PHYSICAL REVIEW A VOLUME 36, NUMBER 2 JULY 15, 1987

Statistical dynamics of stable processes

Frank E. Peseckis
RD 1, Box 55, Haddock Hill Road, Old Chatham, New York 12136
(Received 15 December 1986)

We consider the Markovian evolution in phase space of a distribution of particles subject both to a regular external field and to stochastic forces whose transition probabilities are Lévy stable densities. An integro-differential equation is derived which naturally generalizes the Fokker-Planck equation much as the stable densities generalize the normal. Its compact expression employs a new notation of vector fractional derivatives. Solutions are obtained for equilibrium conditions as well as for field-free and harmonically bound particle forces. The real-space Smoluchowski-equation limit of the Fokker-Planck equation is also generalized. It is suggested that the general evolution equation and stable densities may describe statistically the strange or fractal behavior of turbulent systems in the neighborhood of their critical points in phase space, with generally noninteger-order derivative laws governing the turbulent diffusion.

I. INTRODUCTION

Consider the motion of a body in a regular external field of force additionally subjected to irregular forces from neighboring bodies. Its description has begun historically with a Langevin equation in configuration space $\mathbf{r}$ and time t for the particle momentum $\mathbf{p}$:

$$\mathbf{p}=[-\beta\mathbf{p}+\mathbf{F}(\mathbf{r},t)](\Delta t)+\mathbf{R}(\Delta t) , \qquad (1)$$

in which $\mathbf{p}$ is the momentum change in time Δt, $\mathbf{F}(\mathbf{r},t)$ is the external force, $\mathbf{R}$ is the stochastic force from neighbors, and the frictional coefficient β is assumed to account for regular drag on the particle produced by its surroundings. In the nonrelativistic limit

$$\mathbf{r}=\mathbf{p}\Delta t/m=\mathbf{u}\Delta t , \qquad (2)$$

with m the particle mass and $\mathbf{u}$ the particle velocity. $\mathbf{R}(t)$ is presumed to vary independently of $\mathbf{u}$ and much more rapidly.

The only physically meaningful time-asymptotic distribution of velocities which derives from (1) has always been thought to be of Maxwell-Gauss, or normal, form:

$$P(\mathbf{u})\rightarrow(m/2\pi k_B T)^{3/2}\exp\left[-\left(\frac{m\,|\,\mathbf{u}\,|^2}{2k_B T}\right)\right] , \quad \text{as } t\rightarrow\infty , \qquad (3)$$

in which T is the temperature of the surrounding medium and k_B is Boltzmann's constant. This is the time-asymptotic form for $P(\mathbf{u})$—however, only if the stochastic forces are similarly distributed. A Markovian statistical description of the dynamics which consequently assumes normal transition probabilities for the increments in phase space results in the Fokker-Planck equation for the particle density $P(\mathbf{r},\mathbf{u},t)$ (for the standard derivation see Chandrasekhar[1]):

$$\frac{\partial P}{\partial t}+\mathbf{u}\cdot\frac{\partial P}{\partial \mathbf{r}}+\mathbf{K}\cdot\frac{\partial P}{\partial \mathbf{u}}=\beta\frac{\partial}{\partial \mathbf{u}}\cdot(\mathbf{u}P)+q\left(\frac{\partial}{\partial \mathbf{u}}\cdot\frac{\partial}{\partial \mathbf{u}}\right)P , \qquad (4)$$

$$\mathbf{K}=\mathbf{F}(\mathbf{r})/m , \qquad (5a)$$

$$q=\beta k_B T/m \qquad (5b)$$

(in which the arguments of $P(\mathbf{r},\mathbf{u},t)$ have been suppressed).

The Maxwell-Gauss distribution is but one of a broader class of time-asymptotic distributions permitted by the Langevin equation,[2] the symmetric cases of the stable distributions named after Lévy.[3] For a space of n dimensions the Lévy distributions $P(\mathbf{u},t)$ are of the form[4]

$$P(\mathbf{u},t)=\frac{1}{(2\pi)^n}\int_{-\infty}^{\infty} d\mathbf{k}\,\phi(\mathbf{k},t)exp(-i\mathbf{k}\cdot\mathbf{u}) , \qquad (6a)$$

$$\phi(\mathbf{k},t)=\exp\left[-bt\,|\,\mathbf{k}\,|^{\gamma}\left(1+i\omega(\mathbf{k},\gamma)\frac{\mathbf{c}\cdot\mathbf{k}}{|\,\mathbf{k}\,|}\right)\right] , \qquad (6b)$$

with the parameter restrictions

$$0<\gamma\le 2,\quad b\ge 0,\quad -1\le|\,\mathbf{c}\,|\le 1 , \qquad (7)$$

and with the definition

$$\begin{aligned}\omega(\mathbf{k},\gamma)&=\tan\frac{\pi\gamma}{2},\quad \gamma\neq 1\\ \omega(\mathbf{k},\gamma)&=(2/\pi)\ln|\,\mathbf{k}bt\,|,\quad \gamma=1 .\end{aligned} \qquad (8)$$

(Holt and Crow[5] present graphical representations as well as an extensive list of properties.) The symmetric cases are of present interest, and have $\mathbf{c}=0$ so that the characteristic function ϕ is simply the exponential of a generally noninteger power of the magnitude of the transform variable. They arise asymptotically from solutions to the Langevin equation if the stochastic forces are similarly distributed, a property noted previously for the normal case $\gamma=2$.[6]

Their statistical-dynamical significance is suggested by the fact that the stable distributions obey a broader version of the central-limit theorem: The appropriately normed sum of n mutually independent variables with a common distribution that possesses singular higher mo-

 ©1987 The American Physical Society

ments (subject to certain restrictions on the behavior of the tails of the distribution) will tend to a non-normal stable distribution, just as a normed sum of similar variables whose common distribution has finite higher moments will tend to the normal distribution.

The presence of singularities in the variance and higher moments of the non-normal stable distributions has been regarded as evidence that they are incorrect as detailed descriptions of real physical processes. Yet such singularities would in the case of the Langevin equation reflect the occurrence of non-oscillatory jump discontinuities in the temporal evolution of the momentum[7,8] and so cusps in the configuration-space path traced by a particle. Insofar as such singularities in the particle motion are real, the singularities in its statistical description are physically meaningful. These cusps are characteristic of the "strange" or fractal behavior in phase space of a chaotic system in the neighborhood of its critical points,[9–11] so that a generalization of the Fokker-Planck equation to the wider class of stable transitions may be expected to describe more accurately the statistics of turbulent or chaotic systems in regions of phase space where they show such "anomalous" behavior.

