Scattering from a magnetic flux line due to the Lorentz force of the return flux

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The differential cross section for the scattering of an electron by an impenetrable, infinitely long magnetic flux line with return magnetic flux uniformly distributed on the surface of a cylinder of radius a is calculated. The scattering is due to the Lorentz force from the magnetic field of the return flux. In the limit as the radius of the return flux goes to infinity ($a \to \infty$), the differential scattering cross section is the same as that obtained by Aharonov and Bohm (AB) for the scattering of an electron by an impenetrable, infinitely long magnetic flux line without return flux. The AB scattering cross section is, however, attributed to the effect of the vector potential in a field-free region so no force acts on the electron.

I. INTRODUCTION

Almost three decades ago Aharonov and Bohm[1] (AB) calculated the scattering cross section for the scattering of an electron by an impenetrable, infinitely long magnetic flux line. To the surprise of the physics community, the differential cross section was not zero and in fact the total cross section was infinite. AB ascribed the scattering as due to the vector potential in a region where the magnetic (and electric) field was zero so no force acts on the electron. Since they do not mention the return magnetic flux they have implicitly assumed that it is outside the region defined by the source and the detector. Even though AB diffraction effects have been experimentally observed,[2] there has been no experimental confirmation of AB scattering. In a previous paper we have criticized AB scattering for not satisfying angular momentum conservation.[3]

In this paper we consider another problem, viz., scattering from an impenetrable, infinitely long magnetic flux line when the return magnetic flux[4] is distributed uniformly on the surface of a cylinder of radius a.[5] In this problem the electron experiences a Lorentz force when it passes through the magnetic field of the return flux. In the limit as $a \to \infty$, the *same* cross section as AB originally calculated is obtained. Our problem is different, however, from that of AB.[6] In our case the return flux is always *inside* the region defined by the source and the detector,[5] while in the AB case the return flux is implicitly assumed to be *outside*.[1,7] In our case the scattering of the electron is due to the Lorentz force from the magnetic field of the return flux, while in the AB case it is ascribed to the vector potential in a field-free region. The infinite total cross section obtained can be understood in our case as arising from the scattering by an object of infinite radius, viz., the cylinder on whose surface the flux returns.[8]

The return magnetic flux could be made to return in ways other than uniformly distributed on the surface of a cylinder. In our problem we have chosen the magnetic flux to return in a cylindrically symmetric way, however, to preserve the symmetry. The AB cross section is obtained as the radius a goes to infinity. For models which take into account the return flux in other ways, this limit may not be obtained.

We prove the optical theorem[8] for two-dimensional scattering from the conservation of particle current.[9,10] The problem considered here of scattering from an impenetrable, infinitely long magnetic flux line with the return flux uniformly distributed on the surface of a cylinder of radius a is shown to satisfy the optical theorem even in the limit where $a \to \infty$. We also show that the total angular impulse due to the Lorentz force is zero, so that angular momentum is conserved.

In Sec. II the scattering of an electron is considered from an impenetrable infinitely long magnetic flux line, with the return flux distributed uniformly on the surface of a cylinder. The optical theorem for two-dimensional scattering is proved and applied in Sec. III. In Sec. IV angular momentum conservation is considered. Finally, the conclusion is given in Sec. V.

II. SCATTERING THEORY

Two-dimensional scattering theory[10] is applied here to the scattering of an electron by an impenetrable, infinitely long magnetic flux line with the return magnetic flux uniformly distributed on the surface of a cylinder of radius a. The source of the electrons and the detector are both outside a. When an electron encounters the magnetic field of the return flux, a Lorentz force acts on the electron which scatters it. In the limit as $a \to \infty$, the Aharonov-Bohm scattering cross section is obtained.

A vector potential for the solenoid of zero radius with magnetic flux Φ oriented along the z axis and return flux uniformly distributed on the surface of a cylinder at radius a is

$$\mathbf{A} = \begin{cases} (\Phi/2\pi r)\hat{\theta} & \text{for } r < a \\ 0 & \text{for } r > a \ . \end{cases} \tag{2.1}$$

The time-independent Schrödinger equation for an electron of mass m and charge $q < 0$ with a vector potential $\mathbf{A}$ is

$$(2m)^{-1}(\mathbf{p}-q\,\mathbf{A}/c)^2\psi=E\psi\ , \tag{2.2}$$

where the energy is $E=\hbar^2k^2/2m$ in terms of the wave vector $\mathbf{k}$. When Eq. (2.1) is used in Eq. (2.2), the Schrödinger equation can be written for $r<a$ as

$$\frac{\partial^2\psi_<}{\partial r^2}+r^{-1}\frac{\partial\psi_<}{\partial r}+r^{-2}\left[\frac{\partial}{\partial\theta}+i\alpha\right]^2\psi_<+k^2\psi_<=0 \tag{2.3}$$

and for $r>a$ as

$$\frac{\partial^2\psi_>}{\partial r^2}+r^{-1}\frac{\partial\psi_>}{\partial r}+r^{-2}\frac{\partial^2\psi_>}{\partial\theta^2}+k^2\psi_>=0\ , \tag{2.4}$$

where $\alpha=-q\Phi/2\pi\hbar c$ is dimensionless. The boundary conditions for the problem are as follows.

