s) - i\theta(s) = I(s) = -\frac{s}{\pi} \int_0^1 \frac{\theta(s')}{s'(s'-s)} ds' , \tag{27}$$

where $I(s)$ is analytic everywhere except on the interval [0,1]. Setting

$$f(s) = e^{i\theta(s)} = q(s) e^{-I(s)} , \tag{28}$$

and introducing the variable z with $dz = e^{-I} ds/s$, Eq. (7) is transformed into

$$-2ik \frac{d^2 f}{dz^2} + f^{-2} + 1 - 2e^I = 0 . \tag{29}$$

Now we assume $\alpha \neq 0$. The Saffman-Taylor type solution is obtained by neglecting the first term, which gives $f = (2e^I - 1)^{-1/2}$. There is a singularity at the point z_0 for which $e^I = \frac{1}{2}$. To zeroth order in k, $z_0 = z(s_0)$. We then perform an expansion around z_0 as $2e^I - 1 = \sum a_n (z - z_0)^n$ and next a change of scale:

$$z - z_0 = (-2ia_1^2 k)^{2/7} \frac{X}{a_1}, \quad f = (-2ia_1^2 k)^{-1/7} g . \tag{30}$$

[The a_n introduced in the expansion of $(2e^I - 1)$ have nothing to do with the eigenvalues we are looking for; to zeroth order $a_1 = -\alpha/8(\alpha+1)$]. All the terms in the expansion of $2e^I - 1$ except the first one turn out to be negligible for the matching between the inner problem and the outer Saffman-Taylor solution. Therefore we are left with the inner equation

$$\frac{d^2 g}{dX^2} + \frac{1}{g^2} = X . \tag{31}$$

We look for a solution which behaves as $f \cong (2e^I - 1)^{-1/2}$, that is $g = X^{-1/2}$ for $|X| \to \infty$.

We set $g = X^{-1/2} + h$. For $|X|$ large, we want $h|X|^{1/2} \ll 1$ and we can expand $1/g^2$ to lowest order which gives

$$\frac{d^2 h}{dX^2} - 2hX^{3/2} = -\tfrac{3}{4} X^{-5/2} . \tag{32}$$

This equation rules the behavior of h for large $|X|$. We show now that it has no solution satisfying $h|X|^{1/2} \ll 1$.

The solutions $H(X)$ of the homogeneous linear equation are easily found in terms of Bessel functions. We choose

$$H(X) = X^{1/2} K_{2/7}(\tfrac{4}{7}\sqrt{2} X^{7/4}) . \tag{33}$$

It diverges exponentially for large $|X|$ when $|\arg X| > 2\pi/7$. The general solution of Eq. (32) is then found in terms of $H(X)$. It diverges, in general, exponentially for large $|X|$, but by a proper choice of the integration constants we obtain the solution which satisfies $hX^{1/2} \to 0$ for $|X| \to \infty$ in the two wedges $|\arg X| < 2\pi/7$ and $2\pi/7 < \arg X < 6\pi/7$. It is given by

$$h(x) = -\tfrac{3}{4} H(x) \int_X^{\infty \exp(4i\pi/7)} \frac{dy}{H^2(y)} \int_y^{\infty} dt\, H(t) t^{-5/2} . \tag{34}$$

However, this solution diverges in the wedge $-6\pi/7 < \arg X < -2\pi/7$ as $-3\beta H(X)/4$ where the constant β is

$$\beta = \int_{\infty \exp(-4i\pi/7)}^{\infty \exp(4i\pi/7)} dy / H^2(y) \int_y^{\infty} dt\, H(t) t^{-5/2} \tag{35}$$

(this constant can be estimated by a saddle-point method, but is has clearly no special reason to be zero). In conclusion, there is no solution of Eq. (31) which can match $X^{-1/2}$ for large $|X|$ and for all directions in the complex plane. We note that one can easily obtain from Eq. (31) an asymptotic expansion of g which satisfies apparently $hX^{1/2} \ll 1$ for $|X| \to \infty$. But this neglects exponentially small terms which produce exponentially large terms when they are continued analytically for all values of $\arg X$.

We turn now to the case $\alpha = 0$. The singularity of the Saffman-Taylor solution is for $s_0 = \infty$ and q_0 behaves as $(-s)^{1/2}$ for large s. We want again to match this behavior to a solution of Eq. (29) for $|s| \to \infty$. For large s, we have the expansion $2e^{I} - 1 \cong 1 - 2\lambda - b/s$. The higher-order terms turn out to be irrelevant. To zeroth order $b = \frac{1}{4}$. Rather than z it is more convenient to use a variable proportional to $\exp[-(1-\lambda)z/2]$ (this is basically $1/s^{1/2}$). After the rescaling

$$\begin{aligned} y^2 &= b[(1-\lambda)^2 k/2]^{-2/3} \exp[-(1-\lambda)z] , \\ f &= -i[(1-\lambda)^2 k/2]^{-1/3} G , \end{aligned} \tag{36}$$

we obtain the inner equation

$$y \frac{d}{dy} \left[y \frac{dG}{dy} \right] + G^{-2} = y^2 + C , \tag{37}$$

where $C = 4(2\lambda - 1)/k^{2/3}$. This is nothing else but Eq. (23) which is recovered by the further change of variables $y = |C|^{1/2} x$ and $G(y) = |C|^{-1/2} Q(x)$.

