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Quantum theory of a squeezed-pump laser

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We analyze a model of a laser pumped by an incoherent source in a squeezed vacuum state. The squeezed pump introduces an anisotropy of phase in the laser output. Above threshold two stable solutions are found, with phases corresponding to the directions along which the noise of the bath is quenched. These solutions are illustrated by the potential function of the laser field. An analysis of fluctuations shows that the laser field has reduced phase fluctuations but not below the quantum limit.

I. INTRODUCTION

Investigation of the field of nonclassical light sources has received considerable impetus lately. There have now been several experiments which have generated squeezed states of light. Squeezed light has fewer fluctuations than the vacuum in one quadrature phase of the electromagnetic field. Other experiments have produced light with reduced number fluctuations compared to coherent laser light. The demonstrations of squeezed-light generation are all in passive systems where there is no pumping of the optical medium.

In contrast to this there has recently been the approach taken by Yamamoto *et al.*[1] who have analyzed the consequences of pump-noise suppression in a laser. They show that if there is reduced amplitude fluctuation in the pump the output from the laser has reduced number fluctuations compared to the usual incoherently pumped laser. Yamamoto *et al.* have recently demonstrated the operation of a semiconductor laser pumped with an electron beam with reduced fluctuations and shown that the output beam has a 7.3% reduction below the standard quantum limit in number fluctuations.[2]

A related system is found in the studies of the micromaser.[3,4] In the micromaser single excited atoms are fed into the laser cavity. If a constant velocity of the atoms and hence constant interaction time is assumed the output of the micromaser is found to have sub-Poissonian statistics. When a velocity distribution and hence a spread of interaction times equivalent to a stochasticity of the pump is included the usual Poisson statistics of a coherent laser are recovered. In this paper we shall investigate a new class of laser which is pumped with squeezed light. This differs from the system considered by Yamamoto where the pump is in a near-photon-number eigenstate.

We shall consider a model for a laser based on N two-level atoms interacting with a single cavity mode. The atoms are pumped with light in a squeezed vacuum,[5] that is, there is no coherent component to the pump light. Note that in this paper we are dealing with a "classical" system inasmuch as the number of atoms N is very large just as in the usual laser—in contrast to the micromaser situation. We will demonstrate how the phase anisotropy in the squeezed-pump fluctuations leads to a phase-dependent gain and, consequently, to a phase-locked steady-state laser field. This contrasts with the laser pumped with a near-photon-number eigenstate, where the phase of the steady-state field still undergoes phase diffusion.

In Sec. II we present the Hamiltonian model and the underlying assumptions and derive a corresponding master and Fokker-Planck equation. Section III treats the semiclassical theory and derives the steady states of the Bloch equations and their stability. We show that well above threshold there is a pair of stable steady states with well-defined phases corresponding to the directions along which the noise in the bath is quenched. These stable phase solutions are clearly seen in a plot of the potential function for the laser field.

In Sec. IV we derive a rotating-wave van der Pol equation (around threshold) with phase-dependent self-excitation for the laser mode for a case in which the atoms can be adiabatically eliminated. The spectroscopic linewidth for the laser near threshold is derived. In Sec. V the noise properties of the laser output are studied. We find that the laser field has reduced phase fluctuations. However, for the type of pumping considered at present the phase fluctuations are not less than the quantum limit.

II. LASER MODEL WITH SQUEEZED PUMPING MECHANISM

A. The model Hamiltonian

Following Haken's[6,7] quantum-mechanical treatment of the laser, we consider a Hamiltonian of the form

$$H = H_{\text{laser}} + H_{\text{bath}} + H_{\text{laser-bath}} , \tag{2.1}$$

where the actual laser system consists of N two-level atoms with transition frequency ω_0 coupled to a resonant single mode running wave field in the laser cavity.

Assuming dipole coupling, the Hamiltonian of the laser system alone reads

$$H_{\text{laser}}=\hbar\omega_0 a^\dagger a+\frac{\hbar\omega_0}{2}S_z+ig\hbar(a^\dagger S_- - aS_+) \tag{2.2}$$

in the rotating-wave approximation (RWA). As usual g denotes the dipole coupling constant and

$$S_z=\sum_{\mu=1}^{N}\sigma_\mu^z\,,\qquad S_\pm=\sum_{\mu=1}^{N}\sigma_\mu^\pm e^{\pm i\mathbf{k}\cdot\mathbf{x}_\mu}\,, \tag{2.3}$$

with σ_μ^z, $\sigma_\mu^\pm=\sigma_\mu^x\pm i\sigma_\mu^y$ being the Pauli matrices (without the factor $\hbar/2$).

The bath Hamiltonian consists of two parts,

$$H_{\text{bath}}=\hbar\int_0^\infty d\omega\, c^\dagger(\omega)c(\omega)-\hbar\int_0^\infty d\omega\, b^\dagger(\omega)b(\omega)\,. \tag{2.2$'$}$$

The first term stands for the reservoir of vacuum radiation modes which is interacting with the laser mode in the cavity, whereas the second term models some incoherent pumping mechanism of the atoms. Following Glauber[8] we use a bath of inverted harmonic oscillators in order to achieve formal pumping of the upper level of the "two-level atoms" (which consist of the two levels of the physical atoms in the active laser medium between which the lasing transition occurs).

This bath can be considered to be the usual bath of harmonic oscillators with the formal replacement

$$\hbar\omega\to-\hbar\omega\,,\qquad b(\omega)\to b^\dagger(\omega)\,, \tag{2.4}$$

hence the negative sign of the second term in (2.2′).

Finally, the couplings of the laser mode a to the radiative reservoir and of the atoms to the pumping bath, respectively, are taken to be linear, that is,

$$H_{\text{laser-bath}}=i\hbar\int_0^\infty d\omega\,\kappa(\omega)[c^\dagger(\omega)a-c(\omega)a^\dagger]+i\hbar\int_0^\infty d\omega\, g(\omega)[b(\omega)S_- - b^\dagger(\omega)S_+]\,. \tag{2.2$''$}$$

Note that due to (2.4) the terms kept within the RWA are bS_- and $b^\dagger S_+$ (rather than $b^\dagger S_-$ and bS_+). We assume $\kappa(\omega)$ and $g(\omega)$ to be slowly varying functions around $\omega=\omega_0$.

The assumptions stated above have all been within models of standard laser theories; now, however, we deviate from the usual procedure which treats the bath coupled to the atoms as being in a thermal state, by assuming that all the bath modes $b(\omega)$ are prepared in a squeezed state ("squeezed white noise"[5]) instead, so that the bath correlations read

$$\begin{aligned}&\langle b(t)\rangle=\langle b^\dagger(t)\rangle=0\,,\\&\langle b^\dagger(t)b(t')\rangle=(n+1)\delta(t-t')\,,\\&\langle b(t)b^\dagger(t')\rangle=n\delta(t-t')\,,\\&\langle b(t)b(t')\rangle=m^* e^{i\omega_0(t+t')}\delta(t-t')\,,\\&\langle b^\dagger(t)b^\dagger(t')\rangle=me^{-i\omega_0(t+t')}\delta(t-t')\,,\end{aligned} \tag{2.5}$$

for some numbers $n\in R_+$, $|m|^2\le n(n+1)$ (the inequality follows from the fact that the expression $\langle[b(t)+\lambda b^\dagger(t)][b^\dagger(t)+\lambda^* b(t)]\rangle$ has to be positive for all complex numbers λ.[5] The equality sign represents the case of a perfect squeezed state in the bath, that is, a minimum-uncertainty state of the Heisenberg uncertainty relation for the bath quadrature operators. In such a state the noise is reduced below the standard quantum limit, since $M>N$ for $M=N(N+1)$.) Note that compared to the correlations given in Ref. 5 the roles of b and $b^\dagger$ are exchanged because of (2.4), since the bath consists of inverted harmonic oscillators.

Our treatment of the pumping mechanism as being a bath of inverted harmonic oscillators has to be regarded as a unifying formal substitute for various possible experimental realizations of pump light with correlations of the form (2.5). The actual mechanism may differ largely, depending on the type of laser system one is dealing with. For the moment, we wish to ignore these experimental considerations, assuming we have some means to squeeze the light pumping our laser atom in a given laser system in such a way that (2.5) holds.

B. The master equation

In this section we will establish a master equation for the laser density operator ρ_{A+F} containing the laser-field mode (F) and the atomic variables (A) in the situation described above.

Gardiner[5] has derived a master equation for the similar situation of a single decaying atom coupled to a squeezed bath. Since—for the sake of simplicity—we wish to treat the N two-level atoms in the cavity as independent, i.e., not interacting directly with each other via collisions, etc., the generalization of Gardiner's master equation is achieved by simply summing over all atoms.