(1) The wave function vanishes at the origin,

$$\psi_<(0,\theta)=0\ , \tag{2.5}$$

because the solenoid is impenetrable.

(2) The asymptotic form of the wave function as $r\to\infty$ is an incoming plane wave and an outgoing cylindrical wave,

$$\psi_>(r,\theta)\to\exp(ikx)+f(\theta)r^{-1/2}\exp(ikr)\ , \tag{2.6}$$

where $x=r\cos\theta$ and $f(\theta)$ is the scattering amplitude.

(3) The wave function is continuous at $r=a$,

$$\psi_>(a,\theta)=\psi_<(a,\theta)\ . \tag{2.7}$$

(4) The derivative of the wave function is continuous at $r=a$,

$$\left.\frac{\partial\psi_>(r,\theta)}{\partial r}\right|_{r=a}=\left.\frac{\partial\psi_<(r,\theta)}{\partial r}\right|_{r=a}\ . \tag{2.8}$$

Boundary conditions (3) and (4) can be obtained by integrating Eq. (2.2) from $a-\varepsilon$ to $a+\varepsilon$ $(\varepsilon>0)$ and taking the limit as $\varepsilon\to 0+$. These are the standard boundary conditions of quantum-mechanical scattering theory.

The interior solution of Eq. (2.3) which satisfies Eq. (2.5) is

$$\psi_<(r,\theta)=\sum_{n=-\infty}^{\infty}a_n\exp(in\theta)J_{|n+\alpha|}(kr)\ , \tag{2.9}$$

where $J_{|n+\alpha|}$ is the Bessel function of the first kind of order $|n+\alpha|$ and a_n are arbitrary constants. The exterior solution of Eq. (2.4) which satisfies Eq. (2.6) is

$$\psi_>(r,\theta)=\sum_{n=-\infty}^{\infty}\exp(in\theta)[i^nJ_n(kr)+b_nH_n(kr)]\ , \tag{2.10}$$

where $H_n\equiv H_n^{(1)}$ is the Hankel function of the first kind of order n. The coefficient of the Bessel function of the first kind in Eq. (2.10) is chosen because[11]

$$\sum_{n=-\infty}^{\infty}i^n\exp(in\theta)J_n(kr)=\exp(ikr\cos\theta)\ , \tag{2.11}$$

which is the first term on the right-hand side of Eq. (2.6). The Hankel function $H_n(kr)$ of the first kind has the asymptotic property that[11]

$$H_n(kr)\to(2/\pi kr)^{1/2}\exp[i(kr-\tfrac{1}{2}\pi n-\pi/4)]\ , \tag{2.12}$$

for $kr\gg 1$, which gives the outgoing cylindrical wave in Eq. (2.6).

The coefficients a_n and b_n are determined from the boundary conditions in Eqs. (2.7) and (2.8) to be

$$a_n(k)=i^n\Delta_n^{-1}[J_n(ka)H_n'(ka)-J_n'(ka)H_n(ka)] \tag{2.13}$$

and

$$b_n(k)=-i^n\Delta_n^{-1}[J_{|n+\alpha|}(ka)J_n'(ka)-J'_{|n+\alpha|}(ka)J_n(ka)]\ , \tag{2.14}$$

where the denominator is

$$\Delta_n(k)=J_{|n+\alpha|}(ka)H_n'(ka)-J'_{|n+\alpha|}(ka)H_n(ka)\ . \tag{2.15}$$

A prime on a function denotes a derivative with respect to its argument. Since $H_n=J_n+iY_n$, where Y_n is the Bessel function of the second kind of order n, Eq. (2.14) may be written as

$$b_n(k)=-i^n(1+i\beta_n)^{-1}\ , \tag{2.16}$$

where β_n is real and is defined as

$$\beta_n=[J_{|n+\alpha|}(ka)Y_n'(ka)-J'_{|n+\alpha|}(ka)Y_n(ka)]\times[J_{|n+\alpha|}(ka)J_n'(ka)-J'_{|n+\alpha|}(ka)J_n(ka)]^{-1}\ . \tag{2.17}$$

The form of Eq. (2.16) is used in verifying the optical theorem.

