Quantum-noise quenching in the correlated spontaneous-emission laser as a multiplicative noise process. I. A geometrical argument

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We show, via simple geometrical arguments, the quantum-noise quenching in a correlated (spontaneous) emission laser (CEL). This noise quenching is a consequence of the correlation between noise sources which results in a multiplicative noise process. The steady-state distribution for the phase difference between the two electric fields in a CEL is compared and contrasted to that of a standard phase-locked laser. Noise quenching is shown to occur in the case of the CEL via an explicit solution of the Fokker-Planck equation.

I. INTRODUCTION AND OVERVIEW

The drive for ultra-high sensitivity in, for example, quantum communication,[1] the ring-laser gyroscope,[2] and gravitational wave detectors[3] has stimulated the study of laser systems which challenge the experimental "limits." Examples of such studies include "squeezing" the vacuum,[4] quantum-nondemolition systems,[5] and more recently the correlated (spontaneous) emission laser[6] (CEL). It has been shown[6,7] that such a CEL device can have a vanishing diffusion coefficient for the relative phase angle ψ between the (complex) electric fields $\mathscr{E}_1$ and $\mathscr{E}_2$ associated with the transitions from two coherently excited states. This quenching of spontaneous emission noise[8] has been supported by further theoretical investigations[9–11] and a recent experiment.[12] Moreover, the CEL has been shown to lead to a possible improvement in the sensitivity of an (idealized) laser gravitational wave detector[3] or a laser gyro.[13]

Nevertheless there appears to be some confusion between CEL operation and the operation of an ordinary two-mode phase-locked laser[14] (PLL). In both the CEL and PLL the two modes are locked to a specific relative phase angle ψ_0. In the PLL case spontaneous emission noise will be reduced relative to that associated with two unlocked lasers.[8] Examples of such PLL noise reduction have been given previously and include (1) phase noise quenching in a laser with an injected signal (symmetry broken laser)[15] and (2) the noise quenching in a locked ring-laser gyroscope.[16] We emphasize, however, that noise is always present in the PLL even though the spectrum may be substantially altered.[16,17]

In the CEL case, however, noise fluctuations in the relative phase angle are, under the appropriate conditions, completely eliminated.[6,7] This occurs for two reasons. First, quantum noise in the two modes is correlated via the CEL effect; second, the system is phase locked as in the PLL. We emphasize that while phase locking is present in the CEL it is not sufficient to lead to the complete suppression of spontaneous emission noise.

In an attempt to better understand and demonstrate the CEL physics we here and in a forthcoming article[18] (referred to as paper II) develop a classical Langevin and Fokker-Planck treatment[8] of the problem. The present analysis emphasizes a geometrical-pictorial derivation of the phase-noise quenching in favor of the previous quantum Langevin[6] and Fokker-Planck approaches.[7] Moreover, the Fokker-Planck equations[19] for the PLL and CEL are compared and contrasted and solved analytically for the steady-state distributions in terms of scalar continued fractions as well as quadratures confirming the noise reduction shown via an approximate (Langevin) treatment. We supplement this geometrical point of view in paper II by a mathematically rigorous treatment of the complete set of CEL equations. Both papers show that the noise reduction in the CEL is intimately related to the concept of multiplicative noise.[20] Moreover, the importance of both mode correlations and mode locking is made especially clear in these treatments.

The present paper is organized as follows. In Sec. II we include spontaneous emission fluctuations in the equation of motion for the relative phase angle ψ of a (two-mode) PLL. We then demonstrate that, depending on the amount of correlation between the two modes, the spontaneous emission fluctuations enter either as additive noise (for the case of no correlation) or as multiplicative noise (for maximum correlation). Similarities and differences between CEL and PLL operation are compared and contrasted in Sec. III A via the Langevin approach[8] by using the first two moments of the phase difference ψ. Quantum noise quenching as a consequence of multiplicative noise is so shown to occur in the CEL but not in the PLL. In the same spirit we, in Sec. III B, present a simple Fokker-Planck analysis of the problem and obtain the corresponding probability densities in terms of scalar continued fractions and quadratures. We so emphasize the difference between ordinary laser operation[8] on one hand, PLL and CEL operation on the other. In Sec. IV we show via geometrical arguments that whereas noise quenching occurs in the phase difference ψ between the fields, the average (or sum) phase angle Ψ undergoes a diffusion process associated with the usual (Schawlow-Townes) rate. Section V is a summary and conclusion.

II. EQUATIONS OF MOTION FOR RELATIVE PHASE DIFFERENCE IN PLL AND CEL

In this section we derive, via geometrical arguments, the equations of motion for the relative phase difference ψ between two modes in a PLL and CEL.

In a CEL configuration[6,7,9–13] two electric fields $\mathcal{E}_1=\rho_1 e^{-i\theta_1}$ and $\mathcal{E}_2=\rho_2 e^{-i\theta_2}$ with slowly varying amplitudes ρ_j and phases θ_j $(j=1,2)$ are locked to a constant relative phase angle ψ_0. This phase locking is described in its simplest form by the familiar Adler equation[14,21] for the phase difference

$$\dot{\psi}=a-b\sin\psi \; , \tag{2.1}$$

where $a=\Omega_1-\Omega_2$ denotes[22] the difference between the eigenfrequencies Ω_j of the two cavities and b is the gain coefficient α. In the application of a CEL as a gravity wave detector[3] we find $a=h\nu$. Here h is the dimensionless amplitude of the gravitational wave ($h\sim 10^{20}$) and ν is the nominal laser frequency. In the laser gyro problem[13] the detuning a reads $a=S\Omega$ where $S=2r/\lambda\!\!\!^{-}$ is the scale factor[2] (r and $\lambda\!\!\!^{-}$ denote the ring radius and the reduced wavelength, respectively).

