## Brief Reports

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# Wave-function collapse due to null measurements: The origin of intermittent atomic fluorescence

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The observation of no photons emitted by a fluorescing multilevel atom dramatically affects its future evolution. This collapse of the quantum state due to measurements with a null result is the cause of intermittent atomic fluorescence even when the exciting field is arbitrarily coherent.

According to the basic principles of quantum mechanics, there are two ways in which a quantum state $|\psi\rangle$ changes in time. For a closed system the Hamiltonian $H$ determines the time development according to Schrödinger's equation $H|\psi\rangle = i\hbar(\partial/\partial t)|\psi\rangle$, but when a measurement is performed, the state collapses to an eigenstate $|i\rangle$ of the operator $Q$ corresponding to the observed eigenvalue $q_i$: $Q|i\rangle = q_i|i\rangle$. Prior to the measurement the system is usually in a superposition of the various states satisfying the eigenvalue equation, but immediately after the measurement the system is found (with certainty) in the state consistent with the particular value observed.

Here we describe the theory of a quantum system where the collapse of $|\psi\rangle$ is brought about by a succession of null observations. In particular the failure of a switched-on photodetector to record outgoing photons from a fluorescing atom will cause the atom's wave function to collapse towards an eigenstate of a forbidden level and thus increase the probability of continued darkness. We have in mind the experiments[1,2] on intermittent fluorescence in a single trapped atom illuminated by lasers tuned to strong $|0\rangle \leftrightarrow |1\rangle$ and weak $|0\rangle \leftrightarrow |2\rangle$ transitions with frequencies $\omega_1, \omega_2$ where $|0\rangle$ is the common ground state. Dehmelt[3] predicted that every now and then the atomic electron would be shelved in $|2\rangle$ with the result that the fluorescence would turn off [see Fig. 1(a)] for a time given roughly by the lifetime $1/\beta_2$ of $|2\rangle$ which is much greater than the lifetime $1/\beta_1$ of $|1\rangle$. Experiments confirmed Dehmelt's intuition that the telegraph could be used to see each individual quantum jump $|0\rangle \leftrightarrow |2\rangle$. The first theory[4] and experiment[1] used an incoherent light source. An assertion[5] that the telegraph should also appear for arbitrarily coherent illumination was criticized[6] on the basis of the principle of superposition applied to the atom. More recent theory[7–12] and experiment[2] support the existence of a telegraph in the coherent case.

By formulating this issue in terms of the projection of the Fock space onto the state with no outgoing photons, we will obtain equations for the time development of this projection which are substantially simpler than the optical Bloch equations yet contain all the statistical information lost in the averaging procedure that yields the Bloch equations. We predict that during the dark period the electron is not shelved in $|2\rangle$ but instead it is in a time-dependent superposition which does not radiate. The critical time for which the observation of no fluorescence implies the beginning of a dark period is $T_c \approx (4/\beta_1)\ln(\beta_1/\Omega_2)$ where $1/\beta_1$ is the lifetime of the strong transition and $\Omega_2$ is the Rabi flopping frequency of the weak transition. The slow increase of $T_c$ as the log of the inverse of the (weak) transition amplitude raises the question of how isolated the atom really is. Measurements on the time scale $T_c$ will show that the "shelving" is not a jump but the limit of a continuous process. We also calculate the large-power dependence of the telegraph as well as the two-time correlation for emission of $\omega_1$ and $\omega_2$ photons. The latter is highly irreversible [as expected from Fig. 1(a)] even for

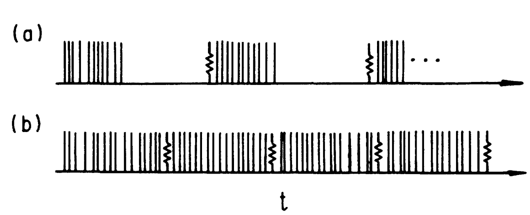


FIG. 1. Schematic of emission events vs time for (a) the telegraph and (b) the simplified superposition description of a three-level atom. A straight line indicates many strong transition photons and a wiggly line indicates a single weak transition photon.