Furthermore, since in an inverted bath b and $b^\dagger$ change roles, the factors $(n+1)$ and n and m and m^*, respectively, have to be exchanged to adapt his equation to our case. Thus we find that the density operator ρ_A of a system of N independent atoms pumped by a bath in squeezed vacuum of inverted harmonic oscillators obeys the following master equation (in a frame rotating at ω_0):

$$\begin{aligned}\left[\frac{\partial\rho_A}{\partial t}\right]=&\ \frac{\gamma n}{2}(2S_-\rho_A S_+-\rho_A S_+S_- -S_+S_-\rho_A)\\&+\frac{\gamma(n+1)}{2}(2S_+\rho_A S_- -\rho_A S_-S_+\\&-S_-S_+\rho_A)\\&-\gamma m(S_-\rho_A S_-)-\gamma m^*(S_+\rho_A S_+)\,,\end{aligned} \tag{2.6}$$

with $S_\pm$ being the total dipole moment operators of the atoms given by (2.3).

From this and from the fact that the squeezed bath couples to the atoms only, we conclude that in order to incorporate the effects of squeezing the bath, we simply have to add the last two terms of (2.6) (which would be absent for an unsqueezed bath) to the atomic part of the otherwise unchanged laser master equation for ρ_{A+F} (as given by Refs. 6 and 7).

Throughout the rest of this paper we will assume $m \in R_+$. This corresponds to assuming that it is the imaginary part of the operator $b(t)e^{-i\omega_0 t}$ which carries the decreased fluctuations, whereas the real part shows increased fluctuations.

C. The Fokker-Planck equation

We wish to convert the two terms proportional to $m \in R_+$ in (2.6) which have been added to the master equation for the laser in the presence of squeezing in the pumping bath to corresponding terms in the Fokker-Planck equation of the laser. Following a standard procedure[6] we make use of the characteristic function

$$\chi(\xi,\xi^*,\zeta,\beta,\beta^*)=\mathrm{Tr}(e^{i\xi^* S_+}e^{i\zeta S_z}e^{i\xi S_-}e^{i\beta^* a^\dagger}e^{i\beta a}) \qquad (2.7)$$

in order to establish a correspondence between operators and c numbers,

$$a \leftrightarrow \alpha, \quad a^\dagger \leftrightarrow \alpha^* ,$$
$$v \leftrightarrow S_-, \quad v^* \leftrightarrow S_+, \quad \frac{D}{2} \leftrightarrow S_z .$$

The Fokker-Planck equation for

$$P(\alpha,\alpha^*,v,v^*,D) = \int d^2\xi\, d\zeta\, d^2\beta\, e^{-i[v\xi+v^*\xi^*+(D/2)\zeta+\alpha\beta+\alpha^*\beta^*]} + \chi(\xi,\xi^*,\zeta,\beta,\beta^*) \qquad (2.8)$$

in a frame rotating at ω_0 is of the form

$$\frac{\partial P}{\partial t}=(L+L_m)P \qquad (2.9a)$$

with

$$L=L_A+L_F+L_{AF} . \qquad (2.9b)$$

In (2.9a) we have split the total Liouville operator into a part L obtained by usual laser theory, consisting of an atomic part L_A, a field part L_F, and an interaction part L_{AF} [given by Eqs. (IV 10.32) to (IV 10.34) in Ref. 6] and an additional contribution

$$L_m=\gamma m\left[\frac{\partial}{\partial v}v^*+\frac{\partial}{\partial v^*}v+\frac{e^{2\partial/\partial D}}{2}\times\left[\frac{\partial^2}{\partial v^2}+\frac{\partial^2}{\partial v^{*2}}\right](N+D)\right] \qquad (2.9c)$$

due to the squeezing.

A truncation of the Fokker-Planck equation (2.9) (which contains derivatives up to fourth order with respect to v and v^* and up to infinite order with respect to D) to second-order derivatives, justified on the grounds that higher-order derivatives are smaller by a factor $1/N \ll 1$ (N being the number of atoms), yields

$$\frac{\partial P}{\partial t}=(\bar{L}_A+\bar{L}_F+\bar{L}_{AF}+\bar{L}_m)P \qquad (2.10a)$$

with

$$\bar{L}_A=\gamma_\perp\left[\frac{\partial}{\partial v}v+\frac{\partial}{\partial v^*}v^*\right]+\frac{\partial}{\partial D}(\gamma_\parallel D-\gamma N) + w_{12}\frac{\partial^2}{\partial v\,\partial v^*}N-2w_{12}\left[\frac{\partial^2}{\partial v\,\partial D}v+\frac{\partial^2}{\partial v^*\partial D}v^*\right] + \frac{\partial^2}{\partial D^2}(\gamma_\parallel N-\gamma D) , \qquad (2.10b)$$

$$\bar{L}_F=\kappa\left[\frac{\partial}{\partial\alpha}\alpha+\frac{\partial}{\partial\alpha^*}\alpha^*\right]+2\kappa\bar{n}_{\rm th}\frac{\partial^2}{\partial\alpha\,\partial\alpha^*} , \qquad (2.10c)$$

$$\bar{L}_{AF}=g\left[-\frac{\partial}{\partial\alpha}v-\frac{\partial}{\partial\alpha^*}v^*-\frac{\partial}{\partial v}D\alpha-\frac{\partial}{\partial v^*}D\alpha^* +2\frac{\partial}{\partial D}(v^*\alpha+v\alpha^*)+\frac{\partial^2}{\partial v^2}v\alpha+\frac{\partial^2}{\partial v^{*2}}v^*\alpha^* -2\frac{\partial^2}{\partial D^2}(v^*\alpha+v\alpha^*)\right] , \qquad (2.10d)$$

and

$$\bar{L}_m=\gamma m\left[\frac{\partial}{\partial v}v^*+\frac{\partial}{\partial v^*}v\right] +\frac{\gamma m}{2}\left[\frac{\partial^2}{\partial v^2}+\frac{\partial^2}{\partial v^{*2}}\right](N+D) \qquad (2.10e)$$

(the coupling constant g has been assumed to be real).

Here we have introduced constants

$$\gamma_\parallel=w_{12}+w_{21} , \quad \gamma_\perp=\tfrac{1}{2}\gamma_\parallel , \qquad (2.11)$$

which will be interpreted in Sec. III. By w_{12} we denote the transition rate $|1\rangle \to |2\rangle$ due to the incoherent pumping; w_{21} stands for the rate $|2\rangle \to |1\rangle$ due to nonlasing transitions (spontaneous emission). The photon number n and the constant γ in (2.6) can be expressed in terms of these rates

$$n=\frac{w_{21}}{w_{12}-w_{21}} , \quad \gamma=w_{12}-w_{21} . \qquad (2.12)$$

In a laser we have $w_{12}>w_{21}$, hence γ is always positive and constitutes the rate at which a population inversion is built up due to the pumping.

For reference in later sections we also introduce a constant

$$M=\frac{m\gamma}{\gamma_\perp} . \tag{2.13}$$

The upper bound of m (cf. Sec. II A), rewritten by using (2.12),

$$m \le m_{\max} ,\qquad m_{\max}=\frac{\sqrt{w_{12}w_{21}}}{w_{12}-w_{21}} , \tag{2.14}$$

imposes $M\in(0,1)$ for $m\in R_+$. Viewing the truncated Fokker-Planck equation (2.10), one notices that there are first-order as well as second-order derivative terms entering the equation if we squeeze the bath, that is, the drift vector as well as the diffusion matrix are changed. Note that we have kept all second-order derivatives in (2.10), in contrast to what is usually done, as we wish to study the laser well above threshold and for large squeezing, where the usual scaling argument for a further truncation[6] does not apply any more. As a consequence, (2.10) is not a proper Fokker-Planck equation, strictly speaking, since the diffusion matrix is not positive definite.

III. OPTICAL BLOCH EQUATIONS AND SEMICLASSICAL STEADY STATES

A. The optical Bloch equations

In order to get a first idea of the qualitative behavior, we investigate the semiclassical equations in this section. Omitting the diffusion terms and studying only the deterministic behavior governed by the drift terms of the Fokker-Planck equation (2.10), we find the following Bloch equations:

$$\begin{aligned}\dot\alpha&=-\kappa\alpha+gv ,\\ \dot v&=-\gamma_\perp v-\gamma m v^*+gD\alpha ,\\ \dot D&=-\gamma_\parallel D+\gamma N-2g(v\alpha^*+v^*\alpha) ,\end{aligned} \tag{3.1}$$

plus the complex conjugate equations for α^* and v^*. Compared to the usual equations, only the equations of motion of v and v^* are changed, inasmuch as v and v^* are now coupled.