We choose a rotating coordinate system in which the fields $\mathcal{E}_1$ and $\mathcal{E}_2$ are slowly varying having a relative phase angle[23] $\psi_0=\arcsin(a/b)$ given by Eq. (2.1) such that $\mathcal{E}_1=\rho_0 e^{-(i/2)\psi_0}$ and $\mathcal{E}_2=\rho_0 e^{+(i/2)\psi_0}$. Here we have assumed a symmetric configuration such that $\rho_1=\rho_2=\rho_0$ and ρ_0 denotes the electric field amplitude at steady state.

Due to spontaneous emission, the electric fields $\mathcal{E}_j$ $(j=1,2)$ fluctuate,

$$\delta\dot{\mathcal{E}}_j=F_j(t) \tag{2.2}$$

with Gaussian noise sources F_j such that

$$\langle F_j\rangle=0 \; , \tag{2.3}$$

$$\langle F_j^*(t)F_k(s)\rangle=2D_{jk}\delta(t-s) \; . \tag{2.4}$$

Note that spontaneous emission events from two coherently excited states are strongly correlated[24] as demonstrated in quantum-beat[25] and Hanle-effect experiments.[26] As a result the cross-correlation diffusion coefficient D_{12} can be made nonvanishing.[6]

We now consider the effect of the fluctuating forces F_1 and F_2 on the relative phase difference ψ shown in Fig. 1. The phase shift $\delta\theta_1$ is caused by a spontaneous-emission event $\delta\mathcal{E}_1$, and is given for $|\delta\mathcal{E}_1|\ll\rho_0$ by

$$\begin{aligned}\delta\theta_1&\cong\frac{|\delta\mathcal{E}_1|}{\rho_0}\sin\left[\delta\phi_1+\frac{\psi}{2}\right]\\&=\frac{1}{\rho_0}[\,|\delta\mathcal{E}_1|\sin(\delta\phi_1)\cos(\psi/2)\\&\qquad+|\delta\mathcal{E}_1|\cos(\delta\phi_1)\sin(\psi/2)]\\&=\frac{1}{\rho_0}[\mathrm{Im}(\delta\mathcal{E}_1)\cos(\psi/2)+\mathrm{Re}(\delta\mathcal{E}_1)\sin(\psi/2)] \; .\end{aligned} \tag{2.5}$$

Similarly we arrive at

$$\begin{aligned}\delta\theta_2&\cong\frac{|\delta\mathcal{E}_2|}{\rho_0}\sin(\delta\phi_2-\frac{\psi}{2})\\&=\frac{1}{\rho_0}[\,|\delta\mathcal{E}_2|\sin(\delta\phi_2)\cos(\psi/2)\\&\qquad-|\delta\mathcal{E}_2|\cos(\delta\phi_2)\sin(\psi/2)]\\&=\frac{1}{\rho_0}[\mathrm{Im}(\delta\mathcal{E}_2)\cos(\psi/2)-\mathrm{Re}(\delta\mathcal{E}_2)\sin(\psi/2)] \; .\end{aligned} \tag{2.6}$$

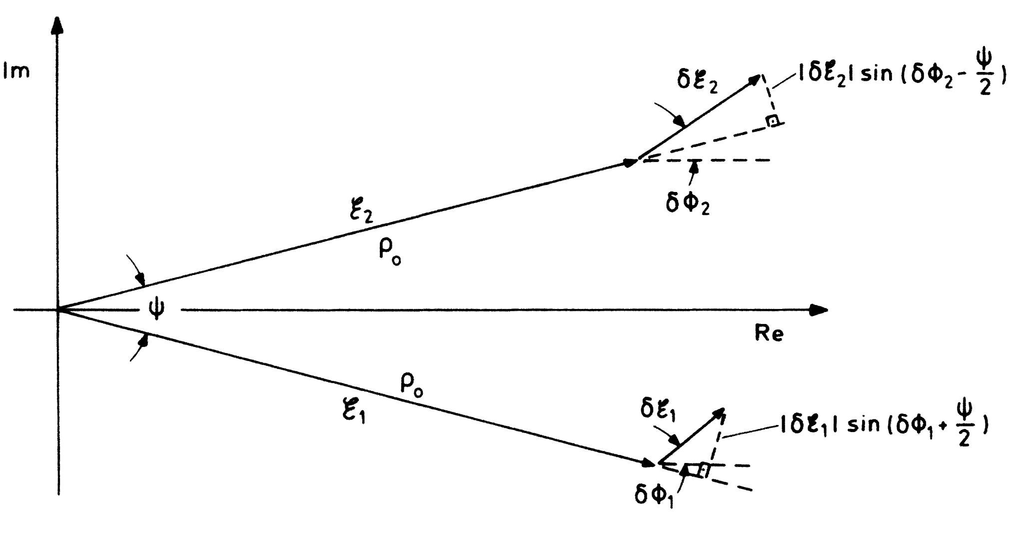

FIG. 1. Phase diagram of spontaneous emission in a phase-locked laser. The electric fields $\mathcal{E}_1$ and $\mathcal{E}_2$ are locked to a phase angle ψ. A fluctuation $\delta\mathcal{E}_j$ $(j=1,2)$ causes a phase change $\delta\theta_j$ given by Eqs. (2.5) and (2.6).

