Explosive instabilities of reaction-diffusion equations

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Explicit solutions are obtained for evolution equations for explosively unstable situations. These solutions include the effects of diffusion with linear or quadratic density dependence of the diffusion coefficient. As a result of balance between the diffusion and nonlinear terms, explosive growth in time can occur with a preservation in shape of certain spatial distributions. The solutions are generalized to cases of two interacting populations.

In physics as well as in various other branches of science such as chemistry, biology, or ecology, the description of populations or densities by means of evolution equations, which are generally nonlinear, plays an important role. As far as physics is concerned, plasma physics, with its applications to fusion and astrophysics, as well as the fields of laser and semiconductor physics, exhibit a variety of phenomena which are governed by such evolution equations.

Recently so-called reaction-diffusion equations have attracted particular attention.[1-4] It is the purpose of the present report to consider such reaction-diffusion equations for situations which are explosively unstable, i.e., where instabilities tend to grow to infinite amplitudes in a limited period of time. In spite of the fact that no general solutions are so far available for such situations in the presence of diffusion, we shall demonstrate that particular solutions which are of practical significance can be obtained. They can indeed be astonishingly simple considering the fact that they correspond to highly nonlinear situations.

Consider the equation

$$\frac{\partial n}{\partial t}=\frac{\partial}{\partial x}\left[D\frac{\partial n}{\partial x}\right]-bn+cn^{p} , \tag{1}$$

where n describes a population density and b and c are constant coefficients, with $c>0$ for explosive cases, p a positive quantity, $p>1$, and D the diffusion coefficient which we assume takes the form $D=an^{\delta}$, where a is a constant coefficient and δ a positive quantity.

It is convenient to introduce new variables of space and time; accordingly,

$$(c/a)^{1/2}x\rightarrow x, \quad ct\rightarrow t .$$

To begin with we consider $b=0$.

The remaining equation can then be written

$$\frac{\partial n}{\partial t}=\frac{\partial}{\partial x}\left[n^{\delta}\frac{\partial n}{\partial x}\right]+n^{p} . \tag{2}$$

Since we are looking for explosive-type solutions, we assume the following similarity form of the solutions, namely,

$$n(x,t)=(t_0-t)^{\mu}\phi(\xi) , \tag{3}$$

$$\xi=x/(t_0-t)^{\nu} . \tag{4}$$

Introducing the expressions (3) and (4) into Eq. (2) and matching powers of (t_0-t) yields

$$\mu=-1/(p-1), \quad \nu=\tfrac{1}{2}(1+\mu\delta) , \tag{5}$$

and the following ordinary differential equation for ϕ:

$$\frac{d}{d\xi}\left[\phi^{\delta}\frac{d\phi}{d\xi}\right]=\frac{1}{p-1}\phi-\phi^{p}+\frac{1}{2}\left[1-\frac{\delta}{p-1}\right]\xi\frac{d\phi}{d\xi} . \tag{6}$$

The cases where $\delta=p-1$, e.g., $p=2$, $\delta=1$ and $p=3$, $\delta=2$, offer possibilities of convenient integration since the coefficient of the last term on the right-hand side of Eq. (6) vanishes. In these cases we have $\nu=0$ in relations (4) and (5), i.e., $\xi=x$.

The remaining two equations become

$$\frac{d}{dx}\left[\phi\frac{d\phi}{dx}\right]=\phi-\phi^{2} \quad (p=2,\ \delta=1,\ \mu=-1) \tag{7}$$

and

$$\frac{d}{dx}\left[\phi^{2}\frac{d\phi}{dx}\right]=\tfrac{1}{2}\phi-\phi^{3} \quad (p=3,\ \delta=2,\ \mu=-\tfrac{1}{2}) . \tag{8}$$

Equations (7) and (8) can be integrated after multiplying both sides of the equations by $\phi(d\phi/dx)$ and $\phi^2(d\phi/dx)$, respectively.

From Eq. (7) we then obtain

$$x-x_0=\pm\frac{\sqrt{6}}{2}\int_{\phi_0}^{\phi}\frac{\phi\, d\phi}{[\phi^3-\phi_c^3-\frac{3}{4}(\phi^4-\phi_c^4)]^{1/2}} , \tag{9}$$

where ϕ_c accounts for a constant of integration. Let us here choose $\phi_c=0$ or $\phi_c=\frac{4}{3}$, which yield identical results.

From relation (9) we then have for $x_0=0$, $\phi_0=\frac{4}{3}$,

$$x=\pm\frac{\sqrt{6}}{2}\int_{4/3}^{\phi}\frac{d\phi}{[\phi(1-\frac{3}{4}\phi)]^{1/2}} , \tag{10}$$

which can be integrated to yield

$$\phi=\tfrac{2}{3}[1+\cos(x/\sqrt{2})] . \qquad (11)$$

From relation (3) and expression (11) it follows that

$$n=\frac{1}{2}\frac{n_0}{1-\frac{3}{4}cn_0t}\left\{1+\cos\left[x\left[\frac{c}{2a}\right]^{1/2}\right]\right\}$$

$$(p=2,\quad \delta=1,\quad \mu=-1) , \qquad (12)$$

where by n_0 we denote $n(x=0, t=0)$.