The constant κ in (3.1) is the cavity damping of the laser mode α; $\gamma_\perp$ and $\gamma_\parallel$, which have been introduced in (2.11), usually constitute the ("transverse") decay rate of the dipole moment v and the ("longitudinal") decay rate of the population inversion D. Reexpressing (3.1) in terms of real and imaginary parts, however, yields

$$\begin{aligned}\dot\alpha_x&=-\kappa\alpha_x+gv_x ,\\ \dot\alpha_y&=-\kappa\alpha_y+gv_y ,\\ \dot v_x&=-\gamma_\perp(1+M)v_x+gD\alpha_x ,\\ \dot v_y&=-\gamma_\perp(1-M)v_y+gD\alpha_y ,\\ \dot D&=-\gamma_\parallel D+\gamma N-4g(v_x\alpha_x+v_y\alpha_y) .\end{aligned} \tag{3.2}$$

We see that the polarization decay involves *two* time scales now, since the real and the imaginary part of the dipole moment decay according to the different rates

$$\begin{aligned}\gamma_x&=\gamma_\perp(1+M) ,\\ \gamma_y&=\gamma_\perp(1-M) .\end{aligned} \tag{3.3}$$

This behavior is closely related to the results found by Gardiner[5] for a single atom coupled to a squeezed reservoir.

Note that in the limit $M\to1$, which corresponds to $m=m_{\max}$ [cf. Eq. (2.14)] and $n\to\infty$ (or, equivalently, $m=m_{\max}$ and $w_{21}\to w_{12}$), the rate γ_y approaches zero. Since this limit requires an infinite number of photons in the bath, it is somewhat unphysical, corresponding to quadrature operator eigenstates in the bath. Finally, we stress that in the case where M approaches 1, the usual adiabatic elimination procedure of the atoms will not be valid any longer, as γ_y may become small enough to be of the order of κ.

B. The semiclassical steady states

Setting the left-hand side of (3.1) to zero and eliminating the steady-state values of v, v^*, and D yields

$$\alpha\left[1-\frac{C}{R(\alpha,\alpha^*)}\right]=-\frac{CM}{R(\alpha,\alpha^*)}\alpha^* \tag{3.4}$$

and the complex-conjugate (c.c.) equation, where

$$R(\alpha,\alpha^*)=\left[1+\frac{|\alpha|^2}{2n_0}\right]^2-\left|M+\frac{\alpha^2}{2n_0}\right|^2 \tag{3.5}$$

is a real function of α and α^*. The constants

$$C=\frac{g^2D_0}{\gamma_\perp\kappa} ,\qquad D_0=\frac{N\gamma}{\gamma_\parallel} ,\qquad n_0=\frac{\gamma_\parallel\gamma_\perp}{4g^2} , \tag{3.6}$$

so far introduced as mere abbreviations, are familiar parameters within the standard laser theory. There C is a pump parameter, D_0 the population inversion below threshold, and n_0 the saturation photon number.

One can easily convince oneself that for $m=0$ one recovers the usual laser threshold condition

$$\alpha\left[1-\frac{C}{1+\frac{|\alpha|^2}{n_0}}\right]=0 . \tag{3.7}$$

Noticing that (3.4) and (3.5) imply that α/α^* has to be a real number of the steady state, we realize a qualitative difference between the threshold conditions (3.4) and (3.7): whereas the usual condition (3.7) is independent of the phase of α, thus allowing fields with any phase in the steady state, this is no longer true with a squeezed pump. The phase dependence in the noise of the squeezed pump light breaks this symmetry. As a consequence only fields which are either purely real or purely imaginary (thus $\alpha/\alpha^* \in R$) may exist (besides the trivial solution $\alpha \equiv 0$) in the steady state.

Calculation shows that (3.4) has the following nontrivial solutions:

$$\alpha = \pm i\sqrt{n_0[C-(1-M)]} \tag{3.8a}$$

and

$$\alpha = \pm\sqrt{n_0[C-(1+M)]} \; . \tag{3.8b}$$

existing for $C > 1+M$ or $C > 1-M$, respectively. Only a stability analysis can reveal which solution among these will be realized in fact.

C. Semiclassical stability analysis

For an equation of motion for $\alpha = u + iv$ of the form

$$\dot{\alpha} = F(\alpha, \alpha^*) \; , \tag{3.9}$$

the stability of a steady state requires all the eigenvalues of the matrix

$$F = \begin{bmatrix} \dfrac{\partial \,\mathrm{Re}F}{\partial u} & \dfrac{\partial \,\mathrm{Re}F}{\partial v} \\ \dfrac{\partial \,\mathrm{Im}F}{\partial u} & \dfrac{\partial \,\mathrm{Im}F}{\partial v} \end{bmatrix} \tag{3.10}$$

evaluated at the steady state under consideration, to be negative. In our case we have

$$\begin{aligned} \dot{u} &= \kappa \left[\frac{C}{R}(1-M) - 1 \right] u \equiv \mathrm{Re}F \; , \\ \dot{v} &= \kappa \left[\frac{C}{R}(1-M) - 1 \right] v \equiv \mathrm{Im}F \; , \end{aligned} \tag{3.11}$$

with the saturation denominator R [given by (3.5)] expressed in terms of u ($\equiv \alpha_x$) and v ($\equiv \alpha_y$):

$$R(u,v) = (1-M^2) + \frac{u^2}{n_0}(1-M) + \frac{v^2}{n_0}(1+M) \; . \tag{3.12}$$

Evaluating the matrix F for the trivial solution $u \equiv v \equiv 0$, we get

$$F(0) = \begin{bmatrix} \dfrac{C}{1+M} - 1 & 0 \\ 0 & \dfrac{C}{1-M} - 1 \end{bmatrix} . \tag{3.13a}$$

Evaluation at $u = 0$, $v = \pm\sqrt{n_0[C-(1-M)]}$ yields

$$F(\alpha_{1,2}) = \begin{bmatrix} \dfrac{1-M}{1+M} - 1 & 0 \\ 0 & -\dfrac{2}{C}[C-(1-M)] \end{bmatrix} \tag{3.13b}$$

and at $u = \pm\sqrt{n_0[C-(1+M)]}$, $v = 0$,

$$F(\alpha_{3,4}) = \begin{bmatrix} -\dfrac{2}{C}[C-(1+M)] & 0 \\ 0 & \dfrac{1+M}{1-M} - 1 \end{bmatrix} . \tag{3.13c}$$

In both of the last two matrices, one eigenvalue depends on M only. Due to a phase choice made earlier, M is a real number between 0 and 1, thus ruling out the two real solutions (3.8b), since

$$\frac{1+M}{1-M} > 1$$

for $M \in (0,1)$.

On the other hand, both eigenvalues of (3.13b) are negative, provided that $C > 1-M$, and the trivial solution proves to be stable for $C < 1-M$. Thus we have a stability exchange between the trivial solution $|\alpha| = 0$ and the two imaginary solutions (3.8a) with $|\alpha|^2 = n_0[C-(1-M)]$ occurring at $C = 1-M$.

If we had chosen $m \in R_-$ (thus decreasing the noise in the *real* quadrature operators of the bath) M would be replaced by $-|M|$ everywhere, and therefore the two real solutions would turn out to be the stable ones for $C > 1-|M|$. In other words, the steady states in phase with the low-noise quadrature in the squeezed bath are always the stable ones. In Sec. IV we will investigate the dynamics of this settling down of the field along the directions with decreased fluctuations in the squeezed bath in some detail.

Inspection of the Bloch equations (3.1) shows that the parameter D_0 introduced in (3.6) still has the meaning of the steady-state population inversion below threshold ($C < 1-M$ implies $\alpha = 0$). Summarizing the stable steady-state values above threshold, we have

$$\begin{aligned} \bar{\alpha} &= \pm i\sqrt{n_0[C-(1-M)]} \; , \\ \bar{v} &= \frac{\kappa}{g}\bar{\alpha} \; , \\ \bar{D} &= \frac{D_0(1-M)}{C} \; . \end{aligned} \tag{3.14}$$

Note that the steady-state inversion $\bar{D}$ is affected by the squeezing: this is why the threshold seems to be shifted away from the usual $C = 1$. In order to gain more insight into this, we note that the continuity of the two solutions for D below and above threshold requires

$$\frac{\bar{D}}{D_0} = 1 \tag{3.15}$$

at threshold. Thus, using (3.14) and (3.15), one may alternatively introduce a new "effective" pump parameter

$$\tilde{C}(M) = \frac{C}{1-M} > C \tag{3.16}$$

and characterize the threshold by $\tilde{C}(M) = 1$. There are two ways to look at the effect of the squeezing: one can either think of the threshold as being shifted compared to the ordinary laser or of the pump parameter as being changed due to the squeezing. In the limit $M \to 1$, the threshold is pushed down towards 0, since, as we have already mentioned, this limit corresponds to $n \to \infty$ and $\gamma \to 0$. Finally, we want to illustrate the "phase-locking" effect of the steady states which we encountered above, by calculating the potential $\Phi(u,v)$ for the equation of motion (3.11). It is easily verified that the potential condition

$$\frac{\partial\,\mathrm{Re}F}{\partial v}=\frac{\partial\,\mathrm{Im}F}{\partial u}$$

holds globally. Thus a potential exists and is by straightforward integration found to be

$$\Phi(u,v)=\frac{\kappa}{2}\{u^2+v^2-cn_0\ln[R(u,v)]\} \tag{3.17}$$

with $R(u,v)$ given by (3.12).