II. EVOLUTION EQUATIONS FOR STABLE DISTRIBUTIONS BY FRACTIONAL DIFFERENTIATION

It will prove useful to obtain the evolution equation for the stable distribution (6)–(8). The $1+1$ dimensional case $P(u,t)$ for one phase-space variable plus time has been treated in part[12] for symmetric distributions with parameter γ a rational number. With γ written in the irreducible ratio form m/n, m and n integers, Seshadri and West have shown that

$$\frac{\partial^n P(u,t)}{\partial t^n}=(-1)^{n+m/2}b^n\frac{\partial^m P(u,t)}{\partial u^m}, \quad m \text{ even} \tag{9a}$$

and

$$\frac{\partial^{2n} P(u,t)}{\partial t^{2n}}=(-1)^{m}b^{2n}\frac{\partial^{2m} P(u,t)}{\partial u^{2m}}, \quad m \text{ odd} . \tag{9b}$$

Let us also first consider the $(1+1)$-dimensional case of a symmetric stable distribution. Simple differentiation of (6)–(8) with respect to t produces

$$\frac{\partial P}{\partial t}=-\frac{b}{2\pi}\int_{-\infty}^{\infty}dk\,|k|^{\gamma}\exp(-iku-bt\,|k|^{\gamma}) . \tag{10}$$

Integrals of this form can be expressed compactly by employing the Fourier representation[13] of what is now known as the (scalar) Weyl fractional derivative.[14,15] The Weyl derivative of order α of a function of $f(x)$ is defined as

$$D^{\alpha}_{x-\infty}f(x)=\frac{(-1)^{\alpha}}{\Gamma(-\alpha)}\int_{x}^{\infty}ds(s-x)^{-1-\alpha}f(s) \tag{11a}$$

or

$$D^{\alpha}_{x+\infty}f(x)=\frac{1}{\Gamma(-\alpha)}\int_{-\infty}^{x}ds(x-s)^{-1-\alpha}f(s) , \tag{11b}$$

whichever is defined, for $\text{Re}\alpha<0$. A more general order derivative can be written with the identity for Riemann-Liouville derivatives:

$$D^{\alpha}_{x-y}f(x)=\frac{d^n}{dx^n}D_{x-y}f(x) \quad \text{for } n-1\le \text{Re}\alpha<n , \tag{12}$$

n a positive integer. In the particular case of the Weyl derivative $y=\pm\infty$. For the Weyl derivative

$$\frac{d^{\alpha}}{dx^{\alpha}}\exp(\beta x)=\beta^{\alpha}\exp(\beta x), \quad \text{Re}\beta\le 0 \tag{13}$$

in which we have used an abbreviated notation for the derivative operator. One can then obtain Fourier's representation of the fractional derivative:

$$\frac{d^{\alpha}}{dx^{\alpha}}f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk(ik)^{\alpha}\exp(ikx) \times\int_{-\infty}^{\infty}dy\,f(y)\exp(-iky) . \tag{14}$$

The αth Weyl derivative with respect to u of the symmetric stable distribution in $1+1$ dimensions is then

$$\frac{\partial^{\alpha}}{\partial u^{\alpha}}P(u,t) =\frac{(-i)^{\alpha-r}}{2\pi}\frac{\partial^r}{\partial u^r}\int_{-\infty}^{\infty}dk\,k^{\alpha-r}\exp(-iku-bt\,|k|^{\gamma}) \quad \text{for } r-1\le\text{Re}\alpha<r , \tag{15}$$

where r is a positive integer.

Another Weyl derivative together with the choice of an appropriate branch of $(-i)^{\alpha}$ then gives

$$\frac{\partial^{\alpha}}{\partial u^{\alpha}}\frac{\partial^{\alpha}}{\partial u^{\alpha}}P(u,t)=\frac{(-1)^{\alpha-r}}{2\pi}\frac{\partial^r}{\partial u^r}\frac{\partial^r}{\partial u^r} \times\int_{-\infty}^{\infty}dk\,|k|^{2(\alpha-r)} \times\exp(-iku-bt\,|k|^{\gamma}) . \tag{16}$$

By a simple identity Eq. (10) can be rewritten for integer r:

$$\frac{\partial}{\partial t}P(u,t)=-(-1)^{-r}\frac{b}{2\pi}\frac{\partial^r}{\partial u^r}\frac{\partial^r}{\partial u^r} \times\int_{-\infty}^{\infty}dk\,|k|^{\gamma-2r}\exp(-iku-bt\,|k|^{\gamma}) . \tag{17}$$

A fractional partial differential equation for $P(u,t)$ follows upon equating $2\alpha=\gamma$, $\gamma\neq1$,

$$\frac{\partial}{\partial t}P(u,t)=-(-1)^{\gamma/2}b\frac{\partial^{\gamma}}{\partial u^{\gamma}}P(u,t) . \tag{18}$$

The derivative on the right-hand side is to be understood as

$$\frac{\partial^\gamma}{\partial u^\gamma}P(u,t)=\left[\frac{\partial}{\partial u}\frac{\partial^{\gamma/2-1}}{\partial u^{\gamma/2-1}}\right]\left[\frac{\partial}{\partial u}\frac{\partial^{\gamma/2-1}}{\partial u^{\gamma/2-1}}\right]P(u,t) \tag{19}$$

since $0<\alpha\le 1$ by (7a) and so $r=1$ by (12b) and (15b).

It is evident that the Gaussian diffusion equation is correctly recovered for $\gamma=2$,

$$\frac{\partial}{\partial t}P(u,t)=b\frac{\partial^2}{\partial u^2}P(u,t) . \tag{20}$$

The same procedure can be applied to the more general case of asymmetric stable distributions with $c\neq 0$ in (6b). The first partial derivative in time of $P(u,t)$ is ($\gamma\neq 1$)

$$\frac{\partial P(u,t)}{\partial t}=-(-1)^r\frac{b}{2\pi}\frac{\partial^r}{\partial x^r}\cdot\frac{\partial^r}{\partial x^r}\int_{-\infty}^{\infty}dk\,|k|^{\gamma-2r}\exp(-ikx-bt|k|^\gamma)\left[1+ic\frac{k}{|k|}\tan\left[\frac{\pi\gamma}{2}\right]\right]$$
$$+(-1)^m\frac{bc}{2\pi}\left[\tan\frac{\pi\gamma}{2}\right]\frac{\partial}{\partial x}\frac{\partial^m}{\partial x^m}\frac{\partial^m}{\partial x^m}\int_{-\infty}^{\infty}dk\,|k|^{\gamma-1-2m}\exp(-ikx-bt|k|^\gamma)$$
$$\times\left[1+ic\frac{k}{|k|}\tan\left[\frac{\pi\gamma}{2}\right]\right] \tag{21}$$

so that by (16) with $\alpha=\gamma/2$ for the first term on the right-hand side of (21) and $\alpha=(\gamma-1)/2$ for the second term,

$$\frac{\partial}{\partial t}P(u,t)=\left[-(-1)^{\gamma/2}b+(-1)^{(\gamma-1)/2}bc\tan\left[\frac{\pi\gamma}{2}\right]\right]\frac{\partial^\gamma}{\partial u^\gamma}P(u,t) . \tag{22}$$

The asymmetry of the distribution does not alter the order of the fractional derivative but does change the complex effective diffusion coefficient seen in Eq. (18).