Because of the asymptotic form of the Hankel function of the first kind in Eq. (2.12), the scattering amplitude in Eq. (2.6) is

$$f(\theta)=(2/\pi ik)^{1/2}\sum_{n=-\infty}^{\infty}(-i)^nb_n\exp(in\theta)\ . \tag{2.18}$$

Consequently, the differential scattering cross section is

$$\sigma(\theta)=|f(\theta)|^2=(2/\pi k)\sum_{n,n'=-\infty}^{\infty}(-1)^{n_i n+n'}b_{n'}^{*}b_n\times\exp[i(n-n')\theta]\ . \tag{2.19}$$

The total scattering cross section is

$$\sigma_T=\int_{-\pi}^{\pi}d\theta\,|f(\theta)|^2=(4/k)\sum_{n=-\infty}^{\infty}|b_n|^2=(4/k)\sum_{n=-\infty}^{\infty}(1+\beta_n^2)^{-1}\ , \tag{2.20}$$

from Eq. (2.16). The total cross section depends on the radius of the return flux a through b_n in Eq. (2.14). The scattering is due to the return flux at $r=a$, since a Lorentz force acts on the electron there. The source of the electrons and the detector is always outside the return flux.

In the limit that $a \rightarrow \infty$, Eq. (2.19) for the differential scattering cross section is equal to the AB cross section.[1] In our case, however, the magnetic flux returns at infinity, but *inside* the source and detector. AB assumed implicitly that the magnetic flux returned at infinity *outside* the source and detector. The asymptotic form of the Bessel function of the first kind is[11]

$$J_n(kr) \rightarrow (2/\pi kr)^{1/2} \cos(kr - \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi) , \tag{2.21}$$

for $kr \gg 1$. When the asymptotic forms of the Bessel and Hankel functions in Eqs. (2.21) and (2.12), respectively, and their derivatives[11] are used in Eqs. (2.14) and (2.15), the result for b_n when $ka \gg 1$ is

$$b_n = \begin{cases} i^{n-1}\sin(\pi\alpha/2)\exp(-i\pi\alpha/2) & \text{for } n \geq -[\alpha] \\ -i^{n-1}\sin(\pi\alpha/2)\exp(i\pi\alpha/2) & \text{for } n < -[\alpha] , \end{cases} \tag{2.22}$$

where $[\alpha]$ is the largest integer less than or equal to α. When this expression is used in Eq. (2.18) for the scattering amplitude, we obtain

$$f(\theta) = -i(2/\pi ik)^{1/2}\sin(\pi\alpha/2)\exp(-i[\alpha]\theta) \times \left[\sum_{n=0}^{\infty} \exp(-i\pi\alpha/2)\exp(in\theta) - \sum_{n=1}^{\infty} \exp(i\pi\alpha/2)\exp(-in\theta) \right] . \tag{2.23}$$

The summation can be performed using the geometrical series,[12,13] which gives the scattering amplitude

$$f(\theta) = (2\pi ik)^{-1/2}\sin(\pi\alpha)\exp(-i\theta/2) \times [\sin(\theta/2)]^{-1}\exp(-i[\alpha]\theta) + (2\pi ik)^{-1/2}2\pi(\cos\pi\alpha - 1)\delta(\theta) , \tag{2.24}$$

which was also obtained by Nagel[12] using another approach. It is understood that the principal value of the first term in Eq. (2.24) is taken. From Eq. (2.19) the differential scattering cross section is

$$\sigma(\theta) = (2\pi k)^{-1}\sin^2(\pi\alpha)[\sin(\theta/2)]^{-2} \quad (\theta \neq 0) , \tag{2.25}$$

which is equal to the AB scattering cross section[1] with the incoming particle coming in from $x = -\infty$. (Incidentally, AB[1] considered their particle coming in from $x = +\infty$.) Therefore, the same scattering cross section as obtained by AB is also obtained in this problem where the scattering is due to the Lorentz force on the electron from the magnetic field of the return flux.

When Eq. (2.22) for b_n in the asymptotic region $ka \gg 1$ is used in Eq. (2.20) for the total scattering cross section, the result is

$$\sigma_T = (4/k)\sin^2(\pi\alpha/2) \sum_{n=-\infty}^{\infty} 1 = \infty . \tag{2.26}$$

It is natural that the total cross section should be infinite for this problem, since in the limit as $a \rightarrow \infty$ the electron is scattered by a cylinder of infinite radius.

III. OPTICAL THEOREM

The optical theorem for scattering theory can be derived in two equivalent ways. Henneberger[8] derived it for two-dimensional scattering from the unitarity of the S matrix.[10] Here we prove it from the conservation of probability or particle number.[9,10] We show that the optical theorem is satisfied for our problem of scattering from an impenetrable, infinitely long magnetic flux line with the return flux uniformly distributed on the surface of a cylinder of radius a.