As before we make a stability analysis. We look for a solution which behaves as $1/y$ for large $|y|$ [this corresponds to the $(-s)^{1/2}$ behavior of q_0]. We set $G(y) = 1/y + H_1(y)$ and investigate if there is a solution H_1 which satisfies $H_1 y \ll 1$ for large y. The linearization of Eq. (37) then yields

$$y \frac{d}{dy} \left[y \frac{dH_1}{dy} \right] - 2y^3 H_1 = C - \frac{1}{y} . \tag{38}$$

We choose the following solution of the linearized equation:

$$K(y) = K_0((2y)^{3/2}/3) , \tag{39}$$

where K_0 is the modified zeroth-order Bessel function. It diverges exponentially for large $|y|$ when $|\arg y| > \pi/3$. The solution of Eq. (38) which satisfies $yH_1 \ll 1$ for large $|y|$ and $-\pi/3 < \arg y < \pi$ is

$$H_1(y) = K(y) \int_y^{\infty \exp(2i\pi/3)} \frac{dz}{zK^2(z)} \times \int_z^{\infty} dt\, K(t) \left[\frac{C}{t} - \frac{1}{t^2} \right] . \tag{40}$$

Again this solution diverges in general for $|y| \to \infty$ and $-\pi < \arg < -\pi/3$. But this time we have the additional parameter C which can be chosen to avoid the divergence. If we choose C in such a way that

$$\int_{\infty \exp(-2i\pi/3)}^{\infty (2i\pi/3)} dz [zK^2(z)]^{-1} \int_z^{\infty} dt\, K(t) \left[\frac{C}{t} - \frac{1}{t^2} \right] = 0 , \tag{41}$$

then H_1 will satisfy $yH_1 \ll 1$ for $|y| \to \infty$ and $-\pi < \arg y < \pi$. This means that we have a solution G of Eq. (37) which behaves as $1/y$ for large $|y|$ and for all directions in the complex plane. Therefore, for this fixed value of C, matching is possible. Unfortunately Eq. (41) is not quantitatively correct (it gives a single value of C, not an infinite set) because nonlinear terms which have been omitted in our linearization cannot be neglected. This matter is taken up in the next sections where the correct values of C are found. We note that our procedure is not the only one which allows us to obtain an approximate linear problem for Eq. (37). We could rewrite Eq. (37) in the form convenient for the large C analysis as we do in Sec. IV. Then this equation can be linearized and leads to an infinite number of solutions. This is the spirit of several approaches.[1,2]

In conclusion we have shown that in the limit $k \to 0$, for $\lambda \neq \frac{1}{2}$ ($\alpha \neq 0$), it is not possible to have a complete matching between a solution of the McLean-Saffman equations and a Saffman-Taylor solution q_0 near the singularity of q_0. On the other hand, complete matching is possible if we choose for q_0 the $\lambda = \frac{1}{2}$ ($\alpha = 0$) Saffman-Taylor solution. This result provides a simple way to understand why the $\lambda = \frac{1}{2}$ Saffman-Taylor solution is selected for $k \to 0$. We note also that, for $\lambda = \frac{1}{2}$, the singularity of the Saffman-Taylor solution $q_0(s)$ is for $s = \infty$. But the McLean-Saffman differential equation has also a singularity at $s = \infty$. Therefore a possible simple way to understand the selection mechanism is to say that, among all the possible zeroth-order solutions $q_0(s)$, the solution whose singularity coincides with the singularity of the differential equation is chosen.

We end up this section by showing that the two boundary conditions that we have used, namely reality of $G(y)$ on the real axis and $G(y) \sim 1/y$ for $|y| \to \infty$ in all directions, are actually equivalent. Indeed if $G(y) \sim 1/y$ for $|y| \to \infty$ in all the directions of the y complex plane, this means that there is no transcendental terms in $G(y)$ (these terms being exponentially small for some directions and exponentially larger for others, owing to the Stokes phenomenon). Therefore $G(y)$ is correctly represented by the asymptotic expansion easily generated from Eq. (37). This expansion has obviously real coefficients and $G(y)$ is real on the real y axis. Conversely if $G(y)$ is real on the real y axis, by analytic continuation the values of $G(y)$ in the upper and lower complex half planes are complex conjugate. Therefore if we choose $G(y) \sim 1/y$ in the upper half plane (we have seen that this is possible; this will be seen again in the next section), we will have the same behavior in the lower half plane. The equivalence between the two boundary conditions will be quite explicit in the next section.

III. SOLUTION OF THE NONLINEAR EIGENVALUE PROBLEM

In this section we obtain an essentially analytical solution of our eigenvalue problem. We show that all the required information is contained in the asymptotic expansion of $G(y)$. More precisely we only need to know the limiting form of the coefficients in this expansion for large order and we reduce the problem to an algebraic solvability condition on this limiting form. Our solution is not completely analytic only because the convergence toward the limiting form happens to be rather slow, and in order to obtain a good precision on the "eigenvalues," it is practically necessary to use numerical methods. But this is not a general limitation of the method, this is only a property of the specific differential equation we have to deal with. We note that one of the advantages of our analytical procedure is to make explicit the appearance of exponentially small terms, as we will see.

We want to find the values of C for which

$$y\frac{d}{dy}\left[y\frac{dG}{dy}\right]+\frac{1}{G^2}=y^2+C \tag{42}$$

has a solution $G(y)$ which behaves as $1/y$ for $|y| \to \infty$, or equivalently is real on the real axis. As we have seen, exponentially growing terms possibly appear when, for large $|y|$, we linearize the equation around $1/y$. If we set

$$G(y)=\frac{1}{y}+H_1(y) ,$$

where $H_1(y)$ is expected to be small, we obtain for $H_1(y)$ the following equation:

$$y\frac{d}{dy}\left[y\frac{dH_1}{dy}\right]-2y^3H_1=C-\frac{1}{y} , \tag{43}$$

and unless we make a special choice of C, this linear equation has exponentially growing solution, as we have seen in Eq. (41) and will follow from the calculations below. However, the trouble with this "linear" approach is that it is inconsistent to retain only the first term $1/y$ in the asymptotic expansion of $G(y)$ as all the higher terms are giving comparable contributions to the exponentially growing mode. We have found two strategies to solve this difficulty. The first one is a natural refinement of the above-mentioned linear approach. Instead of keeping only the first term $1/y$ in the asymptotic expansion of $G(y)$, we keep the first Nth terms $y^{-1}, y^{-2}, y^{-3} \ldots y^{-N}$, and write

$$G(y)=y^{-1}\left[1+\sum_{n=1}^{N-1}a_n y^{-n}\right]+H_N(y) . \tag{44}$$

We can hope, and we will show, that when we go to order y^{-N} in the asymptotic expansion, the corresponding corrections $H_N(y)$ will satisfy an equation like Eq. (43) with a precision which increases indefinitely when $N \to \infty$. If this is so, what we have to do is just to write for H_N, instead of H_1, the condition under which there is no exponentially growing terms for large $|y|$ and let N go to infinity. In this way we will have an exact solution of our problem.