The equation for $P(u,t)$ in more than one phase-space dimension may be written compactly with the new notion of a vector fractional derivative (VFD) introduced in Appendix A. The operator is designed to reduce in one dimension to the scalar Weyl derivative in the form suggested by Fourier.

The first time derivative of a symmetric stable distribution $P(\mathbf{u},t)$ may then be expressed immediately as

$$\frac{\partial}{\partial t}P(\mathbf{u},t)=-(-1)^{\gamma/2}b\left[\frac{\partial}{\partial\mathbf{u}}\cdot\frac{\partial^{\gamma-1}}{\partial\mathbf{u}^{\gamma-1}}\right]P(\mathbf{u},t) . \tag{23}$$

This again reduces to the standard diffusion equation for the Gaussian case $\gamma=2$.

The more general case which includes asymmetric stable forms can be found similarly to be ($\gamma\neq 1$)

$$\frac{\partial}{\partial t}P(\mathbf{u},t)=-(-1)^{\gamma/2}b\left[\frac{\partial}{\partial\mathbf{u}}\cdot\frac{\partial^{\gamma-1}}{\partial\mathbf{u}^{\gamma-1}}\right]P(\mathbf{u},t)$$
$$+(-1)^{(\gamma-1)/2}b\tan\frac{\pi\gamma}{2}\left[\mathbf{c}\cdot\frac{\partial^\gamma}{\partial\mathbf{u}^\gamma}\right]P(\mathbf{u},t) . \tag{24}$$

The second term on the right-hand side may also be written with

$$\left[\mathbf{c}\cdot\frac{\partial^\gamma}{\partial\mathbf{u}^\gamma}\right]P(\mathbf{u},t)=-\left[\frac{\partial}{\partial\mathbf{u}}\cdot\frac{\partial}{\partial\mathbf{u}}\right]\left[\mathbf{c}\cdot\frac{\partial^{\gamma-2}}{\partial\mathbf{u}^{\gamma-2}}\right]P(\mathbf{u},t) \tag{25}$$

to make the connection to the ordinary diffusion equation somewhat more apparent.

III. EVOLUTION EQUATION FOR PROCESSES WITH STABLE TRANSITION PROBABILITIES

Here we will generalize the Fokker-Planck equation to non-normal transition probabilities of stable form, along lines similar to the standard Fokker-Planck derivation in Chandrasekhar.[1]

Suppose that the density of particles $P(\mathbf{r},\mathbf{u},t)$ in phase space subject to regular external and stochastic internal forces evolves as a Markov process with $\Psi(\mathbf{r},\mathbf{u};\Delta\mathbf{r},\Delta\mathbf{u})$ defined as the transition probability that a particle at position $\mathbf{r}$ with velocity $\mathbf{u}$ suffers changes $\Delta\mathbf{r}$ and $\Delta\mathbf{u}$ in a time Δt. Then by definition

$$P(\mathbf{r},\mathbf{u},t+\Delta t)=\int_{-\infty}^{\infty}d(\Delta\mathbf{u})\times\int_{-\infty}^{\infty}d(\Delta\mathbf{r})P(\mathbf{r}-\Delta\mathbf{r},\mathbf{u}-\Delta\mathbf{u},t)\times\Psi(\mathbf{r}-\Delta\mathbf{r},\mathbf{u}-\Delta\mathbf{u};\mathbf{r},\mathbf{u}) . \tag{26}$$

Assume that the individual particle motions are governed by Langevin equations (1)–(2). With a new variable $\psi(\mathbf{r},\mathbf{u};\Delta\mathbf{u})$ defined by

$$\Psi(\mathbf{r},\mathbf{u};\Delta\mathbf{r},\Delta\mathbf{u})=\psi(\mathbf{r},\mathbf{u};\Delta\mathbf{u})\delta(\Delta\mathbf{r}-\mathbf{u}\Delta t) , \tag{27}$$

Eq. (26) becomes

$$P(\mathbf{r}+\mathbf{u}\Delta t,\mathbf{u},t+\Delta t)=\int_{-\infty}^{\infty}d(\Delta\mathbf{u})P(\mathbf{r},\mathbf{u}-\Delta\mathbf{u},t)\times\psi(\mathbf{r},\mathbf{u}-\Delta\mathbf{u};\mathbf{u}) \tag{28}$$

after shifting the space coordinate by $\Delta\mathbf{r}$ and performing the $\Delta\mathbf{r}$ integration.

The expression for ψ can be obtained by assuming that the stochastic accelerations $\mathbf{R}(\Delta t)/m$ in the Langevin equation for the velocity, written in the form

$$\Delta\mathbf{u}=-(\beta\mathbf{u}-\mathbf{K})\Delta t+\mathbf{B}(\Delta t),\quad \mathbf{R}=m\mathbf{B},\quad \mathbf{F}=m\mathbf{K} , \tag{29}$$

are distributed in a symmetric stable manner so that similar distributions describe the time-asymptotic solutions. Hence by Eqs. (6)–(8)

$$\phi(\sigma,\Delta t)=\exp(-b\Delta t\,|\sigma|^{\gamma}) = \int_{-\infty}^{\infty} d[\mathbf{B}(\Delta t)]p[\mathbf{B}(\Delta t)]\exp[i\sigma\cdot\mathbf{B}(\Delta t)]\ , \qquad (30)$$

in which $p[\mathbf{B}(\Delta t)]$ is this distribution of stochastic accelerations.

An equation which is differential in time is derived from the small-Δt limit after expanding (28) and (30) in Taylor series in Δt and retaining terms up to first order. One finds

$$P(\mathbf{r}+\mathbf{u}\Delta t,\mathbf{u},t+\Delta t)\simeq P(\mathbf{r},\mathbf{u},t) + \Delta t\left[\frac{\partial}{\partial t}+\mathbf{u}\cdot\frac{\partial}{\partial \mathbf{r}}\right]P(\mathbf{r},\mathbf{u},t) \qquad (31)$$

so that with (29) and the Fourier inversion of (30) one finds in three dimensions

$$\begin{aligned}\psi(\mathbf{r},\mathbf{u}-\Delta\mathbf{u};\Delta\mathbf{u}) &= \frac{1}{8\pi^3}\int_{-\infty}^{\infty} d\sigma \exp(-i\sigma\cdot\{\Delta\mathbf{u}+[\beta(\mathbf{u}-\Delta\mathbf{u})-\mathbf{K}]\Delta t\}-b\,|\sigma|^{\gamma}\Delta t)\\ &= \frac{1}{8\pi^3}\int_{-\infty}^{\infty} d\sigma[\exp(-i\sigma\cdot\Delta\mathbf{u})]\{1-i\sigma\cdot[\beta(\mathbf{u}-\Delta\mathbf{u})-\mathbf{K}]+b\,|\sigma|^{\gamma}\Delta t\}\\ &= \delta(\Delta\mathbf{u})-\Delta t\frac{1}{8\pi^3}\int_{-\infty}^{\infty} d\sigma[\exp(-i\sigma\cdot\Delta\mathbf{u})]\{i\sigma\cdot[\beta(\mathbf{u}-\Delta\mathbf{u})-\mathbf{K}]+b\,|\sigma|^{\gamma}\}\ . \end{aligned} \qquad (32)$$

To first order in Δt then (28) becomes, suppressing the arguments of $P(\mathbf{r},\mathbf{u},t)$ on the left-hand side,

$$\left[\frac{\partial}{\partial t}+\mathbf{u}\cdot\frac{\partial}{\partial \mathbf{r}}\right]P=-\frac{1}{8\pi^3}\int_{-\infty}^{\infty} d(\Delta\mathbf{u})P(\mathbf{r},\mathbf{u}-\Delta\mathbf{u},t)\int_{-\infty}^{\infty} d\sigma[\exp(-i\sigma\cdot\Delta\mathbf{u})]\{i\sigma\cdot[\beta(\mathbf{u}-\Delta\mathbf{u})-\mathbf{K}]+b\,|\sigma|^{\gamma}\} \qquad (33)$$

since the zeroth-order terms on both sides cancel.