In two dimensions the particle (surface) current density is

$$J_\mu = m^{-1}\mathrm{Re}\psi^*(p_\mu - qA_\mu/c)\psi , \tag{3.1}$$

where $\mu = 1,2$ are the components in the plane. The wave function ψ is normalized over a unit area of the plane. Using the Schrödinger equation, we can derive the equation of continuity

$$\frac{\partial\rho}{\partial t} + \partial_\mu J_\mu = 0 , \tag{3.2}$$

where $\rho = \psi^*\psi$ is the probability per unit area and the repeated Greek indices are summed from to 1 to 2. The integral of Eq. (3.2) over the surface of a circle of radius r as $r \rightarrow \infty$ gives

$$\lim_{r\rightarrow\infty} \int_{-\pi}^{\pi} d\theta\, rJ_r = 0 , \tag{3.3}$$

since probability is conserved. If Eq. (2.6) is substituted into Eq. (3.1), the asymptotic $(r \rightarrow \infty)$ radial surface current density is

$$J_r = (\hbar k/m)\{\cos\theta + |f(\theta)|^2 r^{-1} + \mathrm{Re}\, r^{-1/2} f(\theta) \times (1+\cos\theta)\exp[ikr(1-\cos\theta)]\} , \tag{3.4}$$

where normalization in a unit area is assumed. The first term on the right-hand side of Eq. (3.4) is the current density due to the incoming plane wave, the second term is the current density due to the outgoing cylindrical wave, and third term is due to interference between the plane wave and the outgoing cylindrical wave. When Eq. (3.4) is substituted into Eq. (3.3), the result is

$$\sigma_T = -\lim_{r\rightarrow\infty} \mathrm{Re} \int_{-\pi}^{\pi} d\theta\, r^{1/2} f(\theta)(1+\cos\theta) \times \exp[ikr(1-\cos\theta)] , \tag{3.5}$$

where the total cross section σ_T is given by Eq. (2.20).

Because of the exponential in Eq. (3.5), the main contribution to the integral occurs near $\theta = 0$. If it can be assumed that $f(\theta)$ is slowly varying when θ is near zero, then the scattering amplitude can be evaluated at $\theta = 0$ and taken outside the integral. In this case, Eq. (3.5) becomes

$$\sigma_T = -\lim_{r\rightarrow\infty} \mathrm{Re} f(0) 4r^{1/2} \int_{-1}^{1} d\eta (1-\eta^2)^{1/2} \times \exp(i2kr\eta^2) , \tag{3.6}$$

where $\eta = \sin(\theta/2)$. The trigonometric identities $2\sin^2(\theta/2) = 1-\cos\theta$ and $2\cos^2(\theta/2) = 1+\cos\theta$ have been used in going from Eq. (3.5) to Eq. (3.6). If we make a change of variable to $\xi=(2kr)^{1/2}\eta$ and then take the limit in Eq. (3.6), the result is

$$\sigma_T=-4(2k)^{-1/2}\mathrm{Re}f(0)\int_{-\infty}^{\infty}d\xi\exp(i\xi^2)\ . \tag{3.7}$$

The value of the integral in Eq. (3.7) is $(\pi i)^{1/2}$. When this value is used and the resulting expression is simplified, we obtain

$$\sigma_T=2(\pi/k)^{1/2}[\mathrm{Im}f(0)-\mathrm{Re}f(0)]\ , \tag{3.8}$$

which is the optical theorem for two-dimensional scattering.[10] Equation (3.8) was first obtained by Henneberger[8] using the unitarity of the S matrix. The method used here is essentially the method of stationary phase applied to Eq. (3.5).

Equation (3.8) should be valid for scattering from the return flux for $a<\infty$. In this case both the source and the detector are outside the return flux, and the scattering amplitude should be slowly varying near $\theta=0$. When Eq. (2.18) is substituted into Eq. (3.8), the result is

$$\sigma_T=-(4/k)\sum_{n=-\infty}^{\infty}\mathrm{Re}(-i)^n b_n\ . \tag{3.9}$$

When Eq. (2.16) is substituted into Eq. (3.9), the total cross section is

$$\sigma_T=(4/k)\sum_{n=-\infty}^{\infty}(1+\beta_n^2)^{-1}\ , \tag{3.10}$$

which agrees with Eq. (2.20). The optical theorem is therefore satisfied for scattering from the return flux.

When $ka\gg1$, b_n is given by Eq. (2.22). Equation (3.9) is then equal to Eq. (2.26) for the total cross section. Therefore, the optical theorem is still satisfied in this case.

IV. ANGULAR MOMENTUM

The time rate of change of the expectation value of the kinetic angular momentum is equal to the expectation value of the quantum-mechanical torque operator. By integrating this equation from time $-\infty$ to ∞, the change in the angular momentum is equal to the angular impulse. Even though there is a Lorentz force acting on the electron in our problem, we show that the angular impulse is zero and consequently angular momentum is conserved.