 

coherent illumination.

The main surprise with the telegraph picture is that at first sight it appears to be too classical. According to a simple quantum-mechanical point of view, should not the atom be in a superposition of $|0\rangle$, $|1\rangle$, and $|2\rangle$ prior to the weak emission so that there is no precursor period of darkness? According to the superposition idea one might expect instead a sequence such as that shown in Fig. 1(b) where there are no dark periods but every now and then a weak emission sneaks in.

Let us first motivate this simplified superposition approach by considering the time development of a closed three-level system driven by an electric field $\mathbf{E}=2\mathbf{E}_1\cos\omega_1 t+2\mathbf{E}_2\cos(\omega_2+\Delta)t$ so that

$$|\psi\rangle=\sum c_j\exp(-i\omega_j t)|j\rangle\ ,\qquad i\hbar dc_j/dt=H_{jk}c_k\ ,\tag{1}$$

where $\Omega_2=\mu_2|\mathbf{E}_2|/\hbar<<\Omega_1=\mu_1|\mathbf{E}_1|/\hbar$, $j=0,1,2$, and for the rotating-wave approximation $H_{01}=H_{10}=-\hbar\Omega_1$, $H_{02}=H_{20}^{*}=-\hbar\Omega_2\exp(i\Delta t)$ (all other elements of $H_{ij}$ vanish). The solution has the form $c_j=\sum a_{jk}\exp(\lambda_{jk}t)$ where $\lambda_{jk}$ are determined by (1) and $a_{jk}$ by the initial conditions. For the closed system the $\lambda_{jk}$ are pure imaginary and the amplitude to be in $|2\rangle$ starting from $|0\rangle$ at $t=0$ is $(\Delta=\Omega_1)$

$$c_2\approx\frac{\sqrt{2}}{2}i\sin\left[\frac{\Omega_2 t}{\sqrt{2}}\right]+\frac{\Omega_2}{4\Omega_1}\left[\cos\left[\frac{\Omega_2 t}{\sqrt{2}}\right]-\exp(-2i\Omega_1 t)\right]\ .$$

Now in the simple superposition approach one separates the time development of the atom from the process of emission (detection). The atom is then regarded as developing according to (1) with the probability per second of emitting a strong or weak transition photon being $\beta_1|c_1|^2$ or $\beta_2|c_2|^2$, respectively. The observation of a photon then collapses $|\psi\rangle\to|0\rangle$ (Ref. 13) and resets the time. Such a picture leads to Fig. 1(b). Furthermore, the development of a significant overlap with $|2\rangle$ requires a time $1/\Omega_2$ as determined by the smallest $\lambda_{jk}$. Thus when

$$\beta_1>>\Omega_2\ ,\tag{2}$$

the resets to $|0\rangle$ continuously interrupt (CI) the increase of $|c_2|^2$ so that the percentage of emissions at $\omega_2$ will be down by a factor of $(\Omega_2/\Omega_1)^2$ from the intensity of weak emissions that would occur when only the weak level is driven ($\mathbf{E}_1=0$).

It was in the context of the atomic superposition picture that doubts were raised regarding the existence of a coherently driven telegraph. Since a telegraph in the presence of coherent radiation would be a very sensitive meter for unexpected patterns of order[5] and the presence of "forbidden" processes,[3] it is important to understand precisely what picture is provided by the orthodox quantum mechanics.

The key statistic is the period of darkness.[8–10] From the simple superposition picture the wave function collapses only upon the detection of an emitted photon so that the probability of a period of darkness of length $1/\beta_2$ (starting out from $|0\rangle$) is $\exp(-\beta_1/\beta_2)$, which is absolutely infinitesimal. The fundamental fact needed to resolve the paradox is that even the observation of no photons produces a reduction of the wave function of the atomic system.