Figure 1 shows plots of Φ for several parameters. For $m=0$ we obtain the so-called "Mexican hat" for the laser potential above threshold. For a squeezed bath with $m\neq 0$, containing "reference phases" at which the noise is minimal, the phase symmetry of the problem is broken. How the squeezed bath acts to imprint a particular phase (in our case $\pm\pi/2$) onto the steady state is clearly visible in the deformation of the potential surface with increasing M; we get increasingly deep valleys along $\psi=\pm\pi/2$ and increasingly high ridges at $\psi=0,\pi$. We note the difference from the symmetry breaking which occurs in the laser with injected signal[9] and also in the micromaser,[4] where only one phase is stable.

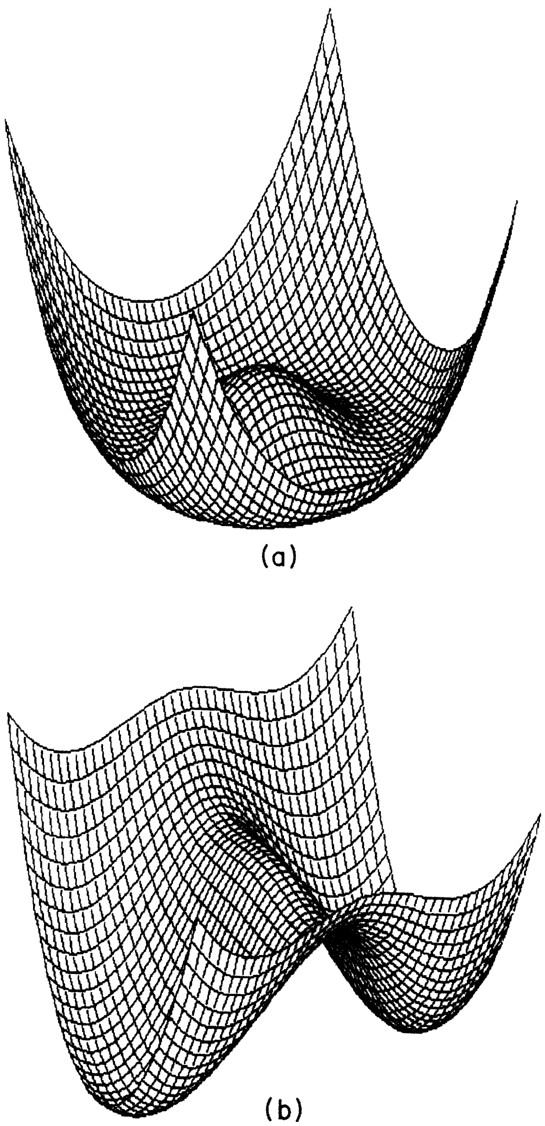

FIG. 1. Potential of the laser above threshold as a function of $u/\sqrt{n_0}\equiv\mathrm{Re}\alpha/\sqrt{n_0}$ and $v/\sqrt{n_0}\equiv\mathrm{Im}\alpha/\sqrt{n_0}$ (n_0 being the saturation photon number) in the region $(u/\sqrt{n_0},v/\sqrt{n_0})\in[-1.5,1.5]\times[-1.5,1.5]$. (a) Ordinary laser with phase diffusion, $M=0$, $C=2$. (b) With squeezing in the pumping bath, $M=0.5$, $C=2$ [thus $\tilde{c}(M)=4$]. It is clearly visible how the potential surface is distorted to form two local minima along the imaginary axis leading to a phase-locking phenomenon.

IV. ADIABATIC ELIMINATION FOR "MILD" SQUEEZING

A. The rotating-wave van der Pol equation with phase-dependent gain

In this section we wish to investigate the case in which the atomic variables v, v^*, and D can be eliminated adiabatically. To this end we assume that both γ_x and γ_y [see Eq. (3.3)] are very much larger than κ, i.e.,

$$\gamma_\perp(1\pm M)\gg\kappa\ .$$

Note that the above requirement does not preclude ideal squeezing in the bath, i.e., $m=m_{\max}$ [and thus $M=\sqrt{4w_{12}w_{21}}/(w_{12}+w_{21})$] if $\gamma_\perp,\gamma_\parallel\gg\kappa$ and $w_{12}\gg w_{21}$, hold, conditions typically found in a laser.

Consistent with the truncated Fokker-Planck equation (2.10), derived in Sec. II, we use the following quantum Langevin equations:

$$\begin{aligned}
\dot\alpha&=-\kappa\alpha+gv+\Gamma_\alpha \quad\text{and c.c. },\\
\dot v&=-\gamma_\perp v-\gamma m v^*+gD\alpha+\Gamma_v \quad\text{and c.c. },\\
\dot D&=-\gamma_\parallel D+\gamma N-2g(v^*\alpha+v\alpha^*)+\Gamma_D\ ,
\end{aligned}\tag{4.1}$$

with

$$\begin{aligned}
\langle\Gamma_\alpha(t)\Gamma_{\alpha^*}(t')\rangle&=\langle\Gamma_{\alpha^*}(t)\Gamma_\alpha(t')\rangle=2\kappa\bar n_{\rm th}\delta(t-t')\ ,\\
\langle\Gamma_v(t)\Gamma_v(t')\rangle&=\langle\Gamma_{v^*}(t)\Gamma_{v^*}(t')\rangle^*\\
&=[\gamma m(N+D)+2gv\alpha]\delta(t-t')\ ,\\
\langle\Gamma_v(t)\Gamma_{v^*}(t')\rangle&=\langle\Gamma_{v^*}(t)\Gamma_v(t')\rangle=w_{12}N\delta(t-t')\ ,\\
\langle\Gamma_v(t)\Gamma_D(t')\rangle&=\langle\Gamma_D(t)\Gamma_v(t')\rangle=-2w_{12}v\delta(t-t')\ ,\\
\langle\Gamma_{v^*}(t)\Gamma_D(t')\rangle&=\langle\Gamma_D(t)\Gamma_{v^*}(t')\rangle=-2w_{12}v^*\delta(t-t')\ ,\\
\langle\Gamma_D(t)\Gamma_D(t')\rangle&=[\,2(\gamma_\parallel N-\gamma D)\\
&\qquad-4g(\alpha^*v+v^*\alpha)]\delta(t-t')\ .
\end{aligned}\tag{4.2}$$

Since the number of thermally excited photons $\bar n_{\rm th}\ll 1$ usually, we have neglected the comparatively small field correlations in the following.

In this section we assume that α and α^* are the complex conjugate of each other, because in our case of small M, implying that the spontaneous emission is the dominant noise source, the diffusion matrix is nearly positive definite somewhat above threshold; hence we need not use a generalized P representation but may treat α and α^* as complex-conjugate variables of a

Glauber P representation to lowest order in M. In Sec. V we will present a more rigorous treatment for arbitrarily strong squeezing using a generalized P representation.

Elimination of v, v^*, and D yields the equation

$$\dot{\alpha}=-\kappa\alpha\left[1-C\frac{1+\frac{|\alpha|^2}{2n_0}}{R}\right]-\kappa\alpha^*\frac{C}{R}\left[M+\frac{\alpha^2}{2n_0}\right]+\frac{g}{\gamma_\perp}\frac{\left[1+\frac{|\alpha|^2}{2n_0}\right]}{R}\Gamma_v-\frac{g}{\gamma_\perp}\frac{\left[M+\frac{\alpha^2}{2n_0}\right]}{R}\Gamma_{v^*}+\left[\frac{g}{\gamma_\perp}\right]^2\frac{\alpha(1+M)}{2R}\Gamma_D \tag{4.3}$$

and its c.c. counterpart. The standard laser theory[6,7] investigates the corresponding equation around threshold. Following this approach, we derive an approximate version of (4.3) for that regime.

