PHYSICAL REVIEW A VOLUME 36, NUMBER 2 JULY 15, 1987

Phase-space–Lagrangian action principle and the generalized K-χ theorem

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(Received 13 February 1987)

The covariant coupled equations for plasma dynamics and the Maxwell field are expressed as a phase-space–Lagrangian action principle. The linear interaction is transformed to the bilinear beat Hamiltonian by a gauge-invariant Lagrangian Lie transform. The result yields the generalized linear susceptibility directly.

The fundamental relation between nonlinear ponderomotive effects and linear plasma response has come to be known as the K-χ theorem. Ponderomotive effects are embodied in the oscillation-center Hamiltonian $K(z)$, introduced by Dewar.[1] It describes the (oscillation-averaged) orbit of a single particle in an oscillatory field, with the dominant effects quadratic in the wave amplitude. On the other hand, the linear susceptibility χ, a functional of the unperturbed particle distribution, describes the oscillatory current density linear in the wave amplitude.

This surprising relation between the quadratic Hamiltonian K_2 and the susceptibility χ, for the case of a single wave, was observed[2] some years ago, and then proved by Johnston and Kaufman[3] and by Cary and Kaufman.[4] The underlying reason for the relation, however, became clear only with the recent development[5] of phase-space–Lagrangian action principles, and the realization that the plasma action term quadratic in the wave amplitude $[-\int f(z)K_2(z)]$ was simultaneously both the oscillation-center energy and the plasma part of the wave Lagrangian. (To be sure, this fact was at least implicit in the earlier work of Dewar[1] and of Johnston and Kaufman.[3]) The importance of this realization, with its embodiment in the action principle, is best exemplified in the recent study,[6] by Similon and co-workers, of self-consistency in the stabilization of a confined plasma by the ponderomotive effects of an electromagnetic wave.

The ponderomotive beat Hamiltonian, introduced by Johnston[7] for the scattering of two waves, and now of especial use for the theory of free-electron lasers and beat-wave accelerators, is a conceptually simple extension of oscillation-center ideas to particles that resonate with the beat of two primary waves. Its utility led Grebogi[8] to the conjecture that it too is related to the linear susceptibility. This Rapid Communication presents a simple proof of that desired relation, and then illustrates it by an explicit calculation.

That calculation, in turn, is based on the use of a powerful new perturbation technique, invented by Littlejohn[9] for a system governed by a phase-space Lagrangian. Whereas the standard Hamiltonian perturbation theories (such as the Hamiltonian Lie transform[10]) preserve the Poisson structure, the new method enables one to perform the desired averaging directly on the Poisson (or symplectic) structure. As a result, the generator of the transform can be made gauge invariant and physically meaningful.

The calculation is outlined here for a field-free background. The extension to the case of a strong background field is conceptually easy, but of course algebraically complex, and will be published later. We begin with the definition of the two-point linear susceptibility tensor,[5,11] as a functional derivative:

$$\chi^{\mu\nu}(x_1,x_2)=\delta j^{\mu}(x_1)/\delta A_{\nu}(x_2) \ . \tag{1a}$$

It is convenient to use covariant notation, with metric $(1,1,1,-1)$ and $c=1$. Thus $x=(\mathbf{x},t)$, $j^{\mu}=(\mathbf{j},\rho)$, and $A_{\nu}=(\mathbf{A},-\phi)$. In terms of the Fourier transforms [e.g., $j^{\mu}(k)=\int d^4x\, j^{\mu}(x)\exp(-ik\cdot x)$, $k_{\mu}=(\mathbf{k},-\omega)$], the susceptibility reads

$$\chi^{\mu\nu}(k_1,k_2)=\delta j^{\mu}(k_1)/\delta A_{\nu}(k_2) \ . \tag{1b}$$

In Eq. (1), j is the linear current response to a perturbing electromagnetic potential A. Since j must be invariant under gauge transformations of A, the susceptibility must satisfy $\chi^{\mu\nu}(k,k')k'_{\nu}=0$. In addition, charge conservation $(\partial j^{\mu}/\partial x^{\mu}=0)$ implies that $k_{\mu}\chi^{\mu\nu}(k,k')=0$. Because each particle responds to the perturbing field independently, the current density is additive in the particles; hence the susceptibility is a linear functional of the unperturbed distribution.