In our case a state of the atom plus E.M. (electromagnetic) field is described by the vector

$$|\Psi\rangle=\sum_{i,\{\mathbf{k}\}}c_{i,\{\mathbf{k}\}}|i,n_{\mathbf{k}_1},n_{\mathbf{k}_2},\ldots\rangle\ ,$$

where $i=0,1,2$ labels the atomic states and $n_{\mathbf{k}}$ is the number of outgoing photons scattered (in directions different from the laser beam) with momentum $\mathbf{k}$.

The Hilbert space of the system is then $\mathcal{H}=\mathcal{H}_{\text{atom}}\otimes\mathcal{H}_{\text{em}}$ ($\mathcal{H}_{\text{em}}$ is the Fock space of the E.M. field). In the limit of perfect quantum efficiency the observation of no photons is described by the operator $P=\sum_i|i,\{0\}\rangle\langle\{0\},i|$ which is the projector over the Fock vacuum for the scattered photons. Starting from $t=0$ the probability of a dark period of length $T'>T$ is obviously the probability of having no scattered photons at time $T$, which is[14] $\langle\Psi|P|\Psi\rangle=\sum_i|c_{i,0}(T)|^2$ where $c_{i,0}\equiv c_{i,\{0\}}$. After such an observation the system is projected in the state $|\Psi'\rangle=P|\Psi\rangle/\langle\Psi|P|\Psi\rangle$. More generally, if the initial density matrix of the system is $\rho$, after observation it becomes $P\rho/\text{tr}\rho P$.[14]

From the $|c_{i,0}|^2$ one can calculate the key statistics of the system. For instance the probability density $D_0(t)$ for the time between emissions is

$$D_0(t)=-dW_0/dt\ ,\tag{3}$$

where $W_0(t)=\sum|c_{i,0}(t)|^2$ for $c_{0,0}(0)=1$.

The evolution of $P|\Psi\rangle$ (i.e., $c_{i,0}$) is given by Eq. (6) below. To derive this result, consider the effective Green function of the system $PG(t)P$ where $G(t)$ is the retarded Green function:

$$G(t)=\int G(E)\exp(-iEt)dE/2\pi\ ,\tag{4}$$

where $G(E)=(E+i\epsilon-H)^{-1}$ and $H$ is the sum of a free Hamiltonian $H_0([H_0,P]=0)$ plus an interaction term $H_I$. For the case of an $N$-level atom

$$H_0=\tfrac{1}{2}\int[\epsilon_0\mathbf{E}^2+(1/\mu_0)\mathbf{B}^2]d^3r+\sum_i\hbar\omega_i\Lambda_{ii}\ ,\qquad H_I=-i\sum_{i,j}(\omega_i-\omega_j)\boldsymbol{\mu}_{ij}\cdot\mathbf{A}(0,t)\Lambda_{ij}\ ,\tag{5}$$

where $i=0,\ldots,N-1$, $\hbar\omega_i$ is the energy of the $i$th level of the atom (located at $r=0$), and $\boldsymbol{\mu}_{ij}$ are the various transition amplitudes. The $\Lambda_{ij}$ are defined by $\Lambda_{ij}=|i\rangle\langle j|\otimes I$ on $\mathcal{H}$ and generate the algebra of $U(N)$:

$$[\Lambda_{ij},\Lambda_{kl}]=\delta_{jk}\Lambda_{il}-\delta_{il}\Lambda_{jk}\ .$$

From (4) it is easy to show that $PG(E)P=[E-PHP-PH_IQ\hat{G}(E)QH_IP]^{-1}$ where $\hat{G}(E)=(E+i\epsilon-QHQ)^{-1}$ and $Q=I-P$. The evolution of $P|\Psi\rangle$ is then described by the effective, non-Hermitian Hamiltonian

$$H_{\text{eff}} = \sum_{\alpha} E_\alpha \,|\, E_\alpha \rangle_R \langle E_\alpha \,|_L \ .$$