Let

$$\mathbf{y}=\mathbf{u}-\Delta\mathbf{u},\quad d(\Delta\mathbf{u})=-d\mathbf{y}\ , \qquad (34)$$

so that the right-hand side is

$$-\frac{1}{8\pi^3}\int_{-\infty}^{\infty} d\mathbf{y}\int_{-\infty}^{\infty} d\sigma\{\exp[-i\sigma\cdot(\mathbf{u}-\mathbf{y})]\} \times\{i\sigma\cdot[\beta\mathbf{y}-\mathbf{K}]+b\,|\sigma|^{\gamma}\}P(\mathbf{r},\mathbf{y},t)\ . \qquad (35)$$

With the aid of some identities presented in Appendix A involving fractional differentiation, one finds that upon performing the integrations over σ and $\mathbf{y}$ Eq. (33) becomes

$$\left[\frac{\partial}{\partial t}+\mathbf{u}\cdot\frac{\partial}{\partial \mathbf{r}}+\mathbf{K}\cdot\frac{\partial}{\partial \mathbf{u}}\right]P =\beta\frac{\partial}{\partial \mathbf{u}}\cdot(\mathbf{u}P)-(-1)^{\gamma/2}b\left[\frac{\partial}{\partial \mathbf{u}}\cdot\frac{\partial^{\gamma-1}}{\partial \mathbf{u}^{\gamma-1}}\right]P\ . \qquad (36)$$

This is the general evolution equation desired. In the Gaussian limit $\gamma=2$ it clearly reduces to the Fokker-Planck equation. As before, the effect of these non-Gaussian symmetric statistics has been to generalize the ordinary Laplacian to an operator of noninteger order.

IV. EQUILIBRIUM SOLUTION OF THE STABLE TRANSITION EQUATION

In equilibrium there is no change in the distribution $P(\mathbf{r},\mathbf{u},t)$ due to collisions. By (36) this change is

$$\left.\frac{\partial P}{\partial t}\right|_{\text{collisions}}=\beta\frac{\partial}{\partial \mathbf{u}}\cdot(\mathbf{u}P)-(-1)^{\gamma/2}b\left[\frac{\partial}{\partial \mathbf{u}}\cdot\frac{\partial^{\gamma-1}}{\partial \mathbf{u}^{\gamma-1}}\right]P\ . \qquad (37)$$

The equilibrium condition is, after one integration over the velocity,

$$\beta\mathbf{u}P(\mathbf{r},\mathbf{u},t)-(-1)^{\gamma/2}b\frac{\partial^{\gamma-1}}{\partial \mathbf{u}^{\gamma-1}}P(\mathbf{r},\mathbf{u},t)=\mathbf{c}_0\ , \qquad (38)$$

in which $\mathbf{c}_0$ is a constant vector.

This equation is best solved for the Fourier transform of $P(\mathbf{r},\mathbf{u},t)$ with respect to $\mathbf{u}$, that is, its characteristic function in velocity space, defined by

$$Q(\mathbf{r},\mathbf{k},t)=\int_{-\infty}^{\infty} d\mathbf{u}\,P(\mathbf{r},\mathbf{u},t)\exp(ik\cdot\mathbf{u})\ . \qquad (39)$$

Transformation of (38) produces the equation for $Q(\mathbf{r},\mathbf{k},t)$:

$$\frac{\partial}{\partial \mathbf{k}}Q(\mathbf{r},\mathbf{k},t)+\frac{b}{\beta}\mathbf{k}\,|\mathbf{k}|^{\gamma-2}Q(\mathbf{r},\mathbf{k},t)=i\frac{(2\pi)^n}{\beta}\mathbf{c}_0\delta(\mathbf{k})\ , \qquad (40)$$

in which the right-hand side contains the Dirac δ distribution.

A formal representation of the solution of (40) is

$$Q(\mathbf{r},\mathbf{k},t)=\left[Q_0(\mathbf{r},t)+i\frac{(2\pi)^n}{\beta}\mathbf{c}_0\int d\mathbf{k}\delta(\mathbf{k})\exp\left[\frac{b}{\gamma\beta}\,|\mathbf{k}|^{\gamma}\right]\right] \times\exp\left[-\frac{b}{\gamma\beta}\,|\mathbf{k}|^{\gamma}\right]\ . \qquad (41)$$

If the arbitrary integration constant $c_0=0$, then (41) is the characteristic function for a stable distribution. Hence, the particles in equilibrium are Lévy distributed in velocity. As usual, the special case $\gamma=2$ correctly reduces to Maxwell-Gauss form. The homogeneous form of (38) with the arguments $\mathbf{r}$ and t suppressed is then an ordinary fractional derivative equation for the single-variable symmetric stable distributions.

V. FIELD-FREE SOLUTIONS OF THE STABLE TRANSITION EQUATION

Here we will treat the solutions of (36) for $\mathbf{K}=0$ separately for the conditions of configuration-space homogeneity and inhomogeneity.

A. Configuration-space homogeneity

By assumption, $\mathbf{K}=0$ and $(\partial/\partial\mathbf{r})P(\mathbf{r},\mathbf{u},t)=0$ in (36) so that in three dimensions

$$\frac{\partial}{\partial t}P=3\beta P+\beta\mathbf{u}\cdot\frac{\partial}{\partial\mathbf{u}}P-i^{\gamma}b\left[\frac{\partial}{\partial\mathbf{u}}\cdot\frac{\partial^{\gamma-1}}{\partial\mathbf{u}^{\gamma-1}}\right]P\ . \tag{42}$$

The term linear in $\mathbf{u}$ can be eliminated by a change of independent variable,

$$\mathbf{v}=\mathbf{u}\exp(\beta t),\quad P=P(\mathbf{r},\mathbf{v},t)\ , \tag{43}$$

to give

$$\frac{\partial}{\partial t}P=3\beta P-i^{\gamma}b[\exp(\gamma\beta t)]\left[\frac{\partial}{\partial\mathbf{v}}\cdot\frac{\partial^{\gamma-1}}{\partial\mathbf{v}^{\gamma-1}}\right]P\ . \tag{44}$$