Hence the total fluctuation $\delta\psi=\delta\theta_1-\delta\theta_2$ in the phase difference ψ reads

$$\delta\psi=\frac{1}{\rho_0}[\cos(\psi/2)\,\mathrm{Im}(\delta\mathcal{E}_1-\delta\mathcal{E}_2)+\sin(\psi/2)\,\mathrm{Re}(\delta\mathcal{E}_1+\delta\mathcal{E}_2)]$$

and in view of Eq. (2.2),

$$\delta\psi=\frac{1}{\rho_0}[\cos(\psi/2)\,\mathrm{Im}(F_1\delta t-F_2\delta t)+\sin(\psi/2)\,\mathrm{Re}(F_1\delta t+F_2\delta t)]\ .$$

Defining $\delta\psi/\delta t\equiv(\partial\psi/\partial t)\,|_{\mathrm{fluct}}$ we thus find

$$\left.\frac{\partial\psi}{\partial t}\right|_{\mathrm{fluct}}=\frac{1}{\rho_0}[\cos(\psi/2)\,\mathrm{Im}(F_1-F_2)+\sin(\psi/2)\,\mathrm{Re}(F_1+F_2)]\ .$$

Adding this to the deterministic equation (2.1) we arrive at the geometrically motivated equation of motion

$$\dot\psi=a-b\,\sin\psi+\cos(\psi/2)F_-+\sin(\psi/2)F_+\ . \tag{2.7}$$

In the last step we have introduced the Gaussian Langevin forces $F_-=(1/\rho_0)\,\mathrm{Im}(F_1-F_2)$ and $F_+=(1/\rho_0)\,\mathrm{Re}(F_1+F_2)$ which according to Eqs. (2.3) and (2.4) have the properties

$$\langle F_-\rangle=\langle F_+\rangle=0 \tag{2.8}$$

and

$$\langle F_-(t)F_-(s)\rangle=\frac{1}{\rho_0^2}[D_{11}+D_{22}-2\,\mathrm{Re}(D_{12})]\delta(t-s)\ , \tag{2.9}$$

$$\langle F_+(t)F_+(s)\rangle=\frac{1}{\rho_0^2}[D_{11}+D_{22}+2\,\mathrm{Re}(D_{12})]\delta(t-s)\ , \tag{2.10}$$

$$\langle F_+(t)F_-(s)\rangle=-\frac{2}{\rho_0^2}\,\mathrm{Im}(D_{12})\delta(t-s)\ . \tag{2.11}$$

Here we have used the fact that $D_{12}=D_{21}^*$ [which follows from Eq. (2.4)]. According to Eqs. (2.9) and (2.10) a correlation of the noise sources F_1 and F_2, that is, $D_{12}\neq0$, leads to a reduction of noise strength of F_- and to a corresponding increase in F_+. Thus the Langevin forces F_- and F_+ are correlated as expressed by Eq. (2.11). Depending on the amount of correlation between F_1 and F_2 either the cosine or the sine contribution in the equation of motion for ψ [Eq. (2.7)] gains more weight.

In the absence of correlation, that is, $D_{12}=0$, the two noise forces F_- and F_+ have equal weight and Eq. (2.7) can be simplified by introducing the Gaussian noise $F=\cos(\psi/2)F_-+\sin(\psi/2)F_+$ which according to Eqs. (2.8)–(2.11) and Appendix A has the properties

$$\langle F\rangle=0\ , \tag{2.12}$$

$$\langle F(t)F(s)\rangle=2\mathcal{D}\delta(t-s)\ , \tag{2.13}$$

where $\mathcal{D}=D/\rho_0^2$. Here and in the remainder of the present article we have set $D_{11}=D_{22}\equiv D$. The equation of motion for ψ [Eq. (2.7)] thus reduces to the familiar equation for the phase-locked laser[14–17,27] (PLL)

$$\dot\psi=a-b\,\sin\psi+F(t)\ . \tag{2.14}$$

We now turn to the case of maximum correlation, that is, $D_{11}+D_{12}=2\,\mathrm{Re}D_{12}$, and thus $\langle F_-(t)F_-(s)\rangle=0$. For the sake of simplicity we assume[28] $\mathrm{Im}D_{12}=0$ and therefore $\langle F_-F_+\rangle=0$. Since $\langle F_-\rangle=0$ and F_- is Gaussian all higher correlation functions are zero as well; therefore $F_-=0$. As a result Eq. (2.7) simplifies to the equation of motion for the phase difference in the correlated spontaneous emission laser (CEL) (Ref. 29)

$$\dot\psi=a-b\,\sin\psi+\sin(\psi/2)F_+(t)\ , \tag{2.15}$$

where according to Eq. (2.10)

$$\langle F_+(t)F_+(s)\rangle=2(2\mathcal{D})\delta(t-s)\ . \tag{2.16}$$

Comparing Eq. (2.16) to Eq. (2.13) we note that due to the noise correlation, the noise strength is twice that associated with F. Moreover, we emphasize that the two equations of motion for ψ, Eqs. (2.14) and (2.15), are distinctly different. Whereas in Eq. (2.14) the noise enters in an additive way, in Eq. (2.15) the noise F_+ is multiplied by a nonlinear function of the stochastic variable ψ.

III. COMPARISON BETWEEN PLL AND CEL OPERATION

The similarities and differences between PLL and CEL operations come to light in the steady-state distributions $P_0^{(a)}$ and $P_0^{(m)}$ of the corresponding Fokker-Planck equations. Here the superscripts a and m denote the case of additive and multiplicative noise, Eqs. (2.14) and (2.15), respectively. We therefore in this section present (formally) exact expressions for $P_0^{(j)}$ in terms of scalar continued fractions and quadratures (Appendix B). Whereas the continued fraction method is well suited for numerical analysis it makes it difficult to get some insight into the functional dependence of $P_0^{(j)}$ on the various parameters such as the detuning a. It is therefore worthwhile to consider approximate analytical expressions (Appendix C) which can be checked against the exact scalar continued-fraction treatment. Moreover, insight into the noise quenching in CEL operation springs from an approximate solution of the two Langevin equations.