The $E_\alpha$ are the zeros of the determinant: $\det[E - PHP - PH_I Q\hat{G}(E) Q H_I P]$ and the subscripts $R,L$ are used to denote the corresponding right and left eigenvectors. The system described by the Hamiltonian (5) can be solved exactly as will be shown in detail in a forthcoming paper. The operators $\Lambda_{ij}^P(t) \equiv |\, i \rangle \langle j \,| \otimes P$ are related to the $c_{i,0}(t)$ through

$$\langle \Psi(0) \,|\, \Lambda_{ij}^P(t) \,|\, \Psi(0) \rangle = \langle i \,|\, P \,|\, \Psi(t) \rangle \langle \Psi(t) \,|\, P \,|\, j \rangle = c_{i,0}(t) c_{j,0}^*(t) \ .$$

And the equation of motion for $\langle \Psi \,|\, \Lambda_{ij}^P \,|\, \Psi \rangle$ are linear and of the form

$$\langle \Psi \,|\, \dot{\Lambda}_{ij}(t) \,|\, \Psi \rangle = i \hat{H}_{ik} \langle \Psi \,|\, \Lambda_{kj} \,|\, \Psi \rangle - i \langle \Psi \,|\, \Lambda_{ik} \,|\, \Psi \rangle \hat{H}^*_{kj} \ .$$

We can now interpret $\hat{H}_{ij}$ as $\langle i \,|\, H_{\text{eff}} \,|\, j \rangle$ so that we finally have for the equations of motion for the $c_{i,0}$

$$i\hbar\, dc_{i,0}/dt = \hat{H}_{ij} c_{j,0} \ , \qquad (6)$$

where $\hat{H}_{ij} = H_{ij} + H'_{ij}$, $H'_{11} = -i\hbar\beta_1$, $H'_{22} = -i\hbar\beta_2$, and the other elements of $H'$ vanish. The $\beta_1$ and $\beta_2$ turn out to be, quite naturally, the Einstein coefficients for the spontaneous decay (to all orders of perturbation theory). We now show that $W_0(t)$ has a slowly decaying part, so that the probability of observing a dark period $T \gg 1/\beta_1$ can be nonvanishing.

The irreversibility exhibited by the effective Hamiltonian $\hat{H}$ of the projected state $P \,|\, \Psi \rangle$ is not due to an ensemble average but rather has its source in the retarded solution to Maxwell's equations and the large density of states for outgoing photons. Relaxation of these conditions (as well as the rotating-wave approximation) could lead to long-time quasiperiodicity[15] and chaos.[16] The decay of $c_{1,0}$ and $c_{2,0}$ as given by (6) is due to the buildup of the amplitude to be in the states $|\, i, n_{\mathbf{k}_1}, \ldots \rangle$, ($\{n\} \neq 0$) at a rate determined by the spontaneous decay coefficients $\beta_1, \beta_2$.

Equation (6) like Eq. (1) has solutions of the form $c_{j,0} = \sum a_{jk} \exp \lambda_{jk} t$, where now the $\lambda_{jk}$ have nonzero real parts. Taking $\Delta = \Omega_1$ and setting $|\, \Psi \rangle = |\, 0,0 \rangle$ at $t = 0$ leads to

$$c_{2,0} = 2i(\Omega_2/\beta_1)[\exp(-\gamma t) - \exp(-\beta_1 t/4)] \ ,$$

where $\gamma = \beta_2/2 + 2\Omega_2^2/\beta_1$. After the emission of a photon the probability that there will be a dark period longer than where $t \gg 4/\beta_1$ is

$$|\, c_{2,0}(t) \,|^2 \to 4(\Omega_2/\beta_1)^2 \exp(-\gamma t) \ . \qquad (7)$$

