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Polarization of 2ω radiation in scattering of circularly polarized light by free electrons

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It is shown that for circularly polarized incident light, the 2ω scattered radiation is circularly polarized for small scattering angles, θ; for $\theta=\pi/2$, the scattered radiation is plane polarized. In the general case, the scattered 2ω light vector traces out an ellipse with eccentricity $\sin\theta$ and is elliptically polarized.

I. INTRODUCTION

Great interest has developed in nonlinear effects in the interaction of electromagnetic radiation since the invention of lasers, which are sources of coherent light, characterized by a high degree of monochromaticity, high directionality, high intensity or brightness, and the advent of optoelectronic techniques for detecting radiation of very low intensity.[1] Englert and Rinehart[2] have recently made measurements of the intensity of second harmonic (2ω) waves in the scattering of intense radiation by free electrons. We suggest that it would be worthwhile first to measure the polarization of the scattered radiation using a polarizer in front of the detector apparatus, as this is easier to do than to detect variations of intensity. Therefore, we have made calculations of the polarization properties of scattered light under different conditions.[3] In this paper we report the polarization properties of the 2ω scattered radiation in the scattering of circularly polarized light by free electrons.

Vachaspati and his collaborators[4–7] pointed out that classical electrodynamics predict the presence of harmonics in the scattering of monochromatic electromagnetic waves by free electrons. Harmonics arise due to nonlinearities in the equation of motion, in the expression for radiation field, and in the retardation effect.[4–11] As pointed out by Vachaspati,[4] if we write the equation of motion of the electron in a plane-wave light field taking account not only of its electric vector **E**, but also the magnetic vector **H**, the latter interaction gives rise to electron oscillations with twice the incident frequency and consequently to a second harmonic in the scattered light. It has been found that for the special case of circularly polarized light, the harmonics are produced due to nonlinearities in the expression for the retarded electromagnetic field due to the scattering electron. Many authors[2,11–14] using different techniques have confirmed the results of Vachaspati *et al.*[4,7]

II. SOLUTION OF ELECTRON EQUATION OF MOTION

The Lorentz equation of motion for a free electron in the field of a circularly polarized electromagnetic wave of frequency ω and wave vector $\mathbf{k}=\omega\mathbf{n}_0$, where $\mathbf{n}_0$ is the unit vector along the direction of propagation, is

$$m\frac{d^2\mathbf{z}}{dt^2}=e\{\mathbf{E}(\mathbf{z},t)+[\dot{\mathbf{z}}\times\mathbf{H}(\mathbf{z},t)]\} \ . \tag{1a}$$

$(\mathbf{z},t)$ is the space-time position coordinate of the electron and $\dot{\mathbf{z}}=d\mathbf{z}/dt$ is the velocity of the electron in a laboratory frame. e and m are, respectively, the charge and the mass of the electron. We take the speed of light as unity and also take the mean position of the electron to be at the origin.

The circularly polarized incident field is given by

$$\mathbf{E}(\mathbf{z},t)=E_0[\mathbf{e}_1\cos(\omega t-\mathbf{k}\cdot\mathbf{z})+\mathbf{e}_2\sin(\omega t-\mathbf{k}\cdot\mathbf{z})] \ , \tag{1b}$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are two mutually perpendicular unit polarization vectors. We call the direction of the electric field as the direction of polarization. We introduce a unit vector $\boldsymbol{\beta}_1$ perpendicular to $\mathbf{n}_0$ in the plane of the paper (the plane containing incident and scattered light) and another unit vector $\boldsymbol{\beta}_2$ perpendicular to the plane of the paper such that the unit vectors $(\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\mathbf{n}_0)$ form a right-handed coordinate system to define the incident light. Explicitly,

$$\boldsymbol{\beta}_1=-\frac{(\mathbf{n}_0\times\mathbf{n})\times\mathbf{n}_0}{\sin\theta}, \quad \boldsymbol{\beta}_2=\frac{\mathbf{n}_0\times\mathbf{n}}{\sin\theta} \ ,$$

i.e.,

$$\boldsymbol{\beta}_1=\boldsymbol{\beta}_2\times\mathbf{n}_0 \ . \tag{2}$$

The expression (1b) for electric field can be written in terms of $\boldsymbol{\beta}_1$, β_2, and $\mathbf{n}_0$ as