A. Approximate Langevin treatment

Oriented towards possible application in gravitational wave detection[3] or in the ring-laser gyroscope[13] we confine our discussion to small detunings, $|a|\ll b$. For our case of weak noise, $\mathcal{D}\ll b$, insight can be won by linearizing the sine function around the stable point

$$\psi_0=\arcsin(a/b)\cong a/b,$$

$$\psi(t)=\frac{a}{b}+\Delta(t) \tag{3.1}$$

yielding for Eq. (2.14)

$$\dot{\Delta}^{(a)}=-b\Delta^{(a)}+F(t)\ , \tag{3.2}$$

whereas Eq. (2.15) simplifies to

$$\dot{\Delta}^{(m)}=-[b-\tfrac{1}{2}F_+(t)]\Delta^{(m)}+\frac{a}{2b}F_+(t)\ . \tag{3.3}$$

From Eq. (3.3) we note a reduction of the effective noise by the factor $a/2b \ll 1$ originating from the multiplicative term $\sin(\psi/2)$ in Eq. (2.15). In particular, for $a=0$, no noise is present in the CEL, whereas in Eq. (3.2) for the PLL the noise strength is independent of the detuning a. This is the central lesson: We recognize noise quenching for the CEL, but *not* for the PLL.

This noise-quenching effect can be seen in more detail by calculating the moments $\langle \Delta^{(k)}\rangle$ and $\langle(\Delta^{(k)})^2\rangle$ for $k=a,m$. Equations (3.2) and (3.3) can be integrated trivially to yield, for the PLL,

$$\Delta^{(a)}(t)=\Delta_0^{(a)}e^{-bt}+\int_0^t dt' e^{-b(t-t')}F(t') \tag{3.4}$$

and, for the CEL,

$$\Delta^{(m)}(t)=\Delta_0^{(m)}e^{-bt+(1/2)\int_0^t dt' F_+(t')} + \frac{a}{2b}\int_0^t dt' e^{-b(t-t')-(1/2)\int_t^{t'} dt'' F_+(t'')}F_+(t')\ . \tag{3.5}$$

Making use of Eq. (2.12) we find in steady state, i.e., for $t\to\infty$, from Eq. (3.4) for the PLL

$$\langle \Delta^{(a)}\rangle=0\ .$$

However, expanding the exponential in Eq. (3.5) for small noise $\mathcal{D}/b \ll 1$ we obtain for the CEL

$$\langle \Delta^{(m)}\rangle=-\frac{1}{2}\frac{a}{b}\frac{\mathcal{D}}{b}\ . \tag{3.6}$$

The origin of this tiny noise-induced shift can be traced back to the multiplicative term $F_+\Delta^{(m)}$ in Eq. (3.3) and thus to the multiplicative noise in Eq. (2.15).

Analogously, the second moments $\langle(\Delta^{(k)})^2\rangle$ follow from Eqs. (3.4) and (3.5) as

$$\langle(\Delta^{(a)})^2\rangle=\frac{\mathcal{D}}{b} \tag{3.7}$$

for the PLL and

$$\langle(\Delta^{(m)})^2\rangle=\frac{1}{2}\left[\frac{a}{b}\right]^2\frac{\mathcal{D}}{b} \tag{3.8}$$

for the CEL. Since $(a/b)^2 \ll 1$ this illustrates CEL noise quenching of the fluctuations in the relative phase angle ψ.

B. Exact Fokker-Planck treatment

In order to find the (steady-state) distribution function $P_0^{(j)}=P_0^{(j)}(\psi)$ not only the first two moments but *all* moments are needed. However, it is well known[20] that linearizing a multiplicative noise process results in divergent higher moments. We therefore turn to the Fokker-Planck equations[19] corresponding to Eq. (2.14), for the PLL,

$$\frac{\partial P^{(a)}}{\partial t}=-\frac{\partial}{\partial\psi}[(a-b\sin\psi)P^{(a)}]+\mathcal{D}\frac{\partial^2 P^{(a)}}{\partial\psi^2}\ , \tag{3.9}$$

and Eq. (2.15), for the CEL,

$$\frac{\partial P^{(m)}}{\partial t}=-\frac{\partial}{\partial\psi}[(a-(b-\tfrac{1}{2}\mathcal{D})\sin\psi)P^{(m)}] + 2\mathcal{D}\frac{\partial^2}{\partial\psi^2}[\sin^2(\psi/2)P^{(m)}]\ , \tag{3.10}$$

directly determining the steady-state distributions $P_0^{(j)}$ subjected to periodic boundary conditions.[29]

Since Eqs. (3.9) and (3.10) are one-dimensional Fokker-Planck equations, analytical expressions for $P_0^{(j)}$ in terms of quadratures can be given (Appendix B).

However, we here use the continued-fraction method[19] to directly obtain the exact steady-state solutions of Eqs. (3.9) and (3.10) equivalent to the integral representations of Appendix B. Substituting the ansatz

$$P_0^{(j)}(\psi)=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty} c_n^{(j)}e^{in\psi} \tag{3.11}$$

into Eqs. (3.9) and (3.10) results in the three-term recurrence relations

$$(ia+n\mathcal{D})c_n^{(a)}-\frac{b}{2}c_{n-1}^{(a)}+\frac{b}{2}c_{n+1}^{(a)}=0\ , \tag{3.12}$$

for the PLL, and

$$(ia+n\mathcal{D})c_n^{(m)}-\left[\frac{b}{2}-\frac{\mathcal{D}}{4}+\frac{n}{2}\mathcal{D}\right]c_{n-1}^{(m)} + \left[\frac{b}{2}-\frac{\mathcal{D}}{4}-\frac{n}{2}\mathcal{D}\right]c_{n+1}^{(m)}=0\ , \tag{3.13}$$

for the CEL. Equations (3.12) and (3.13) can be solved by the iteration

$$c_n^{(j)}=S_n^{(j)}c_{n-1}^{(j)}\ , \tag{3.14}$$

where $S_n^{(j)}$ are given by the scalar continued fractions

$$S_n^{(a)}=\frac{b}{2ia+2n\mathcal{D}+bS_{n+1}^{(a)}}$$

for the PLL and

$$S_n^{(m)}=\frac{b-\frac{\mathcal{D}}{2}+n\mathcal{D}}{2ia+2n\mathcal{D}+(b-\frac{\mathcal{D}}{2}-n\mathcal{D})S_{n+1}^{(m)}}$$

for the CEL. The initial condition for the iteration, Eq. (3.14), follows from Eq. (3.11) and the normalization of $P_0^{(j)}$ to be $c_0^{(j)}=1$. The coefficients $c_n^{(j)}$ for $n<0$ can be obtained from the reality condition of $P_0^{(j)}$, $c_{-n}^{(j)}=c_n^{(j)*}$.