The time rate of change of the expectation value of the kinetic angular momentum operator $\mathbf{L}$ satisfies an Ehrenfest theorem[14]

$$\frac{d\langle\psi\,|\,\mathbf{L}\psi\rangle}{dt}=\langle\psi\,|\,\mathbf{N}\psi\rangle\ . \tag{4.1}$$

The kinetic angular momentum operator is

$$\mathbf{L}=\mathbf{r}\times m\mathbf{v}\ , \tag{4.2}$$

where the velocity operator $\mathbf{v}$ is

$$\mathbf{v}=m^{-1}(\mathbf{p}-q\,\mathbf{A}/c)\ . \tag{4.3}$$

The torque operator $\mathbf{N}$ is

$$\mathbf{N}=\tfrac{1}{2}(\mathbf{r}\times\mathbf{F}-\mathbf{F}\times\mathbf{r})\ , \tag{4.4}$$

where the Lorentz force operator $\mathbf{F}$ is

$$\mathbf{F}=(q/2c)(\mathbf{v}\times\mathbf{B}-\mathbf{B}\times\mathbf{v})\ . \tag{4.5}$$

Equation (4.1) can be proved using only the Schrödinger equation,[14] but we shall explicitly verify it here for scattering from an impenetrable, infinitely long magnetic flux line when the return flux is uniformly distributed on the surface of a cylinder.

When the return flux is taken to be uniformly distributed on the surface of a cylinder of radius a, a torque acts on the electron at $r=a$. If the magnetic induction field is in the z direction $\mathbf{B}=(0,0,B_3)$, then Eq. (4.4) for the z component of the torque N_3 reduces to

$$N_3=-(q/4c)\{B_3,\{x_\mu,v_\mu\}\}\ , \tag{4.6}$$

where $\{,\}$ denotes the anticommutator and the repeated Greek index μ is summed from 1 to 2.

The expectation value of Eq. (4.6) gives

$$\langle\psi\,|\,N_3\psi\rangle=-(q/2c)\mathrm{Re}\langle B_3\psi\,|\,\{x_\mu,v_\mu\}\psi\rangle\ . \tag{4.7}$$

The magnetic field is zero everywhere except on the z axis, from which the electron is excluded, and on the surface of a cylinder of radius a. Since the magnetic flux on the z axis is Φ, the magnetic induction is

$$B_3=\Phi\delta(x)\delta(y)-(2\pi a)^{-1}\Phi\delta(r-a)\ . \tag{4.8}$$

When Eq. (4.8) is substituted into Eq. (4.7), the integral over the radial coordinate can be performed, which gives

$$\langle\psi\,|\,N_3\psi\rangle=(q\Phi/4\pi c)\mathrm{Re}\int_{-\pi}^{\pi}d\theta\,\psi^*\{v_\mu,x_\mu\}\psi\,|_{r=a}\ . \tag{4.9}$$

The wave function ψ in Eq. (4.9) is a wave packet

$$\psi(r,\theta,t)=\int_0^{\infty}dk\,\phi(k)\psi_k(r,\theta)\exp(-i\omega t)\ , \tag{4.10}$$

where the angular frequency is $\omega=\hbar k^2/2m$ and $\phi(k)$ is the probability amplitude in momentum space. The wave function $\psi_k(r,\theta)$ is

$$\psi_k(r,\theta)=\sum_{n=-\infty}^{\infty}\exp(in\theta)R_n(kr)\ , \tag{4.11}$$

where the radial function $R_n(kr)$ is

$$R_n(kr)=\begin{cases}a_nJ_{|n+\alpha|}(kr) & \text{for } r<a\\ i^nJ_n(kr)+b_nH_n(kr) & \text{for } r>a\ ,\end{cases} \tag{4.12}$$

from Eqs. (2.9) and (2.10). Equation (4.11) is thus a solution to the Schrödinger equation in Eqs. (2.3) and (2.4). When Eq. (4.10) is substituted into Eq. (4.9), the expectation value of the torque operator is

$$\langle\psi\,|\,N_3\psi\rangle=-(q\Phi\hbar/mc)\mathrm{Re}\,i\int_0^{\infty}dk'\int_0^{\infty}dk\,\phi^*(k')\phi(k)\sum_{n=-\infty}^{\infty}a_n^*(k')a_n(k)kaJ_{|n+\alpha|}(k'a)J'_{|n+\alpha|}(ka)\times\exp[-i(\omega-\omega')t]\ , \tag{4.13}$$

where $\omega' = \hbar k'^2/2m$. The expectation value of the torque operator is time dependent and involves only the value of the radial coordinate at a.