The ponderomotive Hamiltonian $K_2(z)$ is (by definition) that term of the oscillaton-center Hamiltonian $K(z)$ which is quadratic in the perturbing potential. Its most general form is, thus,

$$K_2(z)=\frac{1}{2}\int d^4x_1\int d^4x_2 A_{\mu}(x_1)A_{\nu}(x_2)K^{\mu\nu}(z;x_1,x_2) \tag{2a}$$

$$=\frac{1}{2}\int d^4k_1\int d^4k_2 A^*_{\mu}(k_1)A_{\nu}(k_2)K^{\mu\nu}(z;k_1,k_2) \ . \tag{2b}$$

[We absorb $(2\pi)^{-4}$ into the element d^4k.]

We may interpret the integrand of (2b) as the contribution to the oscillation-center Hamiltonian of the nonlinear beat between two plane waves with wave vectors k_1 and k_2. The relation we wish to prove, the "generalized K-χ theorem," is

$$K^{\mu\nu}(z;k_1,k_2)=-\delta\chi^{\mu\nu}(k_1,k_2)/\delta f(z) \ . \tag{3}$$

That this relation has not heretofore been observed is probably due to the fact that almost all calculations of χ make specific assumptions on the form of f. However, the functional derivative in (3) requires the susceptibility for completely general f.

One restriction which we do make is that f includes only those particles which are nonresonant with the primary waves k_1, k_2. Hence, χ is Hermitian and K_2 is real. The proper treatment of primary resonances is a large subject in itself, with important contributions, especially by Dewar and co-workers.[12]

The system action S is a functional of the potential field $A(x)$ and the particle orbits in eight-dimensional phase space, denoted $z^\alpha(\tau)\equiv(r^\mu(\tau),\pi_\mu(\tau))$, with $\pi_\mu=(\boldsymbol{\pi},-h)$ the kinetic 4-momentum, h the kinetic energy, and τ an arbitrary orbit parameter. The single-particle action is $S_1=\int[\pi\cdot dr+eA(r)\cdot dr]$. We demand that $\delta S_1=0$ for variation of orbits constrained to the seven-dimensional mass surface $0=H(z)=(\pi^2+m^2)/2m$. With a Lagrangian multiplier $\lambda(\tau)$, we have

$$0=\delta\left[\int[\pi\cdot dr+eA(r)\cdot dr-\lambda(\tau)d\tau H(z)]\right] . \quad (4)$$

Variation with respect to $r(\tau)$ yields $d\pi_\mu=eF_{\mu\nu}dr^\nu$, where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$, while variation with respect to $\pi(\tau)$ yields $dr^\mu=\lambda(\tau)d\tau\pi^\mu/m$. The mass constraint determines $\lambda^2(\tau)=-[dr(\tau)/d\tau]^2$; if one wishes $d\tau$ to represent the particle's proper-time interval, then $\lambda=1$.

The total action is $S=\sum_i S_i+S_m$, where S_i is the action of particle i and $S_m=\int d^4x\,F_{\mu\nu}F^{\nu\mu}/16\pi$ is the Maxwell action. The interaction part of S can be expressed as $\int d^4x\,j^\mu(x)A_\mu(x)$, with $(u^\mu=dr^\mu/d\tau=\pi^\mu/m)$,

$$j^\mu(x)=\sum_i e\int d\tau\,u^\mu(\tau)\delta^4(x-r(\tau))$$
$$=e\int d^8z\,f(z)u^\mu\delta^4(x-r) .$$

We have introduced the particle phase-space density (for each species),

$$f(z)=\sum_i\int d\tau_i\,\delta^4(r-r_i(\tau_i))\delta^4(\pi-\pi_i(\tau_i)) . \quad (5)$$

Variation of S with respect to $A(x)$ yields the Maxwell equation $\partial_\mu F^{\mu\nu}(x)=-4\pi j^\nu(x)$.