Treating $|\alpha|^2/n_0$, M, and the stochastic forces Γ as small quantities of roughly the same order of magnitude and expanding (4.3) with respect to them yields

$$\dot{\alpha}=G(\psi)\alpha-\kappa c\frac{|\alpha|^2}{n_0}\alpha+\frac{g}{\gamma_\perp}\Gamma_v \tag{4.4}$$

to lowest order. Here $G(\psi)$ denotes

$$G(\psi)=\kappa\left[c-1-M\frac{\alpha^*}{\alpha}\right]$$

$$\equiv\kappa(c-1-Me^{-2i\psi}) . \tag{4.5}$$

Setting $M=0$ we get

$$\dot{\alpha}=\kappa(c-1)\alpha-\kappa c\alpha\frac{|\alpha|^2}{n_0}+\frac{g}{\gamma_\perp}\Gamma_v , \tag{4.6}$$

which is the well-known rotating-wave van der Pol equation for the laser as derived by Risken.[10]

For $M=0$, $G=\kappa(c-1)$ represents the gain of the laser; for $G>0$ we get self-excitation of the laser mode (see also Ref. 7). Therefore we interpret $G(\psi)$ as a phase-dependent gain in the case $M\neq 0$. Equation (4.5) shows that the gain—and thus the amplification—is highest for fluctuations around zero which have phase $\psi=\pm\pi/2$. That is why fluctuations of this kind prevail and eventually a steady-state field with $\psi=\pi/2$ or $\psi=-\pi/2$ is built up [of course, the saturation term $\sim|\alpha|^2\alpha/n_0$ in (4.4) prevents unlimited growth]. In Sec. III we have already shown that these two possibilities constitute both stable steady states above threshold, that is, for $G(\pm\pi/2)>0$. Thus (4.4), describing a rotating-wave van der Pol oscillator with phase-sensitive self-excitation, somehow elucidates how one of the two stable steady states is dynamically built up above threshold.

B. The spectroscopic linewidth

We may derive the spectroscopic linewidth of the squeezed-pump laser along the lines of Ref. 7. Since in our case the stable steady states possess well-defined phases $\psi_0=\pm\pi/2$, we may linearize (4.4) in the following way:

$$\alpha=[r_0+\delta r(t)]e^{i[\psi_0+\delta\psi(t)]} . \tag{4.7}$$

Inserting this into (4.4) and keeping only linear terms in the fluctuations, we get

$$\delta\dot{r}+ir_0\delta\dot{\psi}=G(\psi_0)(r_0+\delta r)+2iMr_0\kappa e^{-2i\psi_0}\delta\psi-\frac{\kappa c}{n_0}(r_0^3+3r_0^2\delta r)+\frac{g}{\gamma_\perp}\Gamma_v e^{-i\psi_0}(1-i\delta\psi) .$$

Setting the fluctuations and the stochastic term to zero, we find

$$r_0=\left[\frac{G(\psi_0)}{\kappa C/n_0}\right]^{1/2}=\left[\frac{n_0}{C}[C-(1-M)]\right]^{1/2} . \tag{4.8}$$

Note that, by comparison, this steady-state amplitude r_0 differs by a factor C from the exact result derived earlier [see Eq. (3.8a)]; this is so because we are using an approximate equation of motion derived by expanding (4.3) around threshold, i.e., we have already assumed $C\approx 1$ besides $M\ll 1$.

Splitting (4.11) in real and imaginary parts and making use of (4.12), we find

$$\delta\dot{r}=-2G(\psi_0)\delta r+\mathrm{Re}F , \tag{4.9a}$$

$$\delta\dot{\psi}=-2\kappa M\delta\psi+\frac{1}{r_0}\mathrm{Im}F , \tag{4.9b}$$

with

$$F=+\frac{g}{\gamma_\perp}\Gamma_v e^{-i\psi_0}(1-i\delta\psi) . \tag{4.9c}$$

The relaxation rate of the amplitude (or intensity) fluctuations is essentially the gain

$$\lambda_r=2G(\psi_0) . \tag{4.10a}$$

The phase-locking equation (4.13b) has no counterpart in the usual laser. For $M\neq 0$ we have restoring forces inducing a phase-relaxation rate

$$\lambda_\psi=2\kappa M . \tag{4.10b}$$

From this, it is obvious that if we "switch off" the squeezing by letting $M\to 0$, this rate λ_ψ and the phase-restoring forces approach zero, and we recover the random walk of the phase encountered usually. In order to be sure that a linearization procedure of this type around a particular steady-state phase is appropriate, we demand $\lambda_\psi\simeq\lambda_r$.

Assuming $\delta r(t)\ll r_0$ somewhat above threshold, we may approximate

$$\langle a^\dagger(t)a(0)\rangle\simeq r_0^2\langle e^{-i[\psi_0+\delta\psi(t)]}e^{i\psi_0}\rangle$$

$$=r_0^2\langle e^{-i\delta\psi(t)}\rangle=r_0^2e^{-(1/2)\langle\delta\psi^2(t)\rangle} . \tag{4.11}$$

Thus we identify

$$\gamma_\psi = \frac{1}{2t}\langle \delta\psi(t)^2\rangle \tag{4.12}$$

as the spectroscopic linewidth of the internal laser mode. (It can be shown, using the input-output formalism by Gardiner and Collett[11] that the external linewidth will be given by $2\kappa\gamma_\psi$, provided the cavity has only one port.)

It is worth mentioning that the existence of a well-defined steady-state phase makes the usual Markov assumption on the phase correlations unnecessary for the derivation of (4.16). Integration of (4.13b) for the initial condition $\delta\psi(0)=0$ yields

$$\langle \delta\psi(t)^2\rangle = \int_0^t dt' \int_0^t ds' \langle \mathrm{Im}\bar{\Gamma}(t')\mathrm{Im}\bar{\Gamma}(s')\rangle \tag{4.13}$$

to lowest order, where $\bar{\Gamma}$ stands for

$$\bar{\Gamma}(t) = \frac{g}{\gamma_\perp r_0}\Gamma_v e^{-i\psi_0} . \tag{4.14}$$

Using the correlations (4.2), we finally get

$$\gamma_\psi = \frac{g^2\kappa C}{4\gamma_\perp n_0}\frac{1}{G(\psi_0)}\left[\frac{w_{12}N}{\gamma_\perp} + M(N+D)\right] . \tag{4.15}$$

Setting $M=0$, we obtain

$$\gamma_\psi = \frac{\kappa}{2}\frac{(\bar{n}_{\rm sp}+\bar{n}_{\rm th})}{\bar{n}} ,$$

from (4.19) with the identifications $w_{12}g^2N/2\kappa\gamma_\perp^2 = (\bar{n}_{\rm sp}+\bar{n}_{\rm th})$, the sum of spontaneously emitted photons and thermal photons, and $r_0^3=\bar{n}$ (cf. Ref. 6). This can be brought into the familiar form

$$\gamma_\psi = \frac{\hbar\omega}{P}\kappa^2(\bar{n}_{\rm sp}+\bar{n}_{\rm th}) \tag{4.16}$$

with $P=2\kappa\bar{n}\hbar\omega$ representing the emission power of the laser. Thus (4.19) gives the correct lowest-order result in the limit $M\to 0$, in spite of the fact that because of phase diffusion for $M=0$ a different linearization procedure is actually required. In (4.19) the squeezing affects γ_ψ in two ways: first via the gain which is enhanced $(C-1+M>C-1)$ and thus brings about a narrowing compared to (4.20), and second by addition of a broadening term proportional to M.

In deriving (4.19), however, we had assumed $w_{12}\gg w_{21}$ (and thus $M\ll 1$) and $C\gtrsim 1-M$, and thus the first effect, division by a rather small number, is clearly dominant in the situation considered. Well above threshold $C\gg 1-M$, the gain depends only slightly on M and then a broadening may occur—but even the qualitative validity of (4.19) in that case is doubtful.

V. NOISE SPECTRUM

In this section we investigate the fluctuations of the laser field around a stable steady state in the general case where adiabatic elimination of the atomic variables might not be possible any more. To this end we make use of a generalized (positive) P representation. $P(\alpha,\alpha^\dagger,v,v^\dagger,D)$, that is, we double the number of variables by treating $\alpha^\dagger$ and α and $v^\dagger$ and v as independent variables.