From Eq. (8) we find correspondingly

$$x-x_0=\pm\tfrac{1}{2}\int_{\phi_0}^{\phi}\frac{\phi^2 d\phi}{[\phi^4-\phi_c^4-\frac{4}{3}(\phi^6-\phi_c^6)]^{1/2}} . \qquad (13)$$

Accordingly, by choosing $\phi_c=0$, or alternatively $\phi_c=\sqrt{3}/2$, we obtain

$$x=\pm\tfrac{1}{2}\int_{3/2}^{\phi}\frac{d\phi}{(1-\frac{4}{3}\phi^2)^{1/2}} , \qquad (14)$$

from which follows

$$\phi=\frac{\sqrt{3}}{2}\cos(x/\sqrt{3}) \qquad (15)$$

and

$$n=\frac{n_0}{(1-\frac{4}{3}cn_0^2t)^{1/2}}\cos\left[x\left[\frac{c}{3a}\right]^{1/2}\right]$$

$$(p=3,\quad \delta=2,\quad \mu=-\tfrac{1}{2}) , \qquad (16)$$

where $|x|<\pi/2(3a/c)^{1/2}$. At this stage we can easily extend the solutions (12) and (16) to account for linear dissipation (or growth) as represented by the term $-bn$ in Eq. (1). For $b\neq 0$ we therefore apply a well-known transformation defining the quantities

$$N=n\exp(bt),\quad \tau=b^{-1}[1-\exp(-bt)] . \qquad (17)$$

We then obtain an equation in the new variables N and τ which is formally identical to Eq. (1) provided $\delta=p-1$, as is the case for $\delta=1$, $p=2$ and $\delta=2$, $p=3$, and we also have solutions formally identical to expressions (12) and (16) in terms of N and τ, which define the new solutions for $b\neq 0$ as expressed in n and t by means of transformation (17). The time of explosion for $b\neq 0$ becomes $t_\infty=-b^{-1}\ln(1-b\tau_\infty)$, where by τ_∞ we denote the corresponding time in the absence of linear damping, i.e., for the two cases $\tau_\infty=\frac{4}{3}n_0^{-1}c^{-1}$ and $\tau_\infty=\frac{3}{4}n_0^{-2}c^{-1}$, respectively, as seen from Eqs. (12) and (16). For $b>\tau_\infty^{-1}$ the growth will be limited to such an extent that explosion will not occur in a finite time since in the critical limit $b_c=\tau_\infty^{-1}$ the time of explosion is at infinity.

As a further extension let us consider for the case $\delta=1$, $p=2$ the set of coupled equations describing the evolution of two interacting populations, namely,

$$\frac{\partial n_1}{\partial t}=\frac{\partial}{\partial x}\left[D_1\frac{\partial n_1}{\partial x}\right]+c_1n_1^2+g_1n_1n_2 , \qquad (18)$$

$$\frac{\partial n_2}{\partial t}=\frac{\partial}{\partial x}\left[D_2\frac{\partial n_2}{\partial x}\right]+c_2n_2^2+g_2n_1n_2 , \qquad (19)$$

where we assume

$$D_1=D_2=A(n_1+n_2) . \qquad (20)$$

Looking for solutions of Eqs. (18) and (19) of the form

$$n_1=a_1n,\quad n_2=a_2n , \qquad (21)$$

where n is a solution of the equation

$$\frac{\partial n}{\partial t}=a\frac{\partial}{\partial x}\left[n\frac{\partial n}{\partial x}\right]+cn^2 , \qquad (22)$$

and given by the expression (12), we introduce the relations (20) and (21) into Eqs. (18) and (19) which yields

$$a_1=\frac{c_2-g_1}{c_1c_2-g_1g_2}c,\quad a_2=\frac{c_1-g_2}{c_1c_2-g_1g_2}c , \qquad (23)$$

where in expression (12) and Eq. (22) we substitute a by the quantity $A(a_1+a_2)$. It should be noted that a further extension of the solutions of the coupled system to include linear damping (or growth) terms $-b_1n_1$ and $-b_2n_2$ by a transformation of the form (17) can be done only if $b_1=b_2$ since by necessity the time transformation must be unique for the two equations. Similar solutions and considerations for coupled equations apply also to the case $\delta=2$, $p=3$.

As has been described above, it is possible to construct a class of solutions of the reaction-diffusion equation which exhibits the property of explosive instability with preservation of certain spatial distributions, and to extend the solutions to particular cases of coupled variables. In the example represented by the solution (12) either a certain single pulse or a repetitive standing wave structure in space could grow explosively in time with preservation of shape as governed by the balance between diffusion and nonlinearity. The solutions here obtained may serve as indications of possibilities of finding solutions of an even more general nature.

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