The term linear in P can be eliminated by a change of dependent variable,

$$\chi(\mathbf{r},\mathbf{v},t)=P(\mathbf{r},\mathbf{v},t)\exp(-3\beta t)\ , \tag{45}$$

to yield

$$\frac{\partial}{\partial t}\chi=-i^{\gamma}b[\exp(\gamma\beta t)]\left[\frac{\partial}{\partial\mathbf{v}}\cdot\frac{\partial^{\gamma-1}}{\partial\mathbf{v}^{\gamma-1}}\right]\chi\ . \tag{46}$$

Equation (46) is identical in form to (23) for a symmetric stable distribution, but with a time-dependent generalized diffusion coefficient. This suggests the ansatz (suppressing the configuration-space variable)

$$\chi(\mathbf{v},t)=\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}d\mathbf{k}\exp[-i\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}_0)-a(t)b\,|\,\mathbf{k}\,|^{\gamma}]\ . \tag{47}$$

Substitution of (47) in (46) produces the restriction on $a(t)$:

$$\frac{da}{dt}=\exp(\gamma\beta t)\quad\text{or}\quad a(t)=a_0+\frac{1}{\gamma\beta}\exp(\gamma\beta t)\ , \tag{48}$$

in which a_0 is set by initial conditions.

Upon retracing the transformations, one finds the solution $P(\mathbf{u},t)$ of (42) as

$$P(\mathbf{u},t)=\frac{1}{(2\pi)^3}e^{3\beta t}\times\int_{-\infty}^{\infty}d\mathbf{k}\exp\left[-i\mathbf{k}\cdot(\mathbf{u}e^{\beta t}-\mathbf{u}_0)-b\left[a_0+\frac{1}{\gamma\beta}e^{\gamma\beta t}\right]|\,\mathbf{k}\,|^{\gamma}\right]\ . \tag{49}$$

This result is in agreement with the form obtained by other methods by West and Seshadri[16] for a spatially homogeneous linear system driven by stably distributed external forces. They derived and solved a conventional partial differential equation for the system's characteristic function. The coefficients in this equation contained noninteger powers of the Fourier-transform variable, which as we have seen makes it equivalent to the author's fractional derivative formalism for the probability distribution directly.

In the Gaussian case $\gamma=2$ of (49) simple evaluation of the integral reveals the variance σ_0^2 of the density to be

$$\sigma_0^2=b\left[\frac{1}{\gamma\beta}+a_0e^{-\gamma\beta t}\right]\ . \tag{50}$$

Although there is no finite variance for $\gamma<2$, the expression above may be useful as a pseudovariance for determining a_0 by initial conditions for all γ. Specifically, if the initial distribution is a single velocity, when σ_0^2 at $t=0$ vanishes and so the integration constant is

$$a_0=-\frac{1}{\beta\gamma}\ . \tag{51}$$

For the Gaussian case the integral in (49) then correctly reduces to the well-known Ornstein-Uhlenbeck[17] distribution

$$P(\mathbf{u},t)=\left[\frac{2\pi b}{\beta}[1-\exp(-2\beta t)]\right]^{-3/2}\times\exp\left[\frac{-\beta\,|\,\mathbf{u}-\mathbf{u}_0\exp(-\beta t)\,|^2}{2b\,[1-\exp(-2\beta t)]}\right]\ . \tag{52}$$

B. Configuration-space inhomogeneity

Here it is assumed that in (36) $\mathbf{K}=0$ but the configuration-space derivative of $P(\mathbf{r},\mathbf{u},t)$ need not vanish. One has in three dimensions

$$\frac{\partial}{\partial t}P+\mathbf{u}\cdot\frac{\partial}{\partial\mathbf{r}}P=3\beta P+\beta\mathbf{u}\cdot\frac{\partial}{\partial\mathbf{u}}P-i^{\gamma}b\left[\frac{\partial}{\partial\mathbf{u}}\cdot\frac{\partial^{\gamma-1}}{\partial\mathbf{u}^{\gamma-1}}\right]P\ . \tag{53}$$

Eliminate the term linear in P by transforming to a new P':

$$P'(\mathbf{r},\mathbf{u},t)=P(\mathbf{r},\mathbf{u},t)\exp(-3\beta t) \tag{54}$$

to obtain

$$\frac{\partial}{\partial t}P'+\mathbf{u}\cdot\frac{\partial}{\partial \mathbf{r}}P'=\beta\mathbf{u}\cdot\frac{\partial}{\partial \mathbf{u}}P'-i^{\gamma}b\left[\frac{\partial}{\partial \mathbf{u}}\cdot\frac{\partial^{\gamma-1}}{\partial \mathbf{u}^{\gamma-1}}\right]P' . \quad (55)$$

Without the fractional derivative expression this equation is linear, homogeneous, and first order. Hence, as in the special case $\gamma=2$ treated by Chandrasekhar,[1] its general solution can be written as a function of the Lagrangian system:

$$\frac{d}{dt}\mathbf{u}=-\beta\mathbf{u},\quad \frac{d}{dt}\mathbf{r}=\mathbf{u} , \quad (56)$$

or, equivalently, the new independent variables

$$\mathbf{v}=\mathbf{u}\exp(\beta t) , \quad (57)$$

$$\mathbf{s}=\mathbf{r}+\frac{1}{\beta}\mathbf{u}=\mathbf{r}+\frac{1}{\beta}\mathbf{v}\exp(-\beta t) . \quad (58)$$

As in Sec. V A above, first transform one set of independent variables from $\mathbf{u}$ to $\mathbf{v}$ and so rename $P'(\mathbf{r},\mathbf{u},t)$ as $\mathcal{X}(\mathbf{r},\mathbf{v},t)$ to remove the term linear in $\mathbf{u}$,

$$\frac{\partial}{\partial t}\mathcal{X}+e^{-\beta t}\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{r}}\mathcal{X}=-i^{\gamma}be^{\gamma\beta t}\left[\frac{\partial}{\partial \mathbf{v}}\cdot\frac{\partial^{\gamma-1}}{\partial \mathbf{v}^{\gamma-1}}\right]\mathcal{X} . \quad (59)$$

One expects as before that (59) will be solved by some distribution jointly in $\mathbf{v}$ and $\mathbf{s}$. To find its characteristic function, define a new $\Phi(\mathbf{k},\boldsymbol{l},t)$ with the ansatz

$$\mathcal{X}(\mathbf{r},\mathbf{v},t)=\frac{1}{(2\pi)^6}\int_{-\infty}^{\infty}d\mathbf{k}\int_{-\infty}^{\infty}d\boldsymbol{l}\exp[-i\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}_0)-i\boldsymbol{l}\cdot(\mathbf{s}-\mathbf{s}_0)-\Phi(\mathbf{k},\boldsymbol{l},t)] . \quad (60)$$

Substitution into (59) with the definitions (57) and (58) yields the restriction

$$\frac{\partial}{\partial t}\Phi(\mathbf{k},\boldsymbol{l},t)=b\left|\mathbf{k}\exp(\beta t)+\frac{1}{\beta}\boldsymbol{l}\right| \quad (61)$$

or, equivalently, the time-dependent jointly stable form

$$\Phi(\mathbf{k},\boldsymbol{l},t)=\Phi_0+b\int dt\left|\mathbf{k}\exp(\beta t)+\frac{1}{\beta}\boldsymbol{l}\right| . \quad (62)$$

Φ_0 merely serves to normalize the distribution.