Linearizing the Langevin equations corresponding to the truncated Fokker-Planck equation (2.10) around a stable steady state (again neglecting terms proportional to $\bar{n}_{\rm th}\ll 1$) yields the following equation of motion for $\delta\boldsymbol{\alpha}=[\delta\alpha(t),\delta\alpha^\dagger(t),\delta v(t),\delta v^\dagger(t),\delta D]^T$

$$\delta\dot{\boldsymbol{\alpha}} = -A\,\delta\boldsymbol{\alpha} + \boldsymbol{\Gamma}(t) , \tag{5.1a}$$

with a constant drift matrix given by

$$A = \begin{bmatrix} \kappa & 0 & -g & 0 & 0 \\ 0 & \kappa & 0 & -g & 0 \\ -g\bar{D} & 0 & \gamma_\perp & \gamma m & -g\bar{\alpha} \\ 0 & -g\bar{D} & \gamma m & \gamma_\perp & -g\bar{\alpha}^* \\ 2g\bar{v}^* & 2g\bar{v} & 2g\bar{\alpha}^* & 2g\bar{\alpha} & \gamma_\parallel \end{bmatrix} \tag{5.1b}$$

and

$$\boldsymbol{\Gamma}(t) = [0,0,\Gamma_v(t),\Gamma_v^\dagger(t),\Gamma_D(t)]^T . \tag{5.1c}$$

Here $\bar{\alpha},\bar{\alpha}^\dagger(\equiv\bar{\alpha}^*)$, etc., are the steady-state values as given by Eq. (3.14). We rewrite the δ-correlated stochastic "force" $\boldsymbol{\Gamma}(t)$ as

$$\boldsymbol{\Gamma}(t) = B\,\mathbf{e}(t) \tag{5.2}$$

with $\mathbf{e}$ satisfying $\langle \mathbf{e}(t)\mathbf{e}^T(t')\rangle = \delta(t-t')1(5)$ [$1(5)$ being the five-dimensional identity operator]. Then the constant matrix B in (5.2) satisfies $BB^T=D$, with nonzero matrix elements given by

$$\begin{aligned} &D_{vv} = \gamma m(N+\bar{D}) + 2g\bar{v}\,\bar{\alpha} , \\ &D_{v^\dagger v^\dagger} = \gamma m(N+\bar{D}) + 2g\bar{v}^\dagger\bar{\alpha}^\dagger = D_{vv} , \\ &D_{vv^\dagger} = D_{v^\dagger v} = w_{12}N , \\ &D_{vD} = D_{Dv} = -2w_{12}\bar{v} , \\ &D_{v^\dagger D} = D_{Dv^\dagger} = -2w_{12}\bar{v}^\dagger = -D_{vD} , \\ &D_{DD} = 2(\gamma_\parallel N - \gamma\bar{D}) - 4g(\bar{v}\,\bar{\alpha}^\dagger + \bar{v}^\dagger\bar{\alpha}) , \end{aligned} \tag{5.3}$$

the diffusion matrix of the Fokker-Planck equation (2.10) evaluated at the steady state.

We wish to review briefly the derivation of the noise spectrum $S(\omega)$ which, by standard definition, is given by

$$S(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\, e^{-i\omega t}\langle \delta\boldsymbol{\alpha}(t)\delta\boldsymbol{\alpha}^T(0)\rangle . \tag{5.4}$$

Considering the Fourier transform of $\delta\boldsymbol{\alpha}(t)$,

$$\delta\tilde{\boldsymbol{\alpha}}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\, e^{i\omega t}\delta\boldsymbol{\alpha}(t) , \tag{5.5}$$

satisfying the Fourier transform of Eq. (5.1)

$$[A - i\omega 1(5)]\delta\tilde{\boldsymbol{\alpha}}(\omega) = \tilde{\boldsymbol{\Gamma}}(\omega) \tag{5.6}$$

with

$$\tilde{\boldsymbol{\Gamma}}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\, e^{i\omega t}\boldsymbol{\Gamma}(t) ,$$

we easily derive the relation

$$\langle \delta\boldsymbol{\alpha}(\omega)\delta\boldsymbol{\alpha}^T(\omega')\rangle = \delta(\omega+\omega)S(-\omega) . \tag{5.7}$$

Setting

$$\tilde{\Gamma}(\omega)=B\tilde{\epsilon}(\omega) \qquad (5.8)$$

with the same matrix B as in (5.2) and with $\langle \tilde{\epsilon}(\omega)\tilde{\epsilon}(\omega')\rangle=\delta(\omega+\omega')\mathbf{1}(5)$, consequently, and making use of (5.7), (5.8), and (5.6), we find the well-known expression

$$S(\omega)=[A-i\omega\mathbf{1}(5)]^{-1}D[A^T+i\omega\mathbf{1}(5)]^{-1} \qquad (5.9)$$

with A and D being the constant matrices given in (5.1) and (5.3). Since we are interested in the field fluctuations $\delta\alpha,\delta\alpha^\dagger$ only, we follow a nonadiabatic elimination procedure suggested by Reid.[12] Eliminating the variables $\delta\tilde{v}(\omega),\delta\tilde{v}^\dagger(\omega),\delta\tilde{D}(\omega)$ from Eq. (5.6), the remaining two equations for $\delta\tilde{\alpha}(\omega)$ and $\delta\tilde{\alpha}^\dagger(\omega)$ can be written in the form

$$[\mathcal{A}(\omega)-i\omega\mathbf{1}(2)]\begin{bmatrix}\delta\tilde{\alpha}(\omega)\\ \delta\tilde{\alpha}^\dagger(\omega)\end{bmatrix}=\tilde{\mathcal{F}}(\omega) . \qquad (5.10a)$$

A straightforward calculation yields

$$\mathcal{A}(\omega)=\kappa\begin{bmatrix}1-U(\omega) & -V(\omega)\\ -V(\omega) & 1-U(\omega)\end{bmatrix} ; \qquad (5.10b)$$

introducing abbreviations

$$\Delta(\omega)=1-i\frac{\omega}{\gamma_\perp}\equiv\Delta^*(-\omega) ,$$

$$a(\omega)=2\left[\frac{g\,|\bar{\alpha}|}{\gamma_\perp}\right]^2\frac{1}{1+\Delta} ,$$

the functions U and V in (5.10b) read

$$U=\frac{1}{\mathcal{N}}[(\Delta+a)(1-M-a)+a(a-M)] ,$$

$$V=\frac{1}{\mathcal{N}}[a(\Delta+a)+(1-M-a)(a-M)] ,$$

$$\mathcal{N}=(\Delta+a)^2-(a-M)^2 .$$

The stochastic term in (5.10a) is found to be

$$\begin{bmatrix}\tilde{\mathcal{F}}\alpha(\omega)\\ \tilde{\mathcal{F}}\alpha^{\dagger(\omega)}\end{bmatrix}=\frac{g}{\gamma_\perp\mathcal{N}}\times\begin{bmatrix}(\Delta+a)\tilde{\Gamma}_v(\omega)+(a-M)\tilde{\Gamma}_v^\dagger(\omega)+b\tilde{\Gamma}_D(\omega)\\ (\Delta+a)\tilde{\Gamma}_v^\dagger(\omega)+(a-M)\tilde{\Gamma}_v(\omega)-b\tilde{\Gamma}_D(\omega)\end{bmatrix} \qquad (5.10c)$$

with $b(\omega)=g\bar{\alpha}/\gamma_\perp(\Delta+M/\Delta+1)\equiv-b^*(-\omega)$. Setting

$$\tilde{\mathcal{F}}(\omega)=\mathcal{B}(\omega)\tilde{\epsilon}(\omega) \qquad (5.11)$$

with $\langle\tilde{\epsilon}(\omega)\tilde{\epsilon}(\omega')^T\rangle=\delta(\omega+\omega')\mathbf{1}(2)$, after the model of Eqs. (5.2) and (5.3), we define

$$\mathcal{D}(\omega)=\mathcal{B}(\omega)\mathcal{B}(-\omega)^T . \qquad (5.12)$$

Evaluating the matrix elements of this two-dimensional diffusion matrix, given by $\langle\tilde{\mathcal{F}}_\alpha(\omega)\tilde{\mathcal{F}}_\alpha(-\omega)\rangle$, $\langle\tilde{\mathcal{F}}_\alpha(\omega)\tilde{\mathcal{F}}_\alpha^\dagger(-\omega)\rangle$, etc., yields

$$\mathcal{D}_{\alpha\alpha}(\omega)=\mathcal{D}_{\alpha^\dagger\alpha^\dagger}(\omega)=\left[\frac{g}{\gamma_\perp|\mathcal{N}|}\right]^2\{[|\Delta+a|^2+|a-M|^2]D_{vv}+2\,\mathrm{Re}[(\Delta+a)(a-M)^*]D_{vv^\dagger}+2i\,\mathrm{Im}[b(\Delta+M)^*]D_{vD}-|b|^2D_{DD}\} \qquad (5.13a)$$

and

$$\mathcal{D}_{\alpha^\dagger\alpha}(\omega)=\mathcal{D}_{\alpha\alpha^\dagger}(\omega)=\left[\frac{g}{\gamma_\perp|\mathcal{N}|}\right]^2\{[|\Delta+a|^2+|a-M|^2]D_{vv^\dagger}+2\,\mathrm{Re}[(\Delta+a)(a-M)^*]D_{vv}-2i\,\mathrm{Im}[b(\Delta+M)^*]D_{vD}+|b|^2D_{DD}\} \qquad (5.13b)$$

in terms of the expressions given by Eq. (5.3). Note that $\mathcal{D}$ is real and that $\mathcal{D}(-\omega)\equiv\mathcal{D}^T(\omega)=\mathcal{D}(\omega)$ holds.