The expectation value of the z component of the kinetic angular momentum can also be evaluated in order to verify Eq. (4.1). The expectation value of the z component of the kinetic angular momentum L_3 in Eq. (4.2) is

$$\langle \psi | L_3 \psi \rangle = \left\langle \psi \left| \left[-i\hbar \frac{\partial}{\partial\theta} + \hbar\alpha\Theta(a-r) \right] \psi \right\rangle , \quad (4.14)$$

from Eq. (2.1). The Heaviside function is $\Theta(x)=1$ for $x>0$ and 0 for $x<0$. The expectation value of the canonical angular momentum $-i\hbar\partial/\partial\theta$ in Eq. (4.14) is

$$\left\langle \psi \left| \left[-i\hbar \frac{\partial}{\partial\theta} \right] \psi \right\rangle = 2\pi\hbar \int_0^\infty dk\, k^{-1} |\phi(k)|^2 \sum_{n=-\infty}^{\infty} n s_n(k) , \quad (4.15)$$

which is time independent. The wave function ψ in Eqs. (4.14) and (4.15) is given in Eqs. (4.10)–(4.12). To obtain Eq. (4.15) we used the integral (see Appendix A)

$$\int_0^\infty dr\, r R_n^*(k'r) R_n(kr) = k^{-1} s_n(k) \delta(k-k') , \quad (4.16)$$

where

$$s_n(k) = \left| \int_0^\infty dx\, R_n(x) \right|^2 . \quad (4.17)$$

The expectation of $\hbar\alpha\Theta(a-r)$ in Eq. (4.14) is

$$\langle \psi | \hbar\alpha\Theta(a-r)\psi \rangle = 2\pi\hbar\alpha \,\mathrm{Re} \int_0^\infty dk' \int_0^\infty dk\, \phi^*(k')\phi(k) \sum_{n=-\infty}^{\infty} a_n^*(k') a_n(k) \times \int_0^a dr\, r J_{|n+\alpha|}(k'r) J_{|n+\alpha|}(kr) \exp[-i(\omega-\omega')t] \quad (4.18)$$

when Eqs. (4.10)–(4.12) are used. Equation (4.18) is time dependent. The expectation value of the kinetic angular momentum is obtained by adding Eqs. (4.15) and (4.18).

The time rate of change of the expectation value of the total kinetic angular momentum in Eq. (4.14) is the time rate of change of Eq. (4.18), which gives

$$\frac{d\langle \psi | L_3 \psi \rangle}{dt} = -2\pi\hbar\alpha \,\mathrm{Re}\, i \int_0^\infty dk' \int_0^\infty dk\, \phi^*(k')\phi(k) \sum_{n=-\infty}^{\infty} a_n^*(k') a_n(k) \times (\omega-\omega') \int_0^a dr\, r J_{|n+\alpha|}(k'r) J_{|n+\alpha|}(kr) \exp[-i(\omega-\omega')t] . \quad (4.19)$$

In Appendix B we evaluate the integral in Eq. (4.19), and show that

$$(k^2-k'^2) \int_0^a dr\, r J_{|n+\alpha|}(k'r) J_{|n+\alpha|}(kr) = k'a J'_{|n+\alpha|}(k'a) J_{|n+\alpha|}(ka) - ka J_{|n+\alpha|}(k'a) J'_{|n+\alpha|}(ka) . \quad (4.20)$$

Since $\omega = \hbar k^2/2m$, Eq. (4.19) becomes

$$\frac{d\langle \psi | L_3 \psi \rangle}{dt} = -\left[\frac{q\Phi\hbar}{mc} \right] \mathrm{Re}\, i \int_0^\infty dk' \int_0^\infty dk\, \phi^*(k')\phi(k) \sum_{n=-\infty}^{\infty} a_n^*(k') a_n(k) ka J_{|n+\alpha|}(k'a) J'_{|n+\alpha|}(ka) \exp[-i(\omega-\omega')t] . \quad (4.21)$$

If Eq. (4.21) is compared with Eq. (4.13) we see that Ehrenfest's theorem,

$$\frac{d\langle \psi | L_3 \psi \rangle}{dt} = \langle \psi | N_3 \psi \rangle , \quad (4.22)$$

is indeed satisfied.

If Eq. (4.22) is integrated from $-\infty$ to $+\infty$ we obtain

$$\langle \psi | L_3 \psi \rangle |_{t=+\infty} - \langle \psi | L_3 \psi \rangle |_{t=-\infty} = \int_{-\infty}^{\infty} dt \langle \psi | N_3 \psi \rangle . \quad (4.23)$$

The angular impulse on the right-hand side of Eq. (4.23) is obtained by integrating Eq. (4.13), which gives

$$\int_{-\infty}^{\infty} dt\langle\psi|N_3\psi\rangle=-(2\pi aq\Phi/c)\mathrm{Re}\, i\int_0^{\infty} dk\,|\phi(k)|^2\sum_{n=-\infty}^{\infty}|a_n(k)|^2 J_{|n+\alpha|}(ka)J'_{|n+\alpha|}(ka)=0\;. \tag{4.24}$$

In obtaining Eq. (4.24) we have made use of the integral representation of the δ function,

$$(2\pi)^{-1}\int_{-\infty}^{\infty} dt\,\exp[-i(\omega-\omega')t]=\delta(\omega-\omega')=(m/\hbar k)\delta(k-k')\;. \tag{4.25}$$

Since the right-hand side of Eq. (4.23) vanishes, angular momentum is conserved.