The distribution f satisfies the Vlasov equation[5] $\{f,H\}=0$, in terms of the noncanonical Poisson bracket (PB),

$$\{g_1,g_2\}=J^{\alpha\beta}(z)(\partial g_1/\partial z^\alpha)(\partial g_2/\partial z^\beta) .$$

The Poisson tensor $J(z)$ is the reciprocal of the Lagrange tensor (or symplectic two form),[9] $\omega_{rr}=eF(r)$, $\omega_{\pi r}=-\omega_{r\pi}=I$, and $\omega_{\pi\pi}=0$. Thus, $J^{rr}=0$, $J^{r\pi}=-J^{\pi r}=I$, $J^{\pi\pi}=eF(r)$, and the PB is expressed in the physical variables r, π, F:

$$\{g_1,g_2\}=(\partial g_1/\partial r)\cdot(\partial g_2/\partial\pi)-(\partial g_1/\partial\pi)\cdot(\partial g_2/\partial r)$$
$$+e(\partial g_1/\partial\pi)\cdot F(r)\cdot(\partial g_2/\partial\pi) . \quad (6)$$

For a wave field $F_{\mu\nu}(x)$, oscillations occur in the PB (6) for nonresonant particles. Our aim is to transform away this term, linear in F, by a change of variables from particle coordinates z^α to oscillation-center (OC) coordinates $\bar{z}^\alpha(z;F)$. The linear oscillation induced by F is denoted $\tilde{z}\equiv z-\bar{z}$. We see that $\tilde{z}^\alpha(z)$ is a physically meaningful vector field; it is the generator of the Lagrangian Lie transform.

In terms of the Fourier transform $F_{\mu\nu}(k)$, the linearized particle equations yield the oscillation

$$\tilde{\pi}(\bar{z};F)=e\int d^4k\,F(k)\cdot\bar{u}\,(ik\cdot\bar{u})^{-1}\exp(ik\cdot\bar{r}) ,$$
$$\tilde{r}(\bar{z};F)=-(e/m)\int d^4k\,F(k)\cdot\bar{u}\,(k\cdot\bar{u})^{-2}\exp(ik\cdot\bar{r}) , \quad (7)$$

as a vector field on OC phase space. In order that (7) be well defined, we consider only that portion of phase space which has no primary resonances; i.e., $k\cdot\bar{u}\neq 0$ for all k, such that $F(k)\neq 0$.

Our aim is to make the PB canonical when expressed in OC variables:

$$\{g_1,g_2\}=(\partial g_1/\partial\bar{r})\cdot(\partial g_2/\partial\bar{\pi})-(\partial g_1/\partial\bar{\pi})\cdot(\partial g_2/\partial\bar{r}) . \quad (8)$$

Space limitations permit us only to quote the result of using the Lagrangian Lie transform, which is based on differential-geometric methods.[13] We obtain the OC Hamiltonian $K(\bar{z})=H(\bar{z})+K_2(\bar{z})$ with the ponderomotive term given by the virial[14]

$$K_2(\bar{z};F)=-\tfrac{1}{2}\tilde{r}(\bar{z};F)\cdot[eF(\bar{r})\cdot\bar{u}] . \quad (9)$$

The canonical Hamiltonian equations then yield

$$d\bar{\pi}/d\tau=-\partial K/\partial\bar{r}=-\partial K_2/\partial\bar{r} , \quad (10a)$$

for the ponderomotive force, and

$$d\bar{r}/d\tau=\partial K/\partial\bar{\pi}=\bar{\pi}/m+\partial K_2/\partial\bar{\pi} , \quad (10b)$$

a gauge-invariant expression for the canonical OC momentum $\bar{\pi}$, in terms to the OC velocity $d\bar{r}/d\tau$ and the quadratic term (related to wave momentum). [The mass constraint now reads $0=H(z)=K(\bar{z})$, i.e., the Hamiltonian transforms as a scalar under the coordinate change.]