Since this first approach forced us to contemplate the asymptotic expansion of G to arbitrary high order, we considered if it would be possible to sum the series and get the result. So our second method consists in Borel-summing the asymptotic expansion of G to obtain directly the solvability condition.

A preliminary step is, therefore, to study the asymptotic expansion of G and this is the subject of the next subsection. Following Dashen *et al.*,[9] we shall show that, in contrast to what one might think at first, the behavior of this expansion for large order is simple to obtain. We then present the two methods successively and end this section with our numerical results for the eigenvalues.

A. The asymptotic expansion of G

We plug the full asymptotic expansion into Eq. (42) and we find the coefficients by identification. Let us set $y=1/u$. Equation (42) becomes

$$G^2u\frac{d}{du}\left[u\frac{dG}{du}\right]=\left[\frac{1}{u^2}+C\right]G^2-1 . \tag{45}$$

We write

$$G=u\sum_{n=0}^{\infty}a_n u^n,\quad G^2=u^2\sum_{n=0}^{\infty}A_n u^n ,$$

where we have obviously the nonlinear relation

$$A_n=\sum_{m=0}^{n}a_m a_{n-m} . \tag{46}$$

Again, the a_n we have introduced in the expansion of G are different from the desired eigenvalues of our problem. This should not confuse the reader.

Carrying the expansions for G and G^2 in Eq. (45) and making use of Eq. (46) to obtain a_0, a_1, and a_2 from A_0, A_1, and A_2, we find the following nonlinear recursion relation:

$$A_0=a_0=1,\quad A_1=a_1=0,\quad A_2=2a_2=-C , \tag{47}$$

and for $n \geq 3$

$$A_n = -CA_{n-2} + \sum_{m=0}^{n-3} (m+1)^2 a_m A_{n-m-3} \,. \tag{48}$$

For each n, A_n is obtained from Eq. (48) and then a_n is extracted from Eq. (46).

Now we can see that for large n Eqs. (46) and (48) are consistent with a factorial-type growth for A_n and a_n. Namely, if we assume that the dominant terms are the $m=0$ and $m=n$ terms in Eq. (46) and the $m=n-3$ term in Eq. (48), we have

$$A_n \cong 2a_n, \quad A_n \cong (n-2)^2 a_{n-3} \,. \tag{49}$$

Therefore,

$$a_n = a_{n-3}(n-2)^2/2 \,, \tag{50}$$

which leads to

$$a_n = \frac{A_n}{2} \sim (n!)^{2/3} 2^{-n/3} \,.$$

With this behavior it is easy to check that the terms that we have retained are indeed the dominant ones, since, for example, $a_{n-p}/a_n \sim n^{-2p/3}$ (when $p \ll n$) and moreover that the sum of all the subdominant terms stays negligible compared to the dominant ones.

The major simplification for large n is that the recursion relations [Eqs. (49) and (50)] are now linear so that we can give precise estimates of a_n. Namely,

$$a_{3n+\tau} \sim (\tfrac{9}{2})^n \left[\frac{\Gamma(n+(\tau+1)/3)}{\Gamma((\tau+1)/3)} \right]^2 b_\tau, \quad \tau=0,1,2 \,. \tag{51}$$

The exact value of the b_τ's, which are functions of C only, are of course determined by the complete nonlinear recursion relations. We will return to this point at the end of this section. We now proceed to the solution of our nonlinear eigenvalue problem.

B. Reduction to a renormalized linear problem

Since the recursion relation Eq. (50) is linear for large n, we can solve exactly this relation and find a_n in terms of the "initial conditions" a_N, a_{N+1}, a_{N+2} obtained from the exact equations Eqs. (46)–(48) where N is a large fixed number. Clearly if we let N go to infinity, our result will go to the exact solution. In order to find $H_N(y)$ represented by

$$H_N(y) = y^{-1} \sum_{n=N}^{\infty} a_n y^{-n} \,, \tag{52}$$

we remark that $H_N(y)$ with initial conditions a_N, a_{N+1}, a_{N+2}, is a solution of

$$\frac{y}{2}\frac{d}{dy}\left[y\frac{dH_N}{dy} \right] - y^3 H_N = -a_N y^{2-N} - a_{N+1} y^{1-N} - a_{N+2} y^{-N} \,, \tag{53}$$

as can be seen by inserting directly Eq. (52) into Eq. (53). Precisely $H_N(y)$ is the solution of Eq. (53) which goes to zero for $|y| \to \infty$.

Actually since a_N has a factorial behavior it is more convenient to work with

$$G_N(y) = y^{-1} \sum_{n=0}^{\infty} b_n^{(N)} y^{-n} \,, \tag{54}$$

where for $n \geq N$ the b_n's coincide with the a_n's:

$$b_n^{(N)} = a_n, \quad n \geq N \tag{55}$$

and for $0 \leq n < N$, the b_n's are calculated from a_N, a_{N+1}, a_{N+2} by making a backward use of Eq. (50). Explicitly,

$$b_{N-3p+\tau}^{(N)} = \frac{2^p}{[(N-2+\tau)(N-5+\tau)\cdots(N-3p+\tau+1)]^2} \times a_{N+\tau} \,, \tag{56}$$

where $\tau=0,1,2$ and $0 < 3p-1 \leq N$. Now $G_N(y)$ satisfies

$$\frac{y}{2}\frac{d}{dy}\left[y\frac{dG_N}{dy} \right] - y^3 G_N = -(b_0^{(N)} y^2 + b_1^{(N)} y + b_2^{(N)}) \,, \tag{57}$$

and $\lim_{|y|\to\infty} G_N(y) = 0$ if $\lim_{|y|\to\infty} H_N(y) = 0$ and reciprocally. But $b_0^{(N)}$, $b_1^{(N)}$, and $b_2^{(N)}$ are no longer factorially large. They have as finite limits b_0, b_1, b_2, previously defined in Eq. (51).