In the normal case $\gamma=2$ the time integral can be performed explicitly to give

$$\mathcal{X}(\mathbf{r},\mathbf{v},t)=\frac{e^{\Phi_0}}{(2\pi)^6}\int_{-\infty}^{\infty}d\mathbf{k}\int_{-\infty}^{\infty}d\boldsymbol{l}\exp\left\{-i\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}_0)-i\boldsymbol{l}\cdot(\mathbf{s}-\mathbf{s}_0)-b\left[\left[\frac{1}{2\beta}e^{2\beta t}+c_0\right]|\mathbf{k}|^2+\left[\frac{1}{\beta^2}t+c_2\right]|\boldsymbol{l}|^2+\left[\frac{2}{\beta^2}e^{\beta t}+c_1\right]\mathbf{k}\cdot\boldsymbol{l}\right]\right\} . \quad (63)$$

The initial conditions determine the integration constants c_j. If $\mathbf{v}=\mathbf{v}_0$ and $\mathbf{s}=\mathbf{s}_0$ at $t=0$,

$$c_0=-\frac{1}{2\beta},\quad c_1=-\frac{2}{\beta^2},\quad c_2=0 . \quad (64)$$

Upon performing the integrations, one finds the normalizing factor Φ_0 to be zero, and the closed expression for $\gamma=2$ as

$$\mathcal{X}(\mathbf{s},\mathbf{v},t)=\frac{1}{64\pi^3}(B_0B_2-\tfrac{1}{4}B_1^2)^{-3/2}\exp\left[-\frac{B_2|\mathbf{v}-\mathbf{v}_0|^2-B_1|\mathbf{v}-\mathbf{v}_0|\cdot|\mathbf{s}-\mathbf{s}_0|+B_0|\mathbf{s}-\mathbf{s}_0|^2}{4(B_0B_2-\frac{1}{4}B_1^2)}\right] , \quad (65)$$

with

$$B_0=-\frac{1}{2\beta}[1-\exp(2\beta t)] , \quad (66)$$

$$B_1=-\frac{2}{\beta^2}[1-\exp(\beta t)] , \quad (67)$$

$$B_2=\frac{1}{\beta^2}t , \quad (68)$$

in agreement with the known result for the Fokker-Planck equation.

VI. EXTERNALLY FORCED SOLUTIONS OF THE STABLE TRANSITION EQUATION

The harmonic oscillator is of traditional interest, so we shall first deal with it separately.

A. Harmonic-oscillator force law

For simplicity let us consider the problem in only one dimension. The force law is

$$K(x)=-\omega^2x \quad (69)$$

and (36) in 1D for $P(x,u,t)$ becomes

$$\frac{\partial}{\partial t}P+u\frac{\partial}{\partial x}P-\omega^2x\frac{\partial}{\partial u}P=\beta\frac{\partial}{\partial u}(uP)-i^{\gamma}b\frac{\partial^{\gamma}}{\partial u^{\gamma}}P . \quad (70)$$

The standard transformation of dependent variable to

$$\mathcal{X}(x,u,t)=P(x,u,t)\exp(-\beta t) \quad (71)$$

simplifies (70) to

$$\frac{\partial}{\partial t}\mathcal{X}+u\frac{\partial}{\partial x}\mathcal{X}-\omega^2x\frac{\partial}{\partial u}\mathcal{X}=\beta u\frac{\partial}{\partial u}\mathcal{X}-i^{\gamma}b\frac{\partial^{\gamma}}{\partial u^{\gamma}}\mathcal{X} . \quad (72)$$

The Lagrangian subsystem is

$$\frac{d}{dt}u = -\beta u - \omega^2 x \ , \tag{73}$$

$$\frac{d}{dt}x = u \ , \tag{74}$$

whose first integrals are

$$\rho = (a_0 x - u)\exp(-a_1 t) \ , \tag{75}$$

$$\sigma = (a_1 x - u)\exp(-a_0 t) \ , \tag{76}$$

in which a_0 and a_1 are the two roots of the quadratic

$$a^2 + \beta a + \omega^2 = 0 \ , \tag{77}$$

specifically,

$$a_0 = \tfrac{1}{2}[-\beta + (\beta^2 - 4\omega^2)^{1/2}] \ , \tag{78}$$

$$a_1 = \tfrac{1}{2}[-\beta - (\beta^2 - 4\omega^2)^{1/2}] \ . \tag{79}$$

We again expect the solution of (72) to be a joint distribution in ρ and σ, and so employ the ansatz with the new function $\Phi(k,l,t)$,

$$\chi(\rho,\sigma,t) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dl \exp[-ik(\rho-\rho_0) - il(\sigma-\sigma_0) - \Phi(k,l,t)] \ . \tag{80}$$

Insertion of (80) into (72) gives the restriction on $\Phi(k,l,t)$:

$$\frac{\partial}{\partial t}\Phi(k,l,t) = b[-k\exp(-a_1 t) - l\exp(-a_0 t)]^\gamma \ , \tag{81}$$

or, equivalently, the time-dependent jointly stable form

$$\Phi(k,l,t) = \Phi_0 + b\int dt[-k\exp(-a_1 t) - l\exp(-a_0 t)]^\gamma \ , \tag{82}$$

in which Φ_0 is a normalization constant. This again is equivalent to the characteristic function obtained by other methods by West and Seshadri[16] for a linearly damped harmonic oscillator driven by Lévy stable fluctuations.

It is easily checked by explicit evaluation of the integral in (80) that the $\gamma = 2$ case reduces correctly to the joint Gaussian solution of the Fokker-Planck equation for appropriate initial conditions.

B. General force law

The general method for solving (36) for an arbitrary external force law $\mathbf{K}(\mathbf{r})$ is clear. One finds the first integrals of the motion in n dimensions of

$$\frac{d}{dt}\mathbf{r} = \mathbf{u}, \quad \frac{d}{dt}\mathbf{u} = -\beta\mathbf{u} - \mathbf{K}(\mathbf{r},\mathbf{u},t) \tag{83}$$

and transforms the independent variables from $\mathbf{r}$ and $\mathbf{u}$ to these $2n$ new quantities. One assumes the solution to be a joint distribution in these new variables, which due to the nature of the fractional derivative is most easily Fourier transformed and solved for its characteristic function. This last is an exponential of the first time integral of the γth power of a time-varying weighted sum of the Fourier-transform variables. The distribution is thus of time-dependent jointly stable form.