$$\mathbf{E}(\mathbf{z},t)=E_0[(\cos\phi_0\boldsymbol{\beta}_1+\sin\phi_0\boldsymbol{\beta}_2)\cos(\omega t-\mathbf{k}\cdot\mathbf{z}) + (\sin\phi_0\boldsymbol{\beta}_1+\cos\phi_0\boldsymbol{\beta}_2)\sin(\omega t-\mathbf{k}\cdot\mathbf{z})] \tag{3a}$$

$$\mathbf{e}_1\cdot\boldsymbol{\beta}_1=\mathbf{e}_2\cdot\boldsymbol{\beta}_2=\cos\phi_0, \quad \mathbf{e}_1\cdot\boldsymbol{\beta}_2=\mathbf{e}_2\cdot\boldsymbol{\beta}_1=\sin\phi_0 \ . \tag{3b}$$

The angle which the electric vector makes with $\boldsymbol{\beta}_1$ is called the polarization angle of the incident light and is denoted by ϕ_0 (see Fig. 1).

The coordinate of the electron at any instant t can be expressed in the form of a series as

$$\mathbf{z}=\sum_{n=1}^{\infty}\mathbf{z}^{(n)} \ , \tag{4}$$

wherein $\mathbf{z}^{(1)},\mathbf{z}^{(2)},\mathbf{z}^{(3)},\ldots$ denote successive terms in increasing powers of the magnitude of the applied field E_0.

When this expansion is used in expressions (1a) and (1b) and the first few terms to order E_0 are separately equated to zero, we obtain

$$\omega \mathbf{z}^{(1)} = -\frac{eE_0}{\omega m}[\mathbf{e}_1\cos(\omega t)+\mathbf{e}_2\sin(\omega t)] \; , \qquad \mathbf{z}^{(2)}=0, \quad \mathbf{z}^{(3)}=0 \; . \tag{5}$$

We see that the second- and third-order solutions are not contributing. The second harmonic (2ω) is produced in this case of circularly polarized light due to nonlinearities in the expression for the radiation field due to the motion of the electron (see Sec. III).

III. POLARIZATION OF 2ω RADIATION FIELD

In order to get the electric and magnetic fields generated by an accelerated electron at a retarded space-time point $(\mathbf{x},x_0)$ at large distance $r=|\mathbf{x}|$, we use (for method, see Ref. 4)

$$\mathbf{E}^{\rm scat}=[\mathbf{H}^{\rm scat},\mathbf{n}], \quad \mathbf{H}^{\rm scat}=\left[\frac{e}{r}\right][\mathbf{n},\mathbf{M}] \; , \tag{6a}$$

where $\mathbf{n}=\mathbf{x}/r$ is the unit vector along the direction of scattered light.

On using (5), we get

$$\mathbf{M}=\{(1-\mathbf{n}\cdot\dot{\mathbf{z}})^{-3}\dot{\mathbf{z}}(1-\dot{z}^{2})^{-1}[-(\ddot{\mathbf{z}}\cdot\mathbf{n})+(\ddot{\mathbf{z}}\cdot\dot{\mathbf{z}})-(\mathbf{n}\cdot\dot{\mathbf{z}})(\dot{\mathbf{z}}\cdot\ddot{\mathbf{z}})+\dot{z}^{2}(\mathbf{n}\cdot\ddot{\mathbf{z}})] \\ -(1-\dot{z}^{2})^{-1}(1-\mathbf{n}\cdot\dot{\mathbf{z}})^{-2}[\ddot{\mathbf{z}}-\dot{z}^{2}\ddot{\mathbf{z}}+(\dot{\mathbf{z}}\cdot\ddot{\mathbf{z}})\dot{\mathbf{z}}]\}=\mathbf{M}_{\omega}+\mathbf{M}_{2\omega}+\cdots \; , \tag{6b}$$

where

$$\mathbf{M}_{\omega}=\left[\frac{eE_0}{m}\right]\left[-1+\frac{7}{\delta}\frac{e^2E_0^2}{\omega^2m^2}\right](\mathbf{e}_1\cos\psi+\mathbf{e}_2\sin\psi)\frac{-1}{\delta}\frac{e^3E_0^3}{\omega^2m^3}[(\cos^2\alpha-\cos^2\beta)(\mathbf{e}_1\cos\psi-\mathbf{e}_2\sin\psi) \\ +2\cos\alpha\,\cos\beta(\mathbf{e}_1\sin\psi+\mathbf{e}_2\cos\psi)] \tag{6c}$$