In Fig. 2 we compare the so calculated steady-state distributions $P_0^{(j)}=P_0^{(j)}(\psi)$ for multiplicative noise (solid line) to the corresponding one for additive noise[19] (dashed line). In both cases the detuning is $a/b=0.1$. In order to emphasize the crucial role of multiplicative noise on the width of the distributions we have chosen the same effective noise strengths. For the additive case we have $\mathcal{D}/b=0.1$ while for the multiplicative noise we have, because of Eqs. (2.13) and (2.16), $2\mathcal{D}/b=0.1$. Figure 2 clearly demonstrates the noise quenching in the CEL when compared to the PLL. Moreover, we recall that the standard laser operation not depicted in Fig. 2 shows a constant phase distribution,[8] $P_0(\psi)=\frac{1}{2\pi}$ for $-\pi\leq\psi\leq\pi$, as can be seen from Eqs. (3.11) and (3.12) by setting the coupling $b=0$. Therefore, the PLL exhibits a *noise-quieted* (narrowed) distribution relative to ordinary $(P_0(\psi)=\frac{1}{2\pi})$ laser operation; however, the CEL is noise *quenched* relative to both the PLL and the usual laser case. The inset enlarges the neighborhood of $\psi=a/b=0.1$ emphasizing once again the narrowness of $P_0^{(m)}$ compared to $P_0^{(a)}$. Moreover, the noise-induced shift, Eq. (3.6) is apparent. The approximate distribution

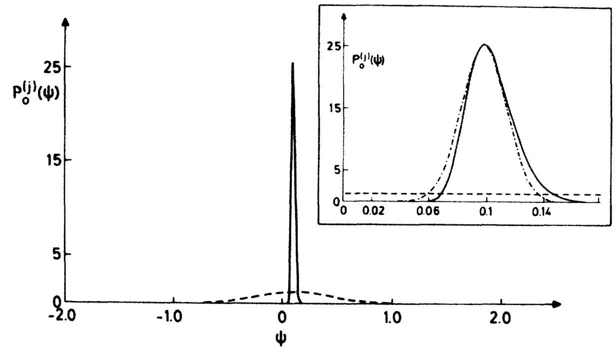

FIG. 2. Comparison between the steady-state distributions $P_0^{(j)}$ of the relative phase angle ψ for a standard phase-locked laser described by Eqs. (2.14) and (3.9) (dashed line) and a correlated spontaneous-emission laser given by Eqs. (2.15) and (3.10) (solid line). For both cases we have chosen $a/b=0.1$ and identical noise strengths, that is, for the additive case $\mathcal{D}/b=0.1$, whereas for the multiplicative case we have $2(\mathcal{D}/b)=0.1$. For comparison the symmetric Gaussian approximation Eq. (3.15) is shown in the inset by the dashed-dotted line.

$$P_0^{(m)}(\psi)\cong\left[\frac{1}{2\pi}\frac{1}{\frac{1}{2}\left[\frac{a}{b}\right]^2\frac{\mathcal{D}}{b}}\right]^{1/2}\exp\left\{-\frac{1}{2}\frac{1}{\frac{1}{2}\left[\frac{a}{b}\right]^2\frac{\mathcal{D}}{b}}\left[\psi-\frac{a}{b}\left[1-\frac{1}{2}\frac{\mathcal{D}}{b}\right]\right]^2\right\} \tag{3.15}$$

derived in Appendix C and shown in the inset by a broken dotted curve is symmetric with respect to $\tilde{\psi}=\frac{a}{b}(1-\frac{1}{2}\frac{\mathcal{D}}{b})$ and thus demonstrates a noise-induced asymmetry of the exact distribution function $P_0^{(m)}$.

In Fig. 3 we show the narrowing of the exact steady distribution $P_0^{(m)}=P_0^{(m)}(\psi)$ for decreasing detuning a/b as expressed by the approximate distribution Eq. (3.15) and the second moment Eq. (3.8). We note that as a approaches zero the distribution function approaches a δ function

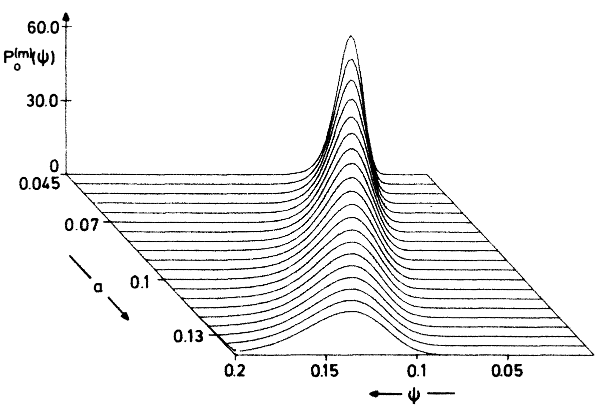

FIG. 3. Narrowing of the steady-state distribution $P_0^{(m)}=P_0^{(m)}(\psi)$ for the multiplicative noise process [Eqs. (2.15) and (3.10)] as a function of the detuning a/b for $2(\mathcal{D}/b)=0.1$.

$$P_0^{(m)}(\psi;a=0)=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}e^{in\psi}=\delta(\psi)\ ,$$

as can be seen from Eq. (3.11) and the fact that $c_n=1$ is a solution of Eq. (3.13) for $a=0$. Moreover, the narrowing of $P_0^{(m)}$ is accompanied by a shift of the maximum towards the origin at the upper right corner.