VII. THE STABLE TRANSITION EQUATION IN CONFIGURATION SPACE FOR LOCAL EQUILIBRIUM

After some duration Δt much greater than the relaxation time $1/\beta$ of the Langevin equation (1), one may expect that equilibrium conditions will exist locally throughout a system of particles governed by the stable transition equation. We have seen that in such a local equilibrium the particles in some suitably small spatial region will be stable distributed in velocity. Here we wish to derive from the stable transition equation another governing the variation in time and configuration space of a distribution which is locally stable in velocity. This will generalize the Smoluchowski-equation[18] limit of the Fokker-Planck equation.

Reorganize the terms in (36) to the form [again suppressing the arguments of $P(\mathbf{r},\mathbf{u},t)$ for simplicity]

$$\begin{aligned}\frac{\partial}{\partial t}P = {} & \left[\beta\frac{\partial}{\partial \mathbf{u}} - \frac{\partial}{\partial \mathbf{r}}\right]\cdot\left[\mathbf{u}P - \frac{1}{\beta}\mathbf{K}P - i^\gamma\frac{b}{\beta}\frac{\partial^{\gamma-1}}{\partial \mathbf{u}^{\gamma-1}}P\right] \\ & - \left[\beta^{\gamma-1}\frac{\partial^{\gamma-1}}{\partial \mathbf{u}^{\gamma-1}} - \frac{\partial^{\gamma-1}}{\partial \mathbf{r}^{\gamma-1}}\right]\cdot\left[i^\gamma\frac{b}{\beta^\gamma}\frac{\partial}{\partial \mathbf{r}}P\right] \\ & - \frac{\partial}{\partial \mathbf{r}}\cdot\left[\frac{1}{\beta}\mathbf{K}P + i^\gamma\frac{b}{\beta^\gamma}\frac{\partial^{\gamma-1}}{\partial \mathbf{r}^{\gamma-1}}P\right] \ ,\end{aligned} \tag{84}$$

Integrate this over all $\mathbf{u}$ along the line

$$\mathbf{r} + \frac{1}{\beta}\mathbf{u} = \mathbf{r}_0 = \text{const} \tag{85}$$

so that

$$d\mathbf{r} = -\frac{1}{\beta}d\mathbf{u}, \quad \beta\frac{\partial}{\partial \mathbf{u}} - \frac{\partial}{\partial \mathbf{r}} = 2\beta\frac{\partial}{\partial \mathbf{u}} \tag{86}$$

and

$$\beta^{\gamma-1}\frac{\partial^{\gamma-1}}{\partial \mathbf{u}^{\gamma-1}} - \frac{\partial^{\gamma-1}}{\partial \mathbf{r}^{\gamma-1}} = [1 + (-1)^\gamma]\beta^{\gamma-1}\frac{\partial^{\gamma-1}}{\partial \mathbf{u}^{\gamma-1}} \ . \tag{87}$$

One finds that

$$\frac{\partial}{\partial t}\int_{\mathbf{r}+\frac{1}{\beta}\mathbf{u}=\mathbf{r}_0} d\mathbf{u}\,P=2\beta\left[\mathbf{u}P-\frac{1}{\beta}\mathbf{K}P-i^{\gamma}\frac{b}{\beta}\frac{\partial^{\gamma-1}}{\partial\mathbf{u}^{\gamma-1}}P\right]\Bigg|_{\mathbf{u}=-\infty}^{\mathbf{u}=+\infty}+[1+(-1)^{\gamma}]i^{\gamma}b\left[\frac{\partial^{\gamma-1}}{\partial\mathbf{u}^{\gamma-1}}P\right]\Bigg|_{\mathbf{u}=-\infty}^{\mathbf{u}=+\infty}$$
$$-\int_{\mathbf{r}+(1/\beta)\mathbf{u}=\mathbf{r}_0} d\mathbf{u}\frac{\partial}{\partial\mathbf{r}}\cdot\left[\frac{1}{\beta}\mathbf{K}P+i^{\gamma}\frac{b}{\beta^{\gamma}}\frac{\partial^{\gamma-1}}{\partial\mathbf{r}^{\gamma-1}}P\right]. \tag{88}$$

For the first integrated term to vanish, $P(\mathbf{u})$ must asymptotically approach zero as $|\mathbf{u}|\to\infty$ faster than $|\mathbf{u}|^{-1}$. This is true only for stable distributions with parameter $\gamma>1$. The other integrated terms vanish for any $\gamma>0$. With the restriction

$$1<\gamma\le 2\ , \tag{89}$$

one then has

$$\int_{\mathbf{r}+(1/\beta)\mathbf{u}=\mathbf{r}_0} d\mathbf{u}\left[\frac{\partial}{\partial t}P+\frac{\partial}{\partial\mathbf{r}}\cdot\left[\frac{1}{\beta}\mathbf{K}P+i^{\gamma}\frac{b}{\beta^{\gamma}}\frac{\partial^{\gamma-1}}{\partial\mathbf{r}^{\gamma-1}}P\right]\right]=0\ . \tag{90}$$

With the assumption of a local stable equilibrium, the velocity dependence of $P(\mathbf{r},\mathbf{u},t)$ can be separated from that on $\mathbf{r}$ and t by

$$P(\mathbf{r},\mathbf{u},t)\simeq U(\mathbf{u})p(\mathbf{r},t)\ , \tag{91}$$

in which $U(\mathbf{u})$ is a stable distribution of order γ. This will be valid within spatial cells of size $\Delta r\simeq|u_M|\Delta t$ or $\Delta r\simeq(b/\beta^3)^{1/2}$ on a side, in which $|u_M|$ is the largest speed whose contribution to the integral (90) is dominant.

The stable generalization of the Smoluchowski equation follows immediately as

$$\frac{\partial}{\partial t}p(\mathbf{r},t)+\frac{\partial}{\partial\mathbf{r}}\cdot\left[\frac{1}{\beta}\mathbf{K}(\mathbf{r})p(\mathbf{r},t)+i^{\gamma}\frac{b}{\beta^{\gamma}}\frac{\partial^{\gamma-1}}{\partial\mathbf{r}^{\gamma-1}}p(\mathbf{r},t)\right]=0\ . \tag{92}$$

The Laplacian in space found in the Gaussian limit is now more broadly an operator involving fractional derivatives, as one would expect.