and

$$\mathbf{M}_{2\omega}=2\frac{e^2E_0^2}{\omega m^2}[(\cos\beta\mathbf{e}_1+\cos\alpha\mathbf{e}_2)\cos2\psi-(\cos\alpha\mathbf{e}_1-\cos\beta\mathbf{e}_2)\sin2\psi] \; , \tag{6d}$$

where $\cos\alpha=-\sin\theta\cos\phi_0$, $\cos\beta=\sin\theta\sin\phi_0$, and $\psi=\omega(x_0-|\mathbf{x}|)$, so that the electric field of the scattered 2ω light can be written as (we omit the terms involving the fundamental frequency ω)

$$\mathbf{E}^{\rm scat}=\left[\frac{2}{r}\right]\frac{e^3E_0^2}{\omega m^2}\{(\cos\beta\mathbf{e}_1+\cos\alpha\mathbf{e}_2-\cos\alpha\,\cos\beta\mathbf{n})\cos2\psi-[\cos\alpha\mathbf{e}_1-\cos\beta\mathbf{e}_2+(\cos^2\beta-\cos^2\alpha)\mathbf{n}]\sin2\psi\} \; . \tag{7}$$

Notice that the electric vector, $\mathbf{E}^{\rm scat}$ is transverse to $\mathbf{n},\mathbf{E}^{\rm scat}\cdot\mathbf{n}=0$, as it should be. Equation (7) may be simplified as

$$\mathbf{E}^{\rm scat}=A\mathbf{S}^{\rm scat}, \quad A=\left[\frac{2}{r}\right]\left[\frac{e^3E_0^2}{\omega m^2}\right] \; ,$$

where

$$\mathbf{S}^{\rm scat}=\sin\theta[(\sin\phi_0\mathbf{e}_1-\cos\phi_0\mathbf{e}_2+\sin\theta\,\sin2\phi_0\mathbf{n})\cos2\psi+(\cos\phi_0\mathbf{e}_1+\sin\phi_0\mathbf{e}_2+\sin\theta\,\cos2\phi_0\mathbf{n})\sin2\psi] \; . \tag{8}$$

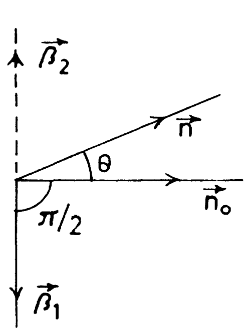

FIG. 1. Electric vector of incident light is resolved along unit vectors ($\boldsymbol{\beta}_1$, β_2, and $\mathbf{n}_0$).

We will write it in a suitable form below to exhibit the ellipticity. We introduce a new phase parameter η such that

$$\mathbf{S}^{\rm scat}=\mathbf{S}_c\cos(2\psi+\eta)+\mathbf{S}_s\sin(2\psi+\eta) \; . \tag{9}$$

Using $\mathbf{S}_c\cdot\mathbf{S}_s=0$, the orthogonality condition to determine the parameter η, we find the relation as

$$\tan(2\eta)=\tan(4\phi_0), \quad \text{i.e.,} \quad \eta=2\phi_0 \; . \tag{10}$$

A coordinate system, similar to the one for the incident light [Eq. (2)] is needed to express the scattered 2ω radiation. We introduce unit vectors $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$, and $\mathbf{n}$ where

$$\boldsymbol{\alpha}_1=\frac{(\mathbf{n}\times\mathbf{n}_0)\times\mathbf{n}}{\sin\theta}, \quad \boldsymbol{\alpha}_2=\frac{\mathbf{n}\times\mathbf{n}_0}{\sin\theta} \quad \text{(downwards)}$$

i.e.,

$$\boldsymbol{\alpha}_1=\boldsymbol{\alpha}_2\times\mathbf{n} \; . \tag{11}$$

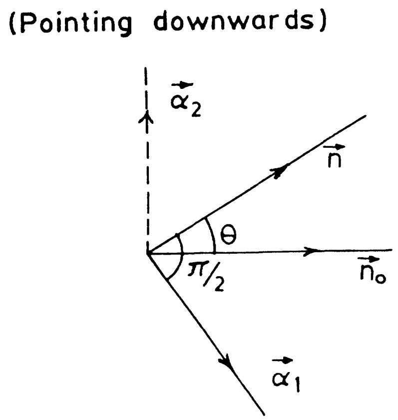

FIG. 2. Electric vector of scattered light is resolved along mutually perpendicular unit vectors ($\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$, and $\mathbf{n}$).