In the limit as $a\to\infty$, Eq. (4.22) remains satisfied. From Eq. (4.21) the expectation value of the torque operator in Eq. (4.13) can be written as Eq. (4.19). Equation (4.19) in the limit as $a\to\infty$ is

$$\begin{aligned}\langle\psi|N_3\psi\rangle\to-2\pi\hbar\alpha\,\mathrm{Re}\, i\int_0^{\infty} dk'\int_0^{\infty} dk\,\phi^*(k')\phi(k)\sum_{n=-\infty}^{\infty}a_n^*(k')a_n(k)(\omega-\omega')\int_0^{\infty} dr\, rJ_{|n+\alpha|}(k'r)J_{|n+\alpha|}(kr)\\ \times\exp[-i(\omega-\omega')t]\\ =0\quad\text{as } a\to\infty\;.\end{aligned} \tag{4.26}$$

To obtain Eq. (4.26), the closure equation for Bessel functions,[15]

$$\int_0^{\infty} dr\, rJ_\nu(k'r)J_\nu(kr)=k^{-1}\delta(k-k')\;, \tag{4.27}$$

is used. In the limit as $a\to\infty$, the expectation value of the kinetic angular momentum operator in Eq. (4.14) becomes

$$\langle\psi|L_3\psi\rangle=\hbar\alpha\ \text{as } a\to\infty\;, \tag{4.28}$$

which is constant. The wave function ψ is assumed normalized, so Eq. (4.18) becomes $\hbar\alpha$. Equation (4.15) approaches zero in this limit, since

$$\sum_{n=-\infty}^{\infty} ns_n(k)\to\sum_{n=-\infty}^{\infty} n=0\quad\text{as } a\to\infty\;. \tag{4.29}$$

In this limit, Eq. (4.17) for $s_n(k)$ becomes

$$s_n(k)\to\left|a_n\int_0^{\infty} dx\, J_{|n+\alpha|}(x)\right|^2\to 1\quad\text{as } a\to\infty \tag{4.30}$$

because the asymptotic form of a_n in Eq. (2.13) gives

$$a_n\to(-1)^n(-i)^{|n+\alpha|}\quad\text{as } a\to\infty\;, \tag{4.31}$$

and the integral in Eq. (4.30) is unity. Therefore, in the limit as $a\to\infty$, Eq. (4.22) reduces to an identity 0=0.

V. CONCLUSION

The problem considered here is that of an electron scattered by an impenetrable, infinitely long magnetic flux line with the return flux uniformly distributed on the surface of a cylinder of radius a. This problem is different from the one considered by AB.[1,16] Aharonov and Bohm[1] consider an electron scattered by an impenetrable, infinitely long magnetic flux line without mention of the return flux. Implicitly they assume that the return flux is *outside* of the region defined by the source and the detector. In our problem the return flux is *inside* the region defined by the source and the detector.

In our problem the scattering is due to the Lorentz force of the magnetic field of the return flux.[17] In the AB problem the scattering is attributed to the vector potential in a field-free region.[18] Remarkably, the differential scattering cross section in our problem is equal to the differential scattering cross section for the AB problem when $a\to\infty$ (or $ka\gg 1$, where k is the wave number). Even when $a\to\infty$, the return flux in our problem is always inside the region defined by the source and detector, which is taken at the start of the calculation to be at infinity. The scattering from an infinitely long cylinder of finite radius has also been studied.[19]

An experiment to measure the differential cross section of an electron scattered by an impenetrable, long magnetic flux line would be of interest. The magnetic flux could be made to return at a given finite, but large, radius a from the magnetic flux line by using a material of high permeability to guide the flux. If the analysis of AB[1] is correct, it should make no difference if the source of the electrons is inside or outside of the return flux. In both cases the differential scattering cross section of Eq. (2.25) should be observed as long as $ka\gg 1$, where k is the wave number. The mechanism of scattering in the two cases is quite different, however. AB scattering is due to the vector potential in a field-free region, whereas in our problem scattering is due to the Lorentz force of the magnetic field of the return flux. Such an experiment would be a test of AB scattering,[20] which is a different physical phenomenon from AB interference.[21]

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APPENDIX A: PROOF OF EQ. (4.16)