Along the same lines as above, we derive

$$S(\omega)=[\mathcal{A}(-\omega)+i\omega\mathbf{1}(2)]^{-1}\mathcal{D}(\omega)[\mathcal{A}^T(\omega)-i\omega\mathbf{1}(2)]^{-1} . \qquad (5.14)$$

In Fig. 2 we plot the elements S_{11} and S_{21} of the two-dimensional spectrum matrix (5.14) as functions of ω/κ for various values of M. [These results may be checked by numerically solving the five-dimensional Eq. (5.9).] It has been assumed that the bath is in an ideal squeezed state, i.e., $m=m_{\max}$, and thus M is given by $M=\sqrt{w_{12}w_{21}}/\gamma_\perp$ [see Eqs. (2.13) and (2.14)]; furthermore the pump parameter $\bar{c}(M)$ has been kept constant, at 20% above threshold, for all curves (this case be achieved by rescaling the coupling constant g for the three curves).

Increasing M, one increases the noise in the system; hence it is necessary to check the validity of the linearization. This has been done by calculating the signal-to-noise ratio in the laser field; since it can be shown that

$$\langle\delta\alpha^\dagger(t')\delta\alpha(t)\rangle=\mathcal{D}_{\alpha^\dagger\alpha}(\omega=0)\delta(t-t') , \qquad (5.15)$$

etc. holds in the steady state, the requirements read

$$|\mathcal{D}_{\alpha^\dagger\alpha}(\omega=0)|/|\bar{\alpha}|^2 \ll 1 ,$$
$$|\mathcal{D}_{\alpha\alpha}(\omega=0)|/|\bar{\alpha}|^2 \ll 1 . \qquad (5.16)$$

In order to satisfy (5.16) when approaching $M=1$ one has to increase the pump parameter $\tilde{c}$ accordingly, i.e., one has to move away from the threshold.

The diagonal elements $S_{11}(\omega)$ and $S_{22}(\omega)$, which are absent in the ordinary laser, are found to be real and negative, thus implying $S_{22}(\omega)\equiv S_{11}(\omega)^* = S_{11}(\omega)$. All four elements of (5.14) are Lorentzians, i.e., symmetric functions of ω, thus we also have $S_{12}(\omega)\equiv S_{21}(-\omega) = S_{21}(\omega)$.

Note that we cannot recover the usual result $S_{11}(\omega)=S_{22}(\omega)=0$ by simply taking the limit $M\rightarrow 0$, since in this case the phase relaxation becomes small, and an altogether different linearization is required. In other words, one can no longer linearize around a stable steady state phase but one has to take the phase diffusion into account.

Finally, we are interested in the amplitude and phase fluctuations of the laser field. Let us consider

$$S_{xx}(\omega)=\tfrac{1}{4}[S_{11}(\omega)+S_{22}(\omega)+S_{12}(\omega)+S_{21}(\omega)]$$
$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\, e^{-i\omega t}\langle \tfrac{1}{2}(\delta\alpha+\delta\alpha^\dagger)\tfrac{1}{2}(\delta\alpha+\delta\alpha^\dagger)\rangle \qquad (5.17a)$$

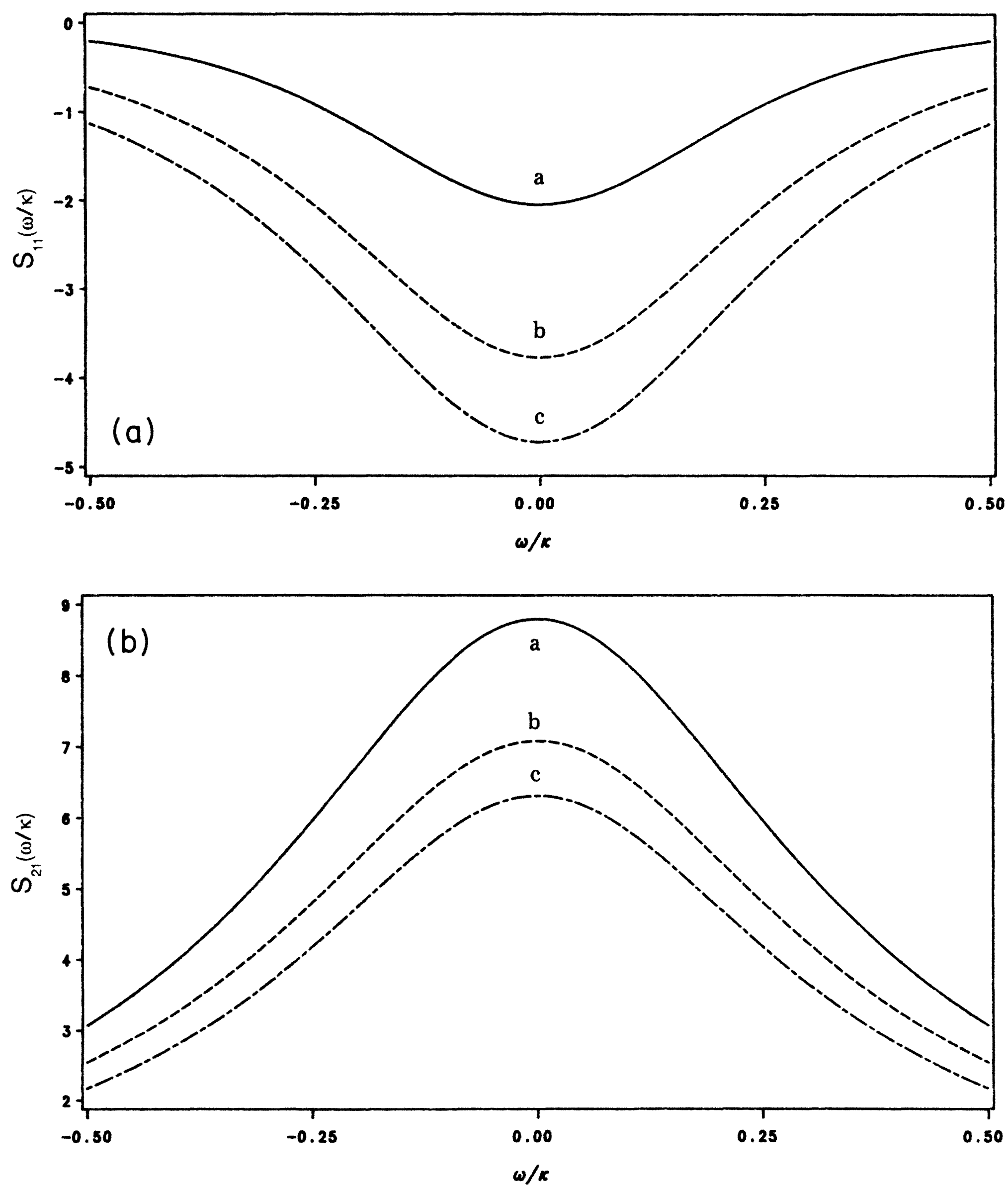

FIG. 2. (a) Plot of $S_{11}(\omega/\kappa)$ for $w_{12}=100\kappa$, $\tilde{c}(M)=1.2$ (=20% above threshold), $M=m_{\max}\gamma/\gamma_\perp=2\sqrt{w_{12}w_{21}}/(w_{12}+w_{21})$ for curve a, $w_{21}=2\kappa$ or $M=0.28$ (solid line); curve b, $w_{21}=4\kappa$ or $M=0.38$ (dashed line); curve c, $w_{21}=8\kappa$ or $M=0.52$ (dot-dashed line). (b) Plot of $S_{21}(\omega/\kappa)$ for the same parameter values.

and since we have chosen a correspondence between c numbers and normally ordered operators, this equals

$$S_{xx}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\, e^{-i\omega t}\langle :\mathrm{Re}(a),\mathrm{Re}(a):\rangle ,$$

where $\langle :A,B:\rangle=\langle :AB:\rangle-\langle :A:\rangle\langle :B:\rangle$ for operator functions A and B of a and $a^\dagger$ and : : denotes normal ordering. Visualizing small fluctuations around the stable steady state with imaginary amplitude $\bar{\alpha}=i\,|\bar{\alpha}|$, we conclude that $S_{xx}(\omega)$ relates to the phase fluctuations of the field.