The angle which the electric vector of the scattered light makes with unit vector $\boldsymbol{\alpha}_1$ is called the angle of polarization of the scattered light and is denoted by ϕ (see Fig. 2).

The scattered electric vector for 2ω radiation can be resolved in terms of $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$, and $\mathbf{n}$. For this, first we decompose the $\mathbf{e}_1$ and $\mathbf{e}_2$ unit vectors along three axes, i.e., $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$, and $\mathbf{n}$, to study the final polarization properties of the 2ω wave:

$$\mathbf{e}_1 = \cos\phi_0 \cos\theta \boldsymbol{\alpha}_1 + \sin\phi_0 \boldsymbol{\alpha}_2 - \sin\theta \cos\phi_0 \mathbf{n} \tag{12a}$$

and

$$\mathbf{e}_2 = -\cos\theta \sin\phi_0 \boldsymbol{\alpha}_1 + \cos\phi_0 \boldsymbol{\alpha}_2 + \sin\theta \sin\phi_0 \mathbf{n} \ . \tag{12b}$$

Making use of Eqs. (12a) and (12b) in Eq. (8), we get

$$\mathbf{S}^{\text{scat}} = \sin\theta[\cos\theta \boldsymbol{\alpha}_1 \sin 2(\psi + \phi_0) - \boldsymbol{\alpha}_2 \cos 2(\psi + \phi_0)] \ . \tag{13}$$

This is of form (9). It represents an ellipse with eccentricity $\sin\theta$. It is clear from this expression that this light is circularly polarized for small angle θ, but it does not remain circularly polarized for large angles. In fact, it becomes linearly polarized at $\theta = \pi/2$. The change from circular to linear polarization should not be difficult to detect experimentally.

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[1]P. A. Franken, A. E. Hill, C. W. Peters, and G. Weinreich, Phys. Rev. Lett. 7, 118 (1961); R. W. Minck, R. W. Terhune, and C. C. Wang, Appl. Opt. **5**, 1955 (1966); R. W. Terhune and P. D. Maker, *Lasers, Vol. 2* (Marcel Dekker, New York, 1968), p. 295.

[2]T. J. Englert and F. A. Rinehart, Phys. Rev. A **28**, 1539 (1983).

[3]V. P. Verma, D. Phil. thesis, Allahabad University (1986).

[4]Vachaspati, Phys. Rev. **128**, 664 (1962); **130**, 2598E (1963).

[5]Vachaspati, Proc. Natl. Inst. Sci., Sect. A **29**, 138 (1963); **2**, 373 (1964).

[6]Vachaspati and S. L. Punhani, Ind. J. Pure Appl. Phys. **1**, 311 (1963).

[7]H. Prakash and Vachaspati, Ind. J. Pure Appl. Phys. **6**, 161 (1968); **5**, 21 (1967); H. Prakash, Ph.D. thesis, Roorkee University (1966), Chap. II.

[8]H. Prakash and Vachaspati, Proc. Phys. Soc. Jpn. **23**, 1427 (1967).

[9]N. D. Sen Gupta, Bull. Math. Soc. Calcutta **39**, 147 (1947); **14**, 187 (1949).

[10]E. S. Sarachik and G. T. Schappert, Nuovo. Cimento Lett. **2**, 7 (1969).

[11]L. S. Brown and T. W. B. Kibble, Phys. Rev. A **133**, 705 (1964).

[12]V. B. Berestetskii, E. M. Liftshitz, and L. P. Pitaevskii, *Relativistic Quantum Theory* (Addison-Wesley, Reading, Mass., 1971).

[13]M. Jafarpour, Ph.D. thesis, University of Wyoming (1975).

[14]P. Bozrikov and G. Kopytov, Sov. Phys. J. **20**, 70 (1977).