We see that G_N satisfies an equation quite analogous to Eq. (43). More precisely, from Eq. (43), $\tilde{G}(y) \equiv y^{-1} + H_1(y)$ verifies

$$\tfrac{1}{2} y \frac{d}{dy}\left[y\frac{d\tilde{G}}{dy} \right] - y^3 \tilde{G} = -y^2 + \frac{C}{2} = -(a_0 y^2 + a_1 y + a_2) \,. \tag{58}$$

Therefore, comparing Eqs. (57) and (58), we see that solving exactly up to a_{N+2} and using beyond that Eq. (50) is just equivalent to renormalize the original coefficients a_0, a_1, a_2 in the inhomogeneous terms of Eq. (58) into $b_0^{(N)}, b_1^{(N)}, b_2^{(N)}$.

Now our program is clear: we have to find under which condition on $b_0^{(N)}, b_1^{(N)}, b_2^{(N)}$ we have $\lim_{|y|\to\infty} G_N(y) = 0$. Then if we let $N \to \infty$, that is if we replace $b_0^{(N)}$, $b_1^{(N)}$, $b_2^{(N)}$ by their limits b_0, b_1, b_2 we will have the exact solution of our problem.

We have already written the boundary condition when we have studied Eq. (38) [which is the same as Eq. (43)]. With our variable y, we can rewrite the solution of Eq. (57) as

$$G_N(y) = K(y) \int_y^{\infty \exp(2i\pi/3)} dt\, F(t) \,, \tag{59}$$

with

$$F(y) = \frac{2}{yK^2(y)} \int_y^{\infty} dt\, K(t) J(t)/t \,, \tag{60}$$

where $J(y)$ is the right-hand side of Eq. (57) and $K(y) = K_0[(2y)^{2/3}/3]$. The boundary condition Eq. (21) [expressing $\lim_{|y|\to\infty} G_N(y) = 0$] becomes

$$\int_{\infty\exp(-2i\pi/3)}^{\infty\exp(2i\pi/3)} dy\, F(y)=0 \ . \tag{61}$$

If we take the path from $\infty e^{-2i\pi/3}$ to $\infty e^{2i\pi/3}$ symmetric with respect to the real axis, it is easy to see that, because $F(y^*)=F^*(y)$, Eq. (61) is equivalent to

$$\mathrm{Im}\int_{y_0}^{\infty\exp 2i\pi/3} F(y)dy=0 \ , \tag{62}$$

where y_0 is anywhere on the real axis [but with $y_0>0$ in order to avoid the singularity of $K(y)$ for $y=0$]. This implies in turn that

$$\mathrm{Im}G_N(y_0)=0 \ . \tag{63}$$

Therefore, $G_N(y_0)$ is real on the real axis which shows explicitly the equivalence between the two boundary conditions.

In order to make this condition explicit, it is more convenient to rewrite $G_N(y)$ in terms of two independent solutions of the homogeneous equation [this amounts to performing a by-part integration in Eq. (59)]. We choose $K(y)$ and $I(y)-iK(y)/\pi$ where $I(y)=I_0((2y)^{3/2}/3)$. For $y\to+\infty$, $K(y)\to 0$ while $I(y)-iK(y)/\pi$ diverges. For $y\to\infty e^{2i\pi/3}$, $K(y)$ diverges and $I(y)-iK(y)/\pi\to 0$. Since the Wronskian[11] of $K(y)$ and $I(y)-iK(y)/\pi$ is $3/(2y)$, $G_N(y)$ is given by

$$G_N(y)=\frac{4}{3}\left[I-\frac{iK}{\pi}\right]\int_{\infty}^{y} dt[KJ/t]-\tfrac{4}{3}K\int_{\infty\exp(2i\pi/3)}^{y} dt[(I-iK/\pi)J/t] \ , \tag{64}$$

where the boundaries of the integrals have been chosen to ensure $G_N(y)\to 0$ when $y\to\infty$ or $\infty e^{2i\pi/3}$. For y_0 real positive

$$\mathrm{Im}G_N(y_0)=-\frac{4K(y_0)}{3\pi}\left[\int_{\infty}^{y} dt(KJ/t)+\mathrm{Im}\int_{\infty\exp(2i\pi/3)}^{y_0} dt[(\pi I-iK)J/t]\right] . \tag{65}$$

Therefore, condition Eq. (63) implies that the large parenthesis on the right-hand side of Eq. (65) must vanish. Now we let $y_0\to\infty$ in this parenthesis which leads to the simple boundary condition

$$\mathrm{Im}\int_{\infty}^{\infty\exp(2i\pi/3)} dt[\pi I(t)-iK(t)]J(t)/t=0 \ . \tag{66}$$

We note that $\mathrm{Im}(\pi I-iK)\to 0$ exponentially for both boundaries of the integral. This is the motivation for our choice of this combination.

Replacing $J(t)$ by its explicit expression, we are left with various integrals involving Bessel functions. These can be done analytically. The details are given in the Appendix. The resulting condition is

$$b_0^{(N)}+Vb_1^{(N)}+Wb_2^{(N)}=0 \ , \tag{67}$$

where

$$V=6^{1/3}\frac{\Gamma^4}{4\pi^2}, \quad W=(\tfrac{2}{9})^{2/3}\Gamma^2, \quad \Gamma=\Gamma(\tfrac{1}{3}) \ . \tag{68}$$

Numerically $V\cong 2.370\,69\ldots$ and $W\cong 2.632\,99\ldots$. We will now give another derivation of this solvability condition before making use of it in the last part of this section.

C. G from a Borel summation of its asymptotic series

In the previous section, we have obtained a solvability condition that depends only on the behavior of the coefficients of the asymptotic series of $G(y)$. Here, we show that we can obtain this solvability condition directly from the asymptotic series of G.