On the other hand,

$$S_{yy}(\omega)=\tfrac{1}{4}[S_{12}(\omega)+S_{21}(\omega)-S_{11}(\omega)-S_{22}(\omega)]$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\, e^{-i\omega t}\left\langle \frac{1}{2i}(\delta\alpha-\delta\alpha^\dagger)\frac{1}{2i}(\delta\alpha-\delta\alpha^\dagger)\right\rangle$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\, e^{-i\omega t}\langle :\mathrm{Im}(a),\mathrm{Im}(a):\rangle \qquad (5.17\mathrm{b})$$

constitutes the spectrum of amplitude fluctuations. S_{xx} and S_{yy}, which in our case reduce to $S_{xx}=\frac{1}{2}(S_{21}+S_{11})$ and $S_{yy}=\frac{1}{2}(S_{21}-S_{11})$, are depicted in Fig. 3. The phase

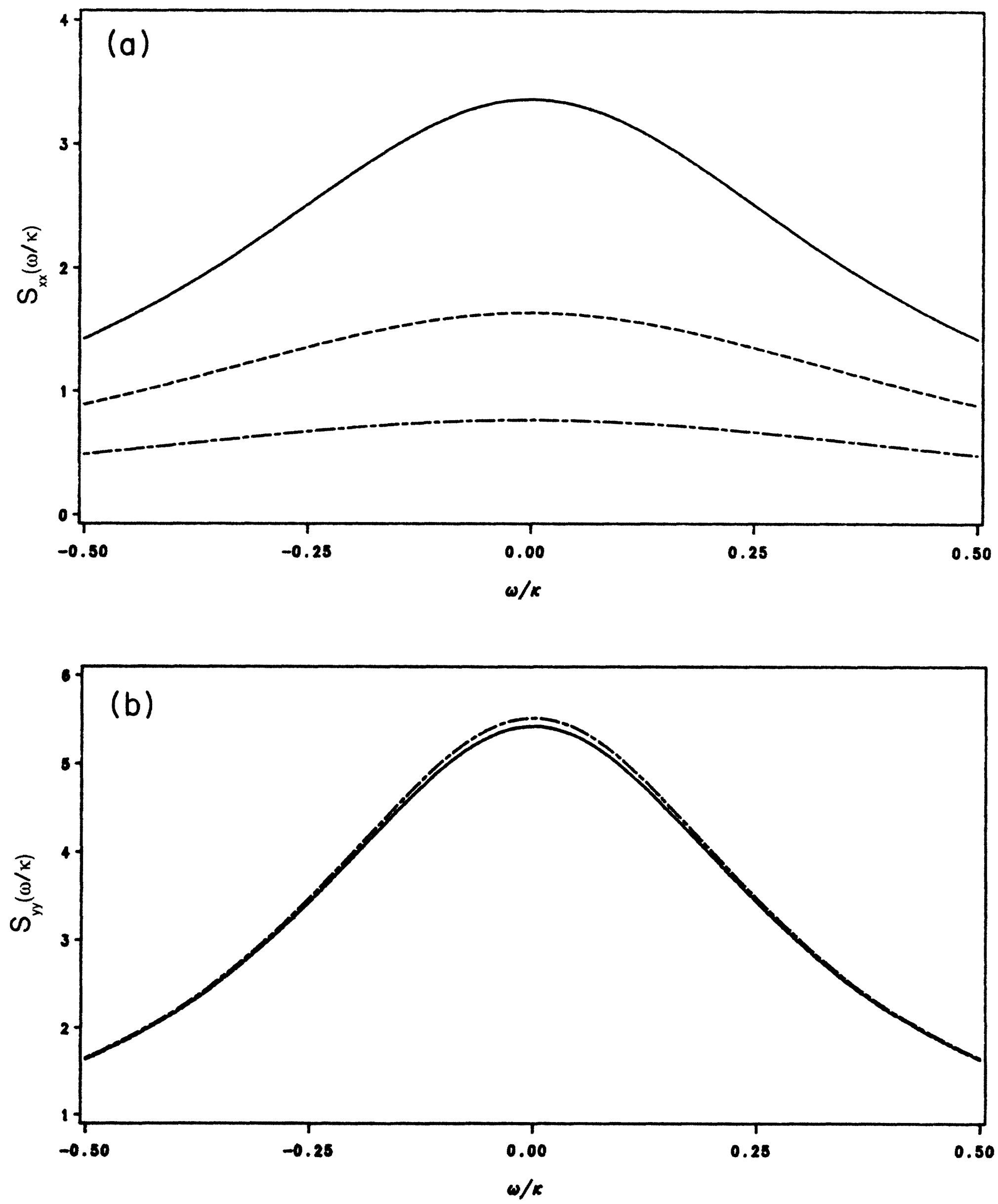

FIG. 3. (a) Plot of $S_{xx}(\omega/\kappa)=\frac{1}{2}[S_{21}(\omega/\kappa)+S_{11}(\omega/\kappa)]$. (b) Plot of $S_{yy}(\omega/\kappa)=\frac{1}{2}[S_{21}(\omega/\kappa)-S_{11}(\omega/\kappa)]$. Parameters as in Fig. 2.

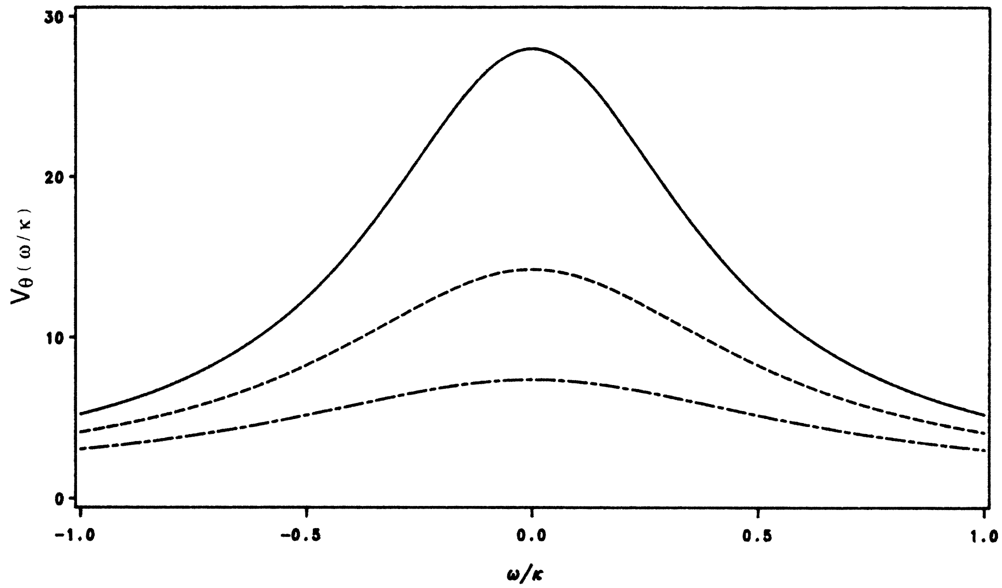

FIG. 4. The squeezing variance $V_\theta(\omega/\kappa)=1+2\kappa[S_{12}(\omega/\kappa)+S_{21}(\omega/\kappa)+e^{-2i\theta}S_{11}(\omega/\kappa)+e^{2i\theta}S_{22}(\omega/\kappa)]$ for $\theta=0$; parameters as in Figs. 2 and 3.

noise, related to S_{xx}, is clearly reduced compared to the noise in the amplitude, related to S_{yy}, in the regime where one can linearize around a steady-state phase. Thus the effect of a pumping bath in a squeezed vacuum state is to produce a steady-state laser field with a well-defined phase and the fluctuations around this phase can be reduced compared to the noise in the other quadrature by letting the state of the bath approach a quadrature operator eigenstate (formally the limit $M \rightarrow 1$).

However, for the case of a squeezed vacuum, treated in the present paper, this reduction of the phase noise is not below the standard quantum limit. It is rather tedious but straightforward to prove from Eqs. (5.10)–(5.14) that the condition $w_{12} > w_{21}$ in the laser and the upper limit of M imply that the spontaneous-emission noise proportional to Nw_{12} (cf. Ref. 6) remains dominant for the phase noise in the regime where the linearized theory is valid. Hence the squeezing variance as defined by[13]

$$V_\theta(\omega)=1+2\kappa[S_{12}(\omega)+S_{21}(\omega)+e^{-2i\theta}S_{11}(\omega) + e^{2i\theta}S_{22}(\omega)] \tag{5.18}$$

stays above the coherent level $V_\theta(\omega)=1$ for $\theta=0$. In Fig. 4 we plot the minimum of the squeezing variance with respect to θ, which occurs at $\theta=0$, for the same parameters as before.

VI. CONCLUSION

In the present paper we have investigated the effects of a lasing medium being pumped incoherently by a bath in a squeezed vacuum state. It has been shown that the anisotropy of the noise in the bath leads to a laser field with a well-defined phase in contrast to the phase diffusion usually encountered. The noise in the quadrature of the laser mode related to the phase is quenched—but no reduction below the standard quantum limit was found. Further investigations describing a laser pumped with squeezed light with reduced amplitude fluctuations are in progress.

Note added in proof. As will be shown in a forthcoming paper, the squeezing variance can drop below the coherent level for the quadrature $\theta=\pi/2$, corresponding to the laser amplitude.

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