The function $R_n(kr)$ in Eq. (4.12) is a solution for $r<a$ to

$$D^2R_n(kr)+r^{-1}DR_n(kr)+[k^2-r^{-2}(n+\alpha)^2]R_n(kr)=0\ , \tag{A1}$$

where $D=d/dr$, and for $r>a$ it is a solution to

$$D^2R_n(kr)+r^{-1}DR_n(kr)+(k^2-r^{-2}n^2)R_n(kr)=0\ . \tag{A2}$$

Equations (A1) and (A2) can be multiplied by $R_n^*(k'r)$. If the complex conjugate term with k and k' interchanged is subtracted, then we get

$$\begin{aligned}(k^2-k'^2)&R_n^*(k'r)R_n(kr)+R_n^*(k'r)D^2R_n(kr)\\&-R_n(kr)D^2R_n^*(k'r)+r^{-1}R_n^*(k'r)DR_n(kr)\\&-r^{-1}R_n(kr)DR_n^*(k'r)=0\ .\end{aligned} \tag{A3}$$

If Eq. (A3) is multiplied by $re^{-\varepsilon r}$ where $\varepsilon>0$, and integrated from zero to infinity, we obtain

$$\begin{aligned}(k^2-k'^2)&\int_0^\infty dr\, re^{-\varepsilon r}R_n^*(k'r)R_n(kr)\\&=-\int_0^\infty dr\, D[re^{-\varepsilon r}R_n^*(k'r)DR_n(kr)-(k\leftrightarrow k')^*]\\&\quad+O(\varepsilon)\ ,\end{aligned} \tag{A4}$$

after integration by parts. The integral in Eq. (A4) can be performed, which gives

$$\begin{aligned}(k^2-k'^2)&\int_0^\infty dr\, re^{-\varepsilon r}R_n^*(k'r)R_n(kr)\\&=\lim_{r\to\infty}[k're^{-\varepsilon r}R_n(kr)R_n'^*(k'r)-(k\leftrightarrow k')^*]+O(\varepsilon)\\&=O(\varepsilon)\ .\end{aligned} \tag{A5}$$

The limit of $re^{-\varepsilon r}R_n(kr)R_n'^*(k'r)$ as $r\to\infty$ gives terms like $\sin(kr)\exp(-\varepsilon r)$ which vanishes as $r\to\infty$. In the limit as $\varepsilon\to 0+$, Eq. (A5) therefore becomes

$$(k^2-k'^2)\int_0^\infty dr\, rR_n^*(k'r)R_n(kr)=0\ . \tag{A6}$$

The integral in Eq. (A6) is defined as the limit of the integral in Eq. (A5) as $\varepsilon\to 0+$ for convergence reasons. Equation (A6) has the solution

$$\int_0^\infty dr\, rR_n^*(k'r)R_n(kr)=k^{-1}s_n(k)\delta(k-k')\ , \tag{A7}$$

where $s_n(k)$ has to be determined. When Eq. (A7) is integrated over k', and assuming that the integrals can be interchanged, we obtain

$$\int_0^\infty dr\int_0^\infty dx\, R_n^*(x)R_n(kr)=k^{-1}s_n(k)\ , \tag{A8}$$

where $x=k'r$. Multiplying Eq. (A8) by k gives Eq. (4.17) for $s_n(k)$.

APPENDIX B: PROOF OF EQ. (4.20)

In order to derive Eq. (4.20), consider the differential equation satisfied by $J_\nu(kr)$,

$$D^2J_\nu(kr)+r^{-1}DJ_\nu(kr)+(k^2-r^{-2}\nu^2)J_\nu(kr)=0\ , \tag{B1}$$

where $D=d/dr$. If Eq. (B1) is multiplied by $J_\nu(k'r)$ and the equation with k and k' interchanged is subtracted, we obtain

$$\begin{aligned}(k^2-k'^2)&J_\nu(k'r)J_\nu(kr)\\&+[J_\nu(k'r)D^2J_\nu(kr)-J_\nu(kr)D^2J_\nu(k'r)\\&\quad+r^{-1}J_\nu(k'r)DJ_\nu(kr)\\&\quad-r^{-1}J_\nu(kr)DJ_\nu(k'r)]=0\ .\end{aligned} \tag{B2}$$

If Eq. (B2) is multiplied by r and integrated from zero to a, we obtain

$$\begin{aligned}(k^2-k'^2)&\int_0^a dr\, rJ_\nu(k'r)J_\nu(kr)\\&=-\int_0^a dr\, D[krJ_\nu(k'r)J_\nu'(kr)-(k\leftrightarrow k')]\ ,\end{aligned} \tag{B3}$$

after integration by parts. When the integral is performed, the lower limit gives zero. The upper limit gives the result of Eq. (4.20) for $\nu=|n+\alpha|$.

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