The one-particle action is now, in the OC representation, including the Hamiltonian constraint,

$$S_1=\int[\bar{\pi}\cdot d\bar{r}-K(\bar{z};F)d\tau] . \quad (11)$$

The terms of $\sum_i S_i$ quadratic in F are, thus,

$$S^{(2)}=-\sum_i\int d\tau_i\,K_2(z_i(\tau_i);F)=-\int d^8z\,f(z)K_2(z;F) . \quad (12)$$

Noting from (9) and (7) that K_2 and $\bar{z}$ are manifestly gauge invariant, we proceed to express K_2 in the desired form (2b), using $F_{\mu\nu}(k)=i(k_\mu A_\nu-k_\nu A_\mu)$; we obtain

$$K^{\mu\nu}(z;k_1,k_2)=(e^2/m)[(k_1\cdot u)^{-2}+(k_2\cdot u)^{-2}]$$
$$\times\mathcal{H}^{\mu\nu}(u;k_1,k_2)\exp[i(k_2-k_1)\cdot r] , \quad (13)$$

with

$$\mathcal{H}^{\mu\nu}(u;k_1,k_2)=k_1\cdot u\,k_2^\mu u^\nu+k_2\cdot u\,u^\mu k_1^\nu$$
$$-k_1\cdot k_2 u^\mu u^\nu-k_1\cdot u\,k_2\cdot u\,g^{\mu\nu} .$$

On substituting (2b) into (12), we obtain

$$S^{(2)}=-\int d^8z\,f(z)\frac{1}{2}\int d^4k_1\int d^4k_2 A_\mu^*(k_1)\times A_\nu(k_2)K^{\mu\nu}(\bar{z};k_1,k_2)\ . \tag{14}$$

Recalling that $j^\mu(x)=\delta\sum_i S_i/\delta A_\mu(x)$, we see from (1a) that

$$\chi^{\mu\nu}(x_1,x_2)=\delta^2 S/\delta A_\mu(x_1)\delta A_\nu(x_2)\ ,$$

or (15)

$$\chi^{\mu\nu}(k_1,k_2)=\delta^2 S/\delta A_\mu^*(k_1)\delta A_\nu(k_2)\ .$$

Applying this to (14), we obtain

$$\chi^{\mu\nu}(k_1,k_2)=-\int d^8z\,f(z)K^{\mu\nu}(\bar{z};k_1,k_2)\ , \tag{16}$$

which is equivalent to the desired theorem (3).

If we set $k_2=k_1$ in (13) and (16), we obtain the covariant form of the single-wave K-χ theorem.[5,15]

In summary, we have indicated that a phase-space transformation from particle to oscillation-center coordinates, using the oscillation vector field as the generator of a Lagrangian Lie transform, converts the Poisson bracket to canonical but gauge-invariant form, and converts the linear interaction to a bilinear form, which simultaneously is the beat Hamiltonian, and expresses the generalized linear susceptibility.

It is a pleasure to acknowledge the essential advice and help of R. Littlejohn, H.-C. Ye, B. Boghosian, and P. Similon. This research was supported by the Offices of Fusion Energy and of Basic Energy Sciences of the U.S. Department of Energy under Contract No. DE-AC03-76SF00098. This research was begun at the Institute for Theoretical Physics, Santa Barbara, supported by the National Science Foundation under Grant No. PHY82-17853, and supplemented by funds from the National Aeronautics and Space Administration.

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