IV. EQUATION OF MOTION FOR AVERAGE PHASE Ψ

We now discuss the influence of the fluctuations in $\mathcal{E}_j$ on the average phase $\Psi=\frac{1}{2}(\theta_1+\theta_2)$ of the electric fields $\mathcal{E}_1$ and $\mathcal{E}_2$. We here pursue the geometrical approach which had been so successful in the case of the phase difference ψ. We add Eqs. (2.5) and (2.6), and make use of Eq. (2.2) which yields

$$\left.\frac{\partial\Psi}{\partial t}\right|_{\text{fluct}}\cong\tfrac{1}{2}[\cos(\psi/2)F_+ +\sin(\psi/2)F_-]\ . \tag{4.1}$$

Comparing this to Eq. (2.7) we note that the roles of F_+ and F_- are interchanged. As a consequence maximal correlation between the spontaneous emission events in $\mathcal{E}_1$ and $\mathcal{E}_2$, that is, $D_{11}+D_{22}=2\,\mathrm{Re}D_{12}$ and the small-angle approximation for ψ [Eq. (3.1)] reduce Eq. (4.1) to

$$\left.\frac{\partial\Psi}{\partial t}\right|_{\text{fluct}}\cong\tfrac{1}{2}F_+\ .$$

Hence there is no noise quenching in the average angle

Ψ. In particular, Ψ undergoes a diffusion process which according to Eq. (2.16) is associated with *twice* the usual Schawlow-Townes rate. In this sense we have shifted the noise from the relative phase angle ψ into the averaged phase angle Ψ.

V. SUMMARY AND CONCLUSIONS

We conclude by emphasizing that the noise quenching in the CEL relies on two effects. First, the correlation[30] of the noise sources F_1 and F_2 allows one to shift all the noise from the cosine contribution to the sine term in the equation of motion for ψ, Eq. (2.7). This noise, however, can be quenched by the use of small phase angles $|\psi_0| \cong |a/b| \ll 1$. We emphasize again that the noise reduction is an immediate consequence of the fact that in contrast to the standard phase-locked laser [Eq. (2.14)], the noise in the CEL is multiplicative noise as expressed by Eq. (2.15). The noise quenching in the relative phase angle ψ is at the expense of the averaged phase Ψ which undergoes a conventional diffusion process with twice the usual diffusion constant. These approximate results have been confirmed by exact steady-state solutions of the Fokker-Planck equations of the CEL and PLL in terms of scalar continued fractions as well as quadratures. The distribution in the case of a CEL has been shown to be extremely narrow compared to the one of a PLL.

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APPENDIX A: MOMENTS OF THE LANGEVIN FORCE F

In this appendix we calculate the first two moments $\langle F\rangle$ and $\langle F(t)F(s)\rangle$ of the Langevin force $F(t) = \cos(\psi/2)F_- + \sin(\psi/2)F_+$ where ψ satisfies Eq. (2.7) and the Gaussian noise sources F_- and F_+ are defined by Eqs. (2.8)–(2.11).

Following Ref. 8 we define $t_c \equiv t-\tau$ where $0<\tau\to 0$ and find

$$\langle \cos[\psi(t)/2]F_-(t)\rangle$$

$$= \left\langle F_-(t)\left[\cos[\psi(t_c)/2] + \int_{t_c}^{t} dt' \frac{d}{dt'}\{\cos[\psi(t')/2]\}\right]\right\rangle$$

$$= -\tfrac{1}{2}\left\langle F_-(t)\int_{t_c}^{t} dt' \sin\left[\frac{\psi(t')}{2}\right]\dot{\psi}(t')\right\rangle .$$

In the last step we have used the fact that the noise F_- at time t is uncorrelated to the phase ψ at an earlier time t_c and that $\langle F_-\rangle = 0$ [Eq. (2.8)]. Substituting the equation of motion for ψ, Eq. (2.7), into the above expression and performing the averages with the help of Eqs. (2.9)–(2.11) we arrive at

$$\langle \cos(\psi/2)F_-\rangle = -\frac{1}{8}\frac{1}{\rho_0^2}(D_{11}+D_{22}-2\,\mathrm{Re}D_{12})\langle \sin\psi\rangle + \frac{1}{2\rho_0^2}\,\mathrm{Im}D_{12}\langle \sin^2(\psi/2)\rangle .$$

Here we have followed the convention

$$\int_0^t dt'\delta(t') = \tfrac{1}{2} .$$

Analogously we find

$$\langle \sin(\psi/2)F_+\rangle = \frac{1}{8}\frac{1}{\rho_0^2}(D_{11}+D_{12}+2\,\mathrm{Re}D_{12})\langle \sin\psi\rangle - \frac{1}{2\rho_0^2}\,\mathrm{Im}D_{12}\langle \cos^2(\psi/2)\rangle ,$$

which yields

$$\langle F\rangle = \frac{1}{2}\frac{1}{\rho_0^2}(\mathrm{Re}D_{12}\langle \sin\psi\rangle - \mathrm{Im}D_{12}\langle \cos\psi\rangle) .$$

Note that any correlation between the noises F_1 and F_2 ($D_{12}\neq 0$) results in a nonvanishing mean value of F. Making use of Eqs. (2.9)–(2.11) the second-order correlation function follows to be

$$\langle F(t)F(s)\rangle = \frac{1}{\rho_0^2}[(D_{11}+D_{22}) - 2\,\mathrm{Re}(D_{12})\langle \cos\psi\rangle - 2\,\mathrm{Im}(D_{12})\langle \sin\psi\rangle]\delta(t-s) .$$

Note that this is identical to the diffusion constant calculated in Ref. 6.