We have seen above (III A) that the coefficients of the asymptotic series of $G(y)$ have a factorial type of growth, so that the radius of convergence is zero. In order to proceed, we define, as usual, a new series:

$$B_\tau(z)=\sum_{n=0}^{\infty}\frac{a_{3n+\tau}}{2n!}z^n \ . \tag{69}$$

The coefficients of this new series are no longer growing too quickly and we can find their asymptotic behavior for large n given the estimate of a_n obtained before [Eq. (51)]. That is,

$$\frac{a_{3n+\tau}}{2n!}\sim\frac{\sqrt{\pi}}{\Gamma^2((\tau+1)/3)}b_\tau n^{(4\tau-5)/6}(\tfrac{9}{8})^n \ . \tag{70}$$

$B_\tau(z)$ is therefore a convergent series of $|z|<\frac{8}{9}$ and defines an analytic function there. If we try to find for $B_\tau(z)$ an analytic continuation in the complex plane, Eq. (70) readily shows that the closest singularity to zero of $B_\tau(z)$ is located at $z_s=\frac{8}{9}$ and that the behavior of the singular part of $B_\tau(z)$ near z_s is given by

$$B_{\tau\,\mathrm{sing}}(z)\sim\sqrt{\pi}\frac{\Gamma((4\tau+1)/6)}{\Gamma^2((\tau+1)/3)}b_\tau(1-\tfrac{9}{8}z)^{-(4\tau+1)/6} \ . \tag{71}$$

Since we will be interested below in values of z on the real positive axis, we choose a particular continuation with a cut in the lower z half plane.

In order to go backward, from $B_\tau(z)$ to $G(y)$, we define

$$G_\tau(y)=\frac{1}{y^{1+\tau}}\int_0^\infty dt\, e^{-t}B_\tau\left[\frac{t^2}{y^3}\right]. \quad (72)$$

We then identify $G(y)$ and $\sum_{\tau=0}^{2} G_\tau(y)$ which satisfy the same differential equation and which have the same asymptotic series in some sector of the lower y half plane (for example, $-\pi/2<\arg y<-\pi/6$). Now that we have an explicit representation of $G(y)$, we can find when $G(y)$ is purely real on the positive y axis.

So we analytically continue $G(y)$ from the axis $\arg y=-\pi/3$ to the real positive axis (staying along the way in the lower half plane). On the real positive axis, we find, for each τ

$$G_\tau=\frac{1}{y^{1+\tau}}\int_0^{(8y^3/9)^{1/2}} dt\, e^{-t}B_\tau\left[\frac{t^2}{y^3}\right]+\frac{1}{y^{1+\tau}}\int_{(8y^3/9)^{1/2}}^\infty dt\, e^{-t}B_\tau\left[\frac{t^2}{y^3}\right]. \quad (73)$$

The noticeable point in this seemingly vacuous equation, is that only the second integral can contribute to the imaginary part of G_τ. Moreover, its asymptotic behavior for large y is dominated by the neighborhood of $[8y^3/9]^{1/2}$, so that it is possible to replace b_τ by its singular behavior near this point Eq. (71). We obtain, therefore,

$$\mathrm{Im}G_\tau(y)\sim\mathrm{Im}\left[\frac{b_\tau}{y^{1+\tau}}\frac{\sqrt{\pi}\Gamma((4\tau+1)/6)}{\Gamma^2((\tau+1)/3)}\exp\left[i\pi\frac{4\tau+1}{6}\right]\int_{(8y^3/9)^{1/2}}^\infty dt\, e^{-t}\left[\frac{9}{8}\frac{t^2}{y^3}-1\right]^{-(4\tau+1)/6}\right], \quad (74)$$

and after a straightforward evaluation of the remaining integral

$$\mathrm{Im}G_\tau(y)\sim\frac{b_\tau}{\Gamma^2((\tau+1)/3)}\left(\tfrac{2}{9}\right)^{(\tau+1)/3}\left[\frac{3\pi^3}{\sqrt{2}y^{3/2}}\right]^{1/2}\exp[-(\tfrac{8}{9}y^3)^{1/2}]. \quad (75)$$

(The careful reader might be worried that the integral to be computed seems divergent for $\tau=2$. This happens because in Eq. (73) there are actually three integrals not only two: we have not written the integral along an infinitesimal half circle around the singularity at $t=[8y^3/9]^{1/2}$. This integral is actually negligible for $\tau=0,1$, but it diverges for $\tau=2$ and cancels the diverging part of the integral that we have kept, thus giving the finite result in Eq. (75).)

The first nice point emerging from this calculation is that the functional form (in y) of the imaginary part is the one predicted before by a linearization around a truncation of the asymptotic series. But now we have obtained the coefficient in front of this imaginary part. The requirement that $\mathrm{Im}G=0$ (i.e. $\sum_{\tau=0}^{2}\mathrm{Im}G_\tau=0$) gives the previously obtained solvability condition

$$b_0+b_1 6^{1/3}\frac{\Gamma^4(\frac{1}{3})}{4\pi^2}+b_2(\tfrac{2}{9})^{2/3}\Gamma^2(\tfrac{1}{3})=0.$$

It is interesting to note that all the information we need to express our solvability condition is contained in the asymptotic expansion. This is not so obvious since one might well have thought that any information on transcendentally small terms was beyond the asymptotic expansion.

If we take the $N=0$ values for $b_\tau^{(N)}$ (namely $1,0,-C/2$), we find from Eq. (67) the (single) solution $C_1=2/W\cong 0.759$. This is quite near the numerical results and raises the hope that the convergence as $N\to\infty$ is very fast. This hope is actually completely unfulfilled. The $b_\tau^{(N)}$'s go very slowly to their limits b_τ. Indeed for large N, $b_\tau^{(N)}$ can be represented by a series expansion in powers of $C/N^{1/3}$. Obviously one must go to fairly large values for N in order to obtain a 10^{-n} precision for the limit since one must reach $N\sim C^3 10^{3n}$. The problem gets worse for large C.

D. Numerical evaluation of the solvability condition

Before proceeding to this last step, let us translate in our notation the numerical results of McLean and Saffmann and Vanden-Broeck. McLean and Saffman find only what corresponds to the lowest solution C_1 for our constant C. For $k=0.069$, 0.131, 0.273, and 0.597 they obtain, respectively, $\lambda_1=0.515$, 0.524, 0.537, and 0.557. If we calculate $4(2\lambda-1)/k^{2/3}$, which must go to C_1 for $k\to 0$, we find, respectively, 0.713, 0.744, 0.703, and 0.643. The extrapolation for $k\to 0$ is not clear because there is a maximum. If we extrapolate from the last three figures, the result falls near 0.8. But in this case the first figure is anomalous, perhaps because of imprecision linked to the small value of k. For $k=0.273$, Vanden-Broeck finds also the next two fingers widths $\lambda_2=0.61$ and $\lambda_3=0.67$. This gives 2.09 and 3.23 for C_2 and C_3, respectively. However, since k is rather large, these can only be taken as indicative values.