Appearance of the slow-time-scale $\gamma$ requires imposing (2) which is the key telegraph inequality. In the absence of emission the long-time probability to be in $|\, 1 \rangle$ is down by a factor of $4(\Omega_2/\beta_1)^2$ from the probability to be in $|\, 2 \rangle$. Figure 2 shows the probability to be in $|\, 2 \rangle$ divided by the sum of probabilities to be in $|\, 0 \rangle$ or $|\, 1 \rangle$ as a function of time of darkness. The key time scale or collapse time $T_c \approx (4/\beta_1)\ln(\beta_1/\Omega_2)$ is determined by $|\, c_{1,0} \,|^2 = |\, c_{2,0} \,|^2$. Note that as the coupling to $|\, 2 \rangle$ goes to zero the time for purification of level $|\, 2 \rangle$ through continuous observation of null emission increases only logarithmically. In contrast with other models of null measurement[17] successive measurements on the shelvable atom continuously modify $|\, \Psi \rangle$ as in Fig. 2.

Placing the $c_{i,0}(t)$ into (3) yields the complete emission statistics of a single atom. The emission statistics of a dark period is obtained from setting $|\, \Psi \rangle = |\, 2;\ 0 \rangle$ at $t = 0$. In this case $c_{2,0}(t) \equiv \bar{c}_{2,0}(t) = \exp(-\gamma t)$ so that the probability of the dark period ending between $t$ and $t + dt$ (due to an emission) is $D_2(t)dt$ where $D_2(t) = -(d \,|\, \bar{c}_{2,0} \,|^2/dt) = 2\gamma \exp(-2\gamma t)$. The lifetime $\tau_D$ for the Poisson distribution of dark periods (as well as the average lifetime) is $\tau_D = 1/2\gamma$. The potentially strong modification of the length of the periods of darkness due to the laser power near $\omega_2$ agrees with Refs. 9 and 10, and differs with Ref. 8, which used the full Bloch equations. This effect can be checked experimentally and should also be a diagnostic for a coherently driven telegraph. For $|\, \mathbf{E}_1 \,| = |\, \mathbf{E}_2 \,|$, $\Omega_2^2\beta_1 \sim \Omega_1^2\beta_2$, $\tau_D$ is diminished from $1/\beta_2$ by the factor $\beta_1^2/2\Omega_1^2$ which effect should be easily observed.

Although $|\, c_{2,0}(T_c) \,|^2$ determined by (7) is much smaller than unity, it is huge compared to $\exp(-\beta_1/\beta_2)$. The average time of darkness $P_2$ that the atom spends in $|\, 2 \rangle$ is estimated by the probability of a period of darkness $(\Omega_2/\beta_1)^2$ multiplied by the number of resets per second $\beta_1 P_1$, and the average time of a dark period $\tau_D$. As $P_1 \sim 1$ we find $P_2 \sim \Omega_2^2/[\Omega_2^2 + (\beta_1\beta_2)/4]$ which is typically of order unity. If $\Delta = 0$ as opposed to $\Omega_1$, the probability of weak transitions is diminished by a factor $(\beta_1/\Omega_1)^4$. The detuning width $|\, \Delta - \Omega_1 \,|$ for effectively exciting the telegraph is $\beta_1$ (not $\beta_2$); thus as first emphasized by Arecchi *et al.*,[12] the CW driven atom will not be useful as a time standard.

From the exact solution for the system described by (5) it is in principle possible to evaluate the multitime correlations:

$$\langle \Lambda_{i_1,j_1}(t_1) \cdots \Lambda_{i_n j_n}(t_n) \Lambda_{j_n i_n}(t_n) \cdots \Lambda_{j_1 i_1}(t_1) \rangle$$

which in view of the relation between $\Lambda_{ij}$ and $\mathbf{E}$ (Ref. 2) yields the correlation functions for the intensity of scat-