APPENDIX B: STEADY-STATE DISTRIBUTIONS $P_0^{(j)}$ IN TERMS OF QUADRATURES

In this appendix we derive an analytical expression for the steady-state solution $P_0^{(m)}$ of Eq. (3.10) and compare and contrast it to the familiar steady-state distribution[14,19,27,31] $P_0^{(a)}$ of Eq. (3.9), for the PLL,

$$P_0^{(a)}(\psi) = \mathcal{N}^{(a)} e^{\phi^{(a)}(\psi)} \int_{\psi}^{\psi+2\pi} d\psi' \frac{e^{-\phi^{(a)}(\psi')}}{\mathcal{D}} , \tag{B1}$$

where

$$\phi^{(a)}(\psi) = \int_{-\pi}^{\psi} d\psi' \frac{a-b\sin\psi'}{\mathcal{D}}$$

and

$$\mathcal{N}^{(a)} = \left[\int_{-\pi}^{\pi} d\psi\, P_0^{(a)}(\psi)\right]^{-1} .$$

In the case of multiplicative noise, a mathematical complication arises from the fact that the coefficient in front of the highest derivative in Eq. (3.10) can vanish. This is a common feature[20] of multiplicative noise. This difficulty can be removed by adding a new diffusion constant D_ϵ which at the end of the calculation we let go to zero.

Equation (3.10) in steady state reads then

$$0=-\frac{d}{d\psi}\left[\{a-[b+(\mathcal{D}/2)]\sin\psi\}P_0^{(m)}-[D_\epsilon+2\mathcal{D}\sin^2(\psi/2)]\frac{dP_0^{(m)}}{d\psi}\right].$$

Integrating this equation twice yields

$$P_0^{(m)}(\psi;D_\epsilon)=e^{\phi^{(m)}(\psi)}\left[N-c\int_{-\pi}^{\psi}d\psi'\frac{e^{-\phi^{(m)}(\psi')}}{D_\epsilon+2\mathcal{D}\sin^2(\psi'/2)}\right], \quad \text{(B2)}$$

where

$$\phi^{(m)}(\psi)\equiv\int_{-\pi}^{\psi}d\psi'\frac{a-[b+(\mathcal{D}/2)]\sin\psi'}{D_\epsilon+2\mathcal{D}\sin^2(\psi'/2)} \quad \text{(B3)}$$

and c and N are constants. We now determine c such that $P_0^{(m)}$ satisfies periodic boundary conditions

$$P_0^{(m)}(\psi+2\pi)=P_0^{(m)}(\psi)\ . \quad \text{(B4)}$$

From Eq. (B2) we find

$$P_0^{(m)}(\psi+2\pi;D_\epsilon)=e^{\phi^{(m)}(\psi+2\pi)}\left[N-c\int_{-\pi}^{\pi}d\psi'\frac{e^{-\phi^{(m)}(\psi')}}{D_\epsilon+2\mathcal{D}\sin^2(\psi'/2)}-c\int_{-\pi}^{\psi}d\psi'\frac{e^{-\phi^{(m)}(\psi'+2\pi)}}{D_\epsilon+2\mathcal{D}\sin^2(\psi'/2)}\right] \quad \text{(B5)}$$

and since according to Eq. (B3)

$$\phi^{(m)}(\psi+2\pi)=\phi^{(m)}(\pi)+\phi^{(m)}(\psi)\ , \quad \text{(B6)}$$

Eq. (B5) reduces to

$$P_0^{(m)}(\psi+2\pi)=P_0^{(m)}(\psi)+e^{\phi^{(m)}(\psi)}\left[N\left[e^{\phi^{(m)}(\pi)}-1\right]-ce^{\phi^{(m)}(\pi)}\int_{-\pi}^{\pi}d\psi'\frac{e^{-\phi^{(m)}(\psi')}}{D_\epsilon+2\mathcal{D}\sin^2(\psi'/2)}\right].$$

Periodic boundary conditions, Eq. (B4), are maintained for the choice

$$c=\frac{N(1-e^{-\phi^{(m)}(\pi)})}{\int_{-\pi}^{\pi}d\psi'\frac{e^{-\phi^{(m)}(\psi')}}{D_\epsilon+2\mathcal{D}\sin^2(\psi'/2)}}\ .$$

Hence the steady-state solution, Eq. (B2), simplifies to

$$P_0^{(m)}(\psi;D_\epsilon)=\mathcal{N}^{(m)}e^{\phi^{(m)}(\psi)}\left[\int_{\psi}^{\pi}d\psi'\frac{e^{-\phi^{(m)}(\psi')}}{D_\epsilon+2\mathcal{D}\sin^2(\psi'/2)}+e^{-\phi^{(m)}(\pi)}\int_{-\pi}^{\psi}d\psi'\frac{e^{-\phi^{(m)}(\psi')}}{D_\epsilon+2\mathcal{D}\sin^2(\psi'/2)}\right],$$

where we have combined all constants to one normalization constant $\mathcal{N}^{(m)}$. We shift the integration in the second integral by 2π and with the help of Eq. (B6) we finally arrive at

$$P_0^{(m)}(\psi;D_\epsilon)=\mathcal{N}^{(m)}e^{\phi^{(m)}(\psi)}\int_{\psi}^{\psi+2\pi}d\psi'\frac{e^{-\phi^{(m)}(\psi')}}{D_\epsilon+2\mathcal{D}\sin^2(\psi'/2)}, \quad \text{(B7)}$$

where $\phi^{(m)}$ is given by Eq. (B3) and

$$\mathcal{N}^{(m)}=\left[\int_{-\pi}^{\pi}d\psi\, P_0^{(m)}(\psi;D_\epsilon)\right]^{-1}.$$

Note that the integrals in Eqs. (B1) and (B7) cannot be performed in a closed form and thus have to be evaluated numerically.[32] In the case of Eq. (B7) in addition the limit $D_\epsilon\to 0$ has to be taken.

APPENDIX C: APPROXIMATE EXPRESSIONS FOR $P_0^{(j)}$

Despite the mathematical complexity of the integral expressions for $P_0^{(j)}$, Eqs. (B1) and (B7) simple approximate analytical expression for $P_0^{(j)}$ can be obtained in the limit of weak noise $\mathcal{D}\ll b$ and small detunings $|a/b|\ll 1$.