We have accelerated the convergence by solving analytically for the first three terms of the series as functions of the limits b_τ (but there is no problem in principle to go to any order one wishes). Then the first unknown corrective term is of order $C^4/N^{4/3}$. We have next further accelerated the convergence by making use of the Richardson extrapolation method. Finally, we have found that, after these transformations, the mean of the values for N, $N+1$, and $N+2$ converges markedly more rapidly than the value for N itself. Naturally the calculation of the $b_\tau^{(N)}$'s from Eqs. (46)–(48) and (56) and the acceleration procedures have been implemented on a computer. We have calculated the left-hand side of Eq. (67) for various values of C and found numerically for which C its zero.

In this way we have found the three lowest value of C. We obtain

$$C_0=0.8158,\quad C_1=2.950,\quad C_2=5.63$$

or (remember that $a=C^{3/2}$)

$$a_0=0.7368,\quad a_1=5.067,\quad a_2=13.36\ .$$

Our result for a_0 agrees nicely with the $k\to 0$ extrapolation of McLean and Saffman results (if the $k=0.069$ point is discarded). While writing this paper, we received two preprints where the nonlinear differential equation Eq. (42) defining our internal problem is solved numerically. The results of Ref. 7 are

$$a_0=0.736\,85,\quad a_1=5.05$$

(the notation in Ref. 7 is $\delta=2a$) while those of Ref. 8 are

$$a_0=0.736\,85,\quad a_1=5.070,\quad a_2=13.37,\quad a_3=25.65\ .$$

The agreement between these calculations and ours is therefore very good.

IV. ASYMPTOTIC BEHAVIOR OF THE NONLINEAR EIGENVALUES

It is possible to find the asymptotic behavior of the solutions of the inner problem in the large a or C limit by using a second time the method of Ref. 5. We first show that as $a\to\infty$ it is given by the solution of a new reduced inner problem which is parameterless, as demonstrated previously in I. We then apply our Borel summation procedure to solve this new inner equation.

A. Reduced inner problem of the large-a limit

The internal problem obtained previously (Eq. 23) is

$$a^{-1}x\frac{d}{dx}\left[x\frac{dQ}{dx}\right]+Q^{-2}=1+x^2\ , \tag{76}$$

with the boundary condition $Q\sim 1/x$ for $\mathrm{Im}(x)\to-\infty$.

It is clear that an asymptotic expansion in power of $1/a$ of a solution of this problem can be obtained,

$$Q=Q_0+\frac{1}{a}Q_1+\frac{1}{a^2}Q_2+\cdots\ , \tag{77}$$

with, e.g.,

$$Q_0=\frac{1}{(x^2+1)^{1/2}} \tag{78}$$

(we choose a cut on the imaginary axis between $-i$ and i).

The Q_n's are odd functions of x. In order to decide whether Q itself is odd, that is, whether the asymptotic expansion is asymptotic for $\mathrm{Re}x\to+\infty$, we examine it in the vicinity of one of its singularities $x=\pm i$ say $x=-i$. In the neighborhood of $x=-i$, $[x=-i(1-y)$, $y\ll 1]$ the internal problem simplifies to

$$a^{-1}\frac{d^2Q}{dy^2}+Q^{-2}=2y\ . \tag{79}$$

As before, the small parameter $1/a$ can be eliminated by a rescaling of the function Q and variable y

$$Q=\left[\frac{a}{4}\right]^{1/7}F,\quad y=\frac{1}{2}\left[\frac{a}{4}\right]^{-2/7}r\ . \tag{80}$$

In terms of the new variable and function we get

$$\frac{d^2F}{dr^2}+\frac{1}{F^2}=r\ . \tag{81}$$

The domain where Eq. (81) is valid defines a new inner region of extension $a^{-2/7}\ll 1$ for large a. It is worth noticing also that we obtained exactly the same equation, directly from the McLean-Saffman equations, when we considered the limit k going to zero, α fixed [Eq. (31)]. The following treatment could also be applied in this limit but in the matching process the WKB solutions [Eq. (86) below] of the internal problem would have to be replaced by the corresponding expressions for the McLean-Saffman equations. The fact that Q should match with Q_0 translates into the boundary condition

$$F\sim\frac{1}{\sqrt{r}}\quad\text{for } \mathrm{Im}(r)\to-\infty\ . \tag{82}$$

The possible exponential asymptotic correction to the algebraic series in the wedge-shaped sector $(-6\pi/7)<\arg(r)<(-2\pi/7)$ is computed by linearization of Eq. (81) around the asymptotic value of F and satisfies

$$\frac{d^2F_1}{dr^2}-2r^{3/2}F_1=0\ . \tag{83}$$

So the possible asymptotic behavior of F_1 in the wedge-shaped sector $(-6\pi/7)<\arg r<(-2\pi/7)$ is

$$F_1\sim\gamma r^{-3/8}\exp(\tfrac{4}{7}\sqrt{2}r^{7/4})\ . \tag{84}$$

$\arg\gamma$ will be obtained below from the solution of Eq. (81) by the Borel summation method. Now that we have obtained the possible form of exponential corrections in the inner region ($|x+i|\ll a^{2/7}$), we do the same in the outer region and match the solutions.