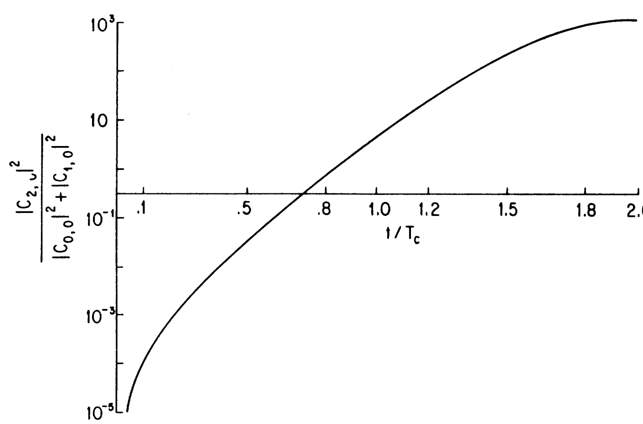


FIG. 2. Probability for the atom to be in the weak level $|\, 2 \rangle$ divided by the probability that it is not in $|\, 2 \rangle$ as a function of the observed time of darkness ($\beta_1/\beta_2 = 10^4$, $\Omega_2/\beta_1 = 10^{-2}$).

tered radiation from the transition $i \to j$. More details are to be given elsewhere but here we notice that as for the two level system[13] the correlations factorize; for instance,

$$\langle \Lambda_{ij}(t)\Lambda_{kl}(t')\Lambda_{lk}(t')\Lambda_{ji}(t)\rangle = \langle \Lambda_{ii}(t)\rangle \langle \Lambda_{kk}(t'-t)\rangle_j ,$$

where $\langle \ \rangle_j$ means average on a state with the atom in the $j$th eigenstate and the E.M. field in a coherent state of the laser field. Proof of the factorization starts from the Heisenberg equation

$$\dot{\Lambda}_{ij}(t) = \mathcal{L}_{ijkl}(t)\Lambda_{kl}(t) , \quad (8)$$

where $\mathcal{L}_{ijkl}$ are $c$ numbers when the system is projected on a subspace, corresponding to the EM field in a coherent state: $\mathbf{A}^{+}_{\text{free}}(t)\rho = \mathbf{V}(t)\rho$ with $V(t)$ a $c$ number. This condition is invariant under temporal evolution. As a consequence, the averages $F_{ijkl}(t \mid \tau) \equiv \langle \Lambda_{ij}(t)\Lambda_{kl}(\tau)\Lambda_{ji}(t)\rangle$ obey

$$dF_{ijkl}(t \mid \tau)/d\tau = \mathcal{L}_{klmn}(\tau)F_{ijmn}(t,\tau)$$

with initial condition $F_{ijkl}(t,t) = \langle \Lambda_{ii}(t)\rangle \delta_{kj}\delta_{lj}$ giving the result (for $\tau \geq t$)

$$F_{ijkk}(t,\tau) = \langle \Lambda_{ii}(t)\rangle \langle \Lambda_{kk}(\tau)\rangle_j . \quad (9)$$

This result holds without any assumption of Markovian response, in parallel with the two-level system.[13] If $I_s(t), I_w(t)$ are the intensities of strong, weak radiation, then (9) yields

$$\langle I_s(t)I_w(t')\rangle = f(t-t')\langle I_s(t)\rangle \langle I_w(t')\rangle ,$$

where $f \approx 1 - \exp[(t'-t)/\tau_d]$ for $t'-t \gg 1/\beta_1$ and $f \approx 1$ for $t - t' \gg 1/\beta_1$ and describes the irreversibility characteristic to the telegraph [Fig. 1(a)].

The Bloch equations follow from the average of (8). As such they involve eight eigenvalues instead of three and since all photon occupations are mixed together for each $|i\rangle$ they do not describe the emission statistics determined by the $c_{i,0}$.

When no photons are recorded, one might expect that the atom is indeed a closed system developing according to a unitary transformation (1). To the contrary we see that the fact that one could have recorded a photon had one been emitted converts the atom into a uniquely open system and affects predictions based upon quantum mechanics. The coherently driven telegraph is a consequence of this phenomenon. For fields so strong that $\Omega_2 > \beta_1$ there are no slow time scales and the response approaches the simplified superposition picture [Fig. 1(b)] characteristic to (1).

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