For the case of additive noise, that is, for the PLL [Eq. (B1)], this expression reads,[16]

$$P_0^{(a)}(\psi)\cong\left[\frac{1}{2\pi}\frac{1}{\mathcal{D}/b}\right]^{1/2}\exp\left[-\frac{1}{2}\frac{1}{\mathcal{D}/b}\left[\psi-\frac{a}{b}\right]^2\right]. \quad \text{(C1)}$$

Due to the singularity in the integral expression for $P_0^{(m)}$, Eq. (B7) when $D_\epsilon=0$, the corresponding asymptotic expansion for the case of multiplicative noise is more difficult to obtain. Since we expect $P_0^{(m)}$ to be an extremely narrow distribution we expand $\phi^{(m)}$ into a Taylor series around the phase angle $\tilde{\psi}$ give by the condition

$$0=\left.\frac{d\phi^{(m)}}{d\psi}\right|_{\psi=\tilde{\psi}}.$$

From Eq. (B3) we find

$$0=\frac{a-[b+(\mathcal{D}/2)]\sin\tilde{\psi}}{D_\epsilon+2\mathcal{D}\sin^2(\tilde{\psi}/2)}\ ,$$

which in the limit of $|a/b|\ll 1$ and $\mathcal{D}\ll b$ yields

$$\tilde{\psi}\cong\frac{a}{b}\left[1-\frac{1}{2}\frac{\mathcal{D}}{b}\right].$$

Thus $\phi^{(m)}=\phi^{(m)}(\psi)$ expanded around $\tilde{\psi}$ reads

$$\phi^{(m)}(\psi)=\phi^{(m)}(\tilde{\psi})+\frac{1}{2}\frac{d^2\phi^{(m)}}{d\psi^2}\bigg|_{\psi=\tilde{\psi}}(\psi-\tilde{\psi})^2+\cdots$$

$$\cong\phi^{(m)}(\tilde{\psi})-\frac{1}{2\frac{D_\epsilon}{b}+\left[\frac{a}{b}\right]^2\frac{\mathcal{D}}{b}}(\psi-\tilde{\psi})^2\ . \qquad \text{(C2)}$$

We substitute Eq. (C2) into Eq. (B7) to find

$$P_0^{(m)}(\psi;D_\epsilon)=\mathcal{N}^{(m)}e^{\phi^{(m)}(\tilde{\psi})}\exp\left[\frac{-(\psi-\tilde{\psi})^2}{2\frac{D_\epsilon}{b}+\left[\frac{a}{b}\right]^2\frac{\mathcal{D}}{b}}\right]$$

$$\times\int_\psi^{\psi+2\pi}d\psi'\frac{e^{-\phi^{(m)}(\psi')}}{D_\epsilon+2\mathcal{D}\sin^2(\psi'/2)}\ . \qquad \text{(C3)}$$

Since $(a/b)^2\mathcal{D}/b\ll 1$ and since we let $D_\epsilon/b\rightarrow 0$ at the end of the calculation the Gaussian function is very narrow. We thus can replace ψ at the boundaries of the integral in Eq. (C3) by $\tilde{\psi}$. Combining all constants we arrive at

$$P_0^{(m)}(\psi;D_\epsilon)\cong\mathcal{N}\exp\left[-\frac{1}{2\frac{D_\epsilon}{b}+\left[\frac{a}{b}\right]^2\frac{\mathcal{D}}{b}}(\psi-\tilde{\psi})^2\right].$$

From the normalization condition we find

$$\mathcal{N}=\left[\frac{1}{\pi\left[2\frac{D_\epsilon}{b}+\left[\frac{a}{b}\right]^2\frac{\mathcal{D}}{b}\right]}\right]^{1/2}.$$

We can now perform the limit $D_\epsilon\rightarrow 0$ which yields, for the CEL,

$$P_0^{(m)}(\psi)\cong\left[\frac{1}{2\pi}\frac{1}{\frac{1}{2}\left[\frac{a}{b}\right]^2\frac{\mathcal{D}}{b}}\right]^{1/2}$$

$$\times\exp\left\{-\frac{1}{2}\frac{1}{\frac{1}{2}\left[\frac{a}{b}\right]^2\frac{\mathcal{D}}{b}}\times\left[\psi-\frac{a}{b}\left[1-\frac{1}{2}\frac{\mathcal{D}}{b}\right]\right]^2\right\}. \qquad \text{(C4)}$$

Comparing Eqs. (C1) and (C4) we note that in both cases the distribution function is approximately a Gaussian which is centered at a/b for the PLL and at $\tilde{\psi}=(a/b)(1-\frac{1}{2}\mathcal{D}/b)$ for the CEL, indicating a tiny noise-induced shift in the latter. More important, however, is the fact that the width of the distributions differ considerably. Whereas in the case of the PLL the width is governed by $(\mathcal{D}/b)^{1/2}$ and is thus independent of a, in the CEL there is a crucial dependence on a, namely, $(a/b)(\mathcal{D}/b)^{1/2}$. Thus for $|a/b|\ll 1$ the CEL phase distribution is considerably narrowed compared to that of the PLL. We emphasize that these widths and moments are identical to those calculated in Sec. III A within the Langevin approach, Eqs. (3.6)–(3.8).

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[23]The actual equation of motion for ψ, for example, in a holographic laser including saturation effects (Ref. 10) is similar to Eq. (2.1) $\dot{\psi}=a-(\alpha-2\beta\rho_0^2)\sin\psi+\beta\rho_0^2\sin(2\psi)$. Here β is the saturation parameter and $\rho_0^2=(2\alpha-\gamma)/8\beta$. The steady-state value ψ_0 is thus different from the one obtained from Eq. (2.1). Oriented towards applications, however, only small locking angles are of interest. Then the above equation yields $\psi_0=2a/\gamma$. Since $\gamma=2\alpha$ [threshold condition (Ref. 8)] this is identical to the one of Eq. (2.1).

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