In the outer region, the form of transcendental corrections may be found by a WKB analysis. We linearize Eq. (76) around the first term of the asymptotic expansion in $1/a$, $Q_0=(x^2+1)^{-1/2}$ and have to solve

$$\frac{1}{a}x\frac{d}{dx}\left[x\frac{d}{dx}h\right]-2(x^2+1)^{3/2}h=0\ . \tag{85}$$

The only acceptable solution (i.e., exponentially decreasing for $\mathrm{Re}x\to+\infty$) is

$$h=h_0\frac{1}{(x^2+1)^{3/8}}\exp\left[-\sqrt{2a}\int_{x_0}^{x}(x'^2+1)^{3/4}\frac{dx'}{x'}\right]\ , \tag{86}$$

where x_0 is an arbitrary point on the real positive axis. The possible exponential correction of the internal problem [Eq. (39) of Sec. I] and h match. So, the searched eigenvalues are obtained when h_0 is real. In order to find the phase of h_0 we have to match F_1 and h. For $x=-i(1-y)$ near $-i$ (i.e., $y\ll 1$) h can be written as

$$h \sim h_0 \exp\left[-\sqrt{2a}\int_{x_0}^{-i}(x'^2+1)^{3/4}\frac{dx'}{x'}\right]$$
$$\times\frac{1}{y^{3/8}}\exp(\sqrt{2a}\,\tfrac{4}{7}2^{3/4}y^{7/4}) .$$

This is exactly the asymptotic form of F_1 [Eq. (84)], considering relation (80) between y and r. So the phase of h_0 is deduced from the phase of γ by subtracting the imaginary part of

$$-\sqrt{2a}\int_{x_0}^{-i}(x'^2+1)^{3/4}\frac{dx'}{x'} ,$$

which is equal to $+\pi\sqrt{a/2}$. [We evaluate the integral along a contour consisting of the segment $[-i,0)$, an infinitesimal quarter of circle around 0 and the real positive axis from 0 to x_0. Then only the infinitesimal quarter of circle contributed to the integral and the result is easily obtained.]

This can be written as

$$\left[\frac{a}{2}\right]^{1/2}=\frac{1}{\pi}(\arg h_0-\arg\gamma) ,$$

and finally the requirement that h_0 is purely real gives for the eigenvalues

$$a_n=2\left[n+\frac{\arg\gamma}{\pi}\right]^2,\quad n\in N . \tag{87}$$

This is, of course, an asymptotic evaluation for large n but we will see below that surprisingly, even a_0,a_1,a_2 are accurately described by this formula. We now proceed to the solution of the reduced inner problem and the computation of $\arg\gamma$.

B. Solution of the reduced inner problem by the Borel-resummation method

Let us recall the strategy. We search for F, solution of Eq. (81), and boundary condition (82) from its asymptotic series. So, we first study the asymptotic series and then sum it. The computation of the first few terms of this series leads one to think that it involves only the powers $r^{-(7n+1)/2}$. So, we look for F as

$$F=\frac{1}{\sqrt{r}}g(r^{7/2}) . \tag{88}$$

Using this expression into Eq. (81) it is easy to obtain an equation for g ($X=r^{7/2}$):

$$3g/X+21g'+49g''X=4\left[1-\frac{1}{g^2}\right] , \tag{89}$$

with the boundary condition

$$g(X)\to 1 \quad\text{as } X\to-\infty+i0^- .$$

We plug the full asymptotic expansion of g

$$g(X)=\sum_{n=0}^{\infty}\frac{g_n}{X^n},\quad g_0=1 \tag{90}$$

into Eq. (89) and find the coefficients by identification. The first important property is that all the g_n's are real since g_0 is real, and that Eq. (89) is a differential equation with real coefficients, which therefore gives rise to a real recursion relation between the g_n's.

The second important property is that, as before, for large n, only the linear part of the recursion relation between the g_n's matters. That is

$$\frac{g_{n+1}}{g_n}\sim\tfrac{49}{8}(n+\tfrac{1}{7})(n+\tfrac{3}{7}) . \tag{91}$$

So we have the estimate for g_n:

$$g_n\sim\tilde{g}(\tfrac{49}{8})^n\Gamma(n+\tfrac{1}{7})\Gamma(n+\tfrac{3}{7}) . \tag{92}$$

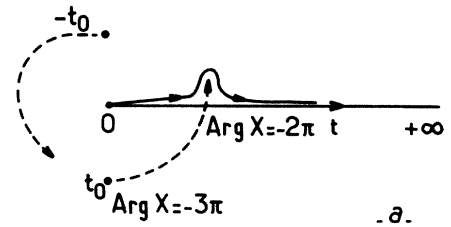
The value of the constant $\tilde{g}$ is of course determined by the complete nonlinear recursion relation. Actually, the convergence of the g_n's toward their asymptotic form appears to be very quick and the computation of the first few g_n's should suffice to obtain a precise estimate of $\tilde{g}$. As it will be seen below, the knowledge of the actual value of $\tilde{g}$ is not necessary for our purpose. The main point besides the asymptotic form Eq. (92) is that $\tilde{g}$ is real as are the g_n's.

Now that we have seen that the coefficients of the asymptotic series of $g(x)$ have a factorial type of growth, we follow the path of Sec. III C to resum the asymptotic series. We define a new series

$$H(Y)=\sum_{n=0}^{\infty}\frac{g_n}{2n!}Y^n . \tag{93}$$

The asymptotic behavior of the coefficients of this new series is [using Eq. (92)]

$$\frac{g_n}{2n!}\sim\tilde{g}\sqrt{\pi}n^{-13/14}(\tfrac{49}{32})^n . \tag{94}$$



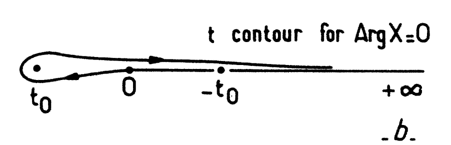


FIG. 5. (a) Evolution of the singularities $\pm t_0$ of the integrand of Eq. (96) as a function of arg$X$. (b) Final t contour for arg$X=0$.

The singularity of $H(y)$ closest to zero is located at $Y=\frac{32}{49}$ and is given by

$$H(Y)_{\rm sing}\sim\tilde{g}\sqrt{\pi}\Gamma(\tfrac{1}{14})(1-\tfrac{49}{32}Y)^{-1/14}\ . \tag{95}$$

In order to go backward from $H(Y)$ to $g(X)$ we use the representation

$$g(X)=\int_0^{+\infty}dt\,e^{-t}H\left[\frac{t^2}{X}\right]. \tag{96}$$

On the Stokes line $\arg r=-6\pi/7$ (or $\arg X=-3\pi$) $g(X)$ is purely real, as it should, and is exactly given by its asymptotic expansion. Now when one rotates in the complex r plane and crosses the anti-Stokes line $\arg r=-4\pi/7$ ($\arg X=-2\pi$), the singularity of the integrand in Eq. (96) at $t_0=-\frac{4}{7}\sqrt{2}\sqrt{X}$ hits the real positive t axis from below [see Fig. 5(a)]. Therefore, in order to analytically continue $g(X)$ up to the next Stokes line $\arg X=-\pi$ (where exponential corrections become oscillating and thus noticeable) one has to deform the integration path in the t variable, so that it goes around t_0 on the left side. It is even more simple to do the calculation of γ on the next anti-Stokes line $\arg X=0$ or $\arg r=0$ because the exponential correction becomes the dominant term. Note, however, that this is here only a matter of technical convenience. The matching to the WKB outer solution given by Eq. (86) still has to be done in the wedge-shaped sector $-6\pi/7\le\arg r\le-2\pi/7$. When one reaches the line $\arg X=0$, the original t contour has been transformed as shown in Fig. 5(b). For large X, the major contribution to $g(X)$ in Eq. (96) comes from the vicinity of t_0 and one easily obtains

$$g(X)\sim e^{2i\pi/14}X^{1/28}(1-e^{-2i\pi/14})\times\left[\left[\frac{4\sqrt{2}}{7}\right]^{1/14}\tilde{g}\sqrt{\pi}\Gamma\left[\frac{1}{14}\right]\times\int_0^{+\infty}e^{-u}u^{-1/14}du\right]\exp\left[\frac{4\sqrt{2}}{7}\sqrt{X}\right].$$

or

$$g(X)\sim ie^{i\pi/4}X^{1/28}\left[2^{13/14}\tilde{g}\pi^{3/2}\left[\frac{4\sqrt{2}}{7}\right]^{1/14}\right]\times\exp\left[\frac{4\sqrt{2}}{7}\sqrt{X}\right]. \tag{97}$$

Using Eq. (88) and comparing (97) with (84) we get for the phase of γ the result

$$\arg\gamma=+\frac{4\pi}{7}\ . \tag{98}$$

This completes our calculation. We have obtained an explicit formula for $\arg\gamma$ because, surprisingly, it turns out that the nonlinear complexity of Eq. (81) appears in the modulus of γ but not in its phase. As mentioned before, with this value of $\arg\gamma$, the asymptotic formula Eq. (87) works well even for the first eigenvalues. Namely, using Eq. (87) we obtain

$$a_0=0.6531,\quad a_1=4.939,\quad a_2=13.22$$

to be compared to the exact values obtained before

$$a_0=0.7368,\quad a_1=5.067,\quad a_2=13.36\ .$$

Note that our result (98) leads to a value of $\phi=\arg\gamma-\pi/2$, as defined in Ref. 8, equal to $\pi/14=0.2244$, which is slightly different from the numerical estimate given in (Ref. 8), $\phi=0.2325$.

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APPENDIX

When $J(t)$ is written explicitly, condition Eq. (66) becomes

$$\mathrm{Im}\int_\infty^{\infty\exp(2i\pi/3)}dt[\pi I(t)-iK(t)](b_0^{(N)}t+b_1^{(N)}+b_2^{(N)}/t)=0\ , \tag{A1}$$

which gives after the change of variable $u=(2t)^{3/2}/3$

$$\frac{3^{2/3}}{3}b_0^{(N)}E_{1/3}+b_1^{(N)}E_{-1/3}+\frac{2}{3^{2/3}}b_2^{(N)}E_{-1}=0\ , \tag{A2}$$

where

$$E_\nu=\mathrm{Im}\int_\infty^{-\infty}du\,u^\nu[\pi I_0(u)-iK_0(u)]\ . \tag{A3}$$

The contour in Eq. (A3) must avoid the origin by going in the upper complex plane. We decompose this contour in three pieces: C_1 goes on the real axis from ∞ to ϵ, C_2 is a semicircle of radius ϵ centered at the origin and located in the upper complex plane, C_3 goes on the real axis from $-\epsilon$ to $-\infty$.

The integrals over C_1 and C_3 are easily related. $I_0(z)$ is an entire function and satisfies $I_0(-u)=I_0(u)$ for u real positive. The analytic continuation[11] of $K_0(z)$ into the upper complex plane satisfies $K_0(ue^{i\pi})=-i\pi I_0(u)+K_0(u)$ for u real positive. Therefore,

$$\begin{aligned}\mathrm{Im}\int_{C_1}du\,u^\nu(\pi I_0-iK_0)&=\int_\epsilon^\infty du\,u^\nu K_0\ ,\\ \mathrm{Im}\int_{C_3}du\,u^\nu(\pi I_0-iK_0)&=\cos(\pi\nu)\int_\epsilon^\infty du\,u^\nu K_0\ .\end{aligned} \tag{A4}$$

For $\nu=\pm\frac{1}{3}$ the integral converges for $\epsilon\to0$, and the contribution from C_2 vanishes in this limit. Therefore,[11]

$$E_{\pm\frac{1}{3}}=\tfrac{3}{2}\int_0^\infty du\,u^{\pm1/3}K_0(u)=\tfrac{3}{4}\times2^{\pm1/3}\Gamma^2(\tfrac{1}{2}\pm\tfrac{1}{6})\ , \tag{A5}$$

where $\Gamma(x)$ is the Euler gamma function. For $\nu=-1$, the contributions from C_1 and C_3 cancel. For the C_2 contribution we can take[11] $I_0(u)\cong 1$ and $K_0(u)\cong -[\ln(u/2)+\gamma]$ where γ is the Euler constant. An elementary calculation then leads to

$$E_{-1}=\frac{\pi^2}{2} .$$

We finally make use in Eq. (A5) of

$$\Gamma(\tfrac{1}{3})\Gamma(\tfrac{2}{3})=2\pi/\sqrt{3} . \qquad \text{(A6)}$$

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