In the case of a stationary generalized diffusion of current $\mathbf{j}_0$,

$$\mathbf{j}_0=\frac{1}{\beta}\mathbf{K}p(\mathbf{r})+i^{\gamma}\frac{b}{\beta^{\gamma}}\frac{d^{\gamma-1}}{d\mathbf{r}^{\gamma-1}}p(\mathbf{r})\ . \tag{93}$$

An explicit expression for $p(\mathbf{r})$ is obtained in Appendix B in the form in one dimension

$$p(x)=-\frac{\beta^{\gamma}}{b}\phi(x)\frac{d^{1-\gamma}}{dx^{1-\gamma}}\frac{j_0}{\phi(x)}\ , \tag{94}$$

where $\phi(x)$ is a stable distribution in the variable ρ defined by

$$\rho^{\gamma}=-\Gamma(1+\gamma)\frac{d^{1-\gamma}}{dx^{1-\gamma}}K(x) \tag{95}$$

or for a force law derivable from a potential

$$\rho^{\gamma}=\Gamma(1+\gamma)\frac{d^{2-\gamma}}{dx^{2-\gamma}}V(x)\ . \tag{96}$$

Equation (94) can be rewritten as

$$j_0=-\frac{b}{\beta^{\gamma}}\phi(x)\frac{d^{\gamma-1}}{dx^{\gamma-1}}\frac{p(x)}{\phi(x)}\ , \tag{97}$$

where $b/\beta=(k_BT/m)^{\gamma/2}$. When this is integrated between any two points x_A and x_B, one finds the generalization of the Kramer's relation[1] from the simple Gaussian case to more broadly stable transitions:

$$j_0\int_{x_A}^{x_B}dx\frac{1}{\phi(x)}=-\frac{b}{\beta^{\gamma}}\left[\frac{d^{\gamma-2}}{dx^{\gamma-2}}\frac{p(x)}{\phi(x)}\right]\Bigg|_{x_A}^{x_B}. \tag{98}$$

In more than one dimension this becomes

$$\mathbf{j}_0\int_{\mathbf{r}_A}^{\mathbf{r}_B}d\mathbf{r}\frac{1}{\phi(\mathbf{r})}=-\frac{b}{\beta^{\gamma}}\left[\frac{d^{\gamma-2}}{d\mathbf{r}^{\gamma-2}}\frac{p(\mathbf{r})}{\phi(\mathbf{r})}\right]\Bigg|_{\mathbf{r}_A}^{\mathbf{r}_B}. \tag{99}$$

In the Gaussian case $\gamma=2$, one correctly recovers the result that $\phi(x)$ is a damped exponential in the potential $V(x)$. For the other stable distributions, however, their arguments are by (96) noninteger powers of fractional derivatives of the potential, and their damping for large values of the argument is asymptotically a noninteger power law rather than exponential.

VIII. DISCUSSION

The formalism developed here, based on stably distributed fluctuating forces, should prove useful for describing a variety of apparently chaotic or more generally nonequilibrium processes.

The success of some recent models[19] of turbulent diffusion which rely in part on stable distributions is encouraging, but also points to the need for an approach such as this paper's which is connected more directly to fundamental statistical dynamical law.

In turbulent flows one encounters enhanced diffusion where the relative mean-square displacement $\langle r^2(t)\rangle$ of marked particles as a function of time increases superlinearly,

$$\langle r^2(t)\rangle\sim t^{\alpha},\quad \alpha>1\ . \tag{100}$$

Richardson[20] experimentally found the dependence to be cubic in turbulent air, and proposed as an explanation that a standard diffusion law applied but with a fractional $\frac{4}{3}$-power law for the relative diffusion coefficient,

$$\frac{\partial}{\partial t}P(r,t)=\frac{\partial}{\partial r}D(r)\frac{\partial}{\partial r}P(r,t)\ , \tag{101}$$

$$D(r)=r^{4/3}\ , \tag{102}$$

where $P(r,t)$ is the probability that two particles spatially coincident at $t=0$ will be spatially separate by a distance r at time t. Hentschel and Procaccia[21] have widened this to a fractional law in space and time

$$D(r,t)=r^a t^b \tag{103}$$

in the study of fractal cloud shapes.

Shlesinger, Klafter, and West[19] have used a stably distributed random-walk model together with Kolmogorov's $-\frac{5}{3}$ law to describe this enhanced diffusion, and have derived the appropriate fractional-power-law mean-square displacements (with intermittency corrections).

This paper's noninteger derivative formalism for diffusion based on stable transitions suggests, however, that these successful model results may not be evidence for a fractional-power relative diffusion coefficient, but rather for a fractional generalization of the order of derivatives in the diffusion equation with spatially constant coefficients.[22] One would replace (101) and in part (103) with

$$\frac{\partial}{\partial t}P(r,t)=-i^{\nu}D\frac{\partial^{\nu}}{\partial r^{\nu}}P(r,t) \ . \tag{104}$$

Straightforward use of fractional calculus reproduces the superlinear law (100), with the fractional-power diffusion coefficient laws being lowest-order approximations to (104). Fractional time dependence follows naturally from the broader form

$$\frac{\partial^{\eta}}{\partial t^{\eta}}P(r,t)=(-1)^{\nu-\eta}D\frac{\partial^{\nu}}{\partial r^{\nu}}P(r,t) \ . \tag{105}$$

There are mathematically more rigorous reasons for believing in the utility of our formalism for turbulence. It is well known that real turbulent processes are manifestly non-Gaussian in their statistics.[23–26] This is usually taken to mean that the distributions of various dynamical quantities such as fluid velocity possess nonzero but finite skewness, and a defined kurtosis which deviates from quasinormality. Theoretical models of turbulence based on this perspective are equivalent to perturbative expansions in small non-Gaussianity parameters around a Gaussian base state.[27,28] Observation of the trajectories of fluid elements in turbulent flows,[29] however, show that their paths are filled with cusps, points of discontinuity in fluid velocity. These are hallmark signs of fractal motion,[30] and, as was noted above, imply statistical distributions with singular moments of at least second and higher order, distributions which are maximally non-Gaussian. (Montroll and Shlesinger[31] have also noted that these singular moments reflect an absence of scale and hence the existence of dynamical features on all scales, which lead to the observed intermittent clustering of events as a function of space or time.) Infinite second or higher moments invalidate the perturbative expansions in conventional turbulence theories. They are, however, natural consequences of the Lévy stable generalizations of Gaussian processes examined here. And since the governing equation presented above which generalizes the Fokker-Planck to stable non-Gaussianity is not higher than second order, the probability density function it describes is everywhere positive. This cannot be guaranteed by other attempts such as the Kramers-Moyal finitely non-Gaussian expansion of the Fokker-Planck equation, insofar as it is higher than second order in its phase-space derivatives.

The ease with which the stable density formalisms include the Gaussian, but one example of which is the simple generalization of the Fokker-Planck equation which results with the use of fractional derivatives in Laplacian-like form, strongly suggests that these probability distributions be regarded as the natural basic elements of new and broader statistical theories which encompass both traditional equilibrium and nonequilibrium processes.

ACKNOWLEDGMENT

I would like to thank Lynn Levy Peseckis for countless helpful discussions during the preparation of this manuscript.

APPENDIX A

Define the vector-fractional-derivative (VFD) operator with

$$\frac{\partial^{\alpha}}{\partial \mathbf{x}^{\alpha}}\exp(i\mathbf{k}\cdot\mathbf{x})=i^{\alpha}\mathbf{k}\,|\,\mathbf{k}\,|^{\,\alpha-1}\exp(i\mathbf{k}\cdot\mathbf{x}) \tag{A1}$$

so that the n-dimensional VFD of an arbitrary function $f(\mathbf{x})$ is, if it exists,

$$\frac{\partial^{\alpha}f(\mathbf{x})}{\partial \mathbf{x}^{\alpha}}=\frac{i^{\alpha}}{(2\pi)^n}\int_{-\infty}^{\infty}d\mathbf{k}\,\mathbf{k}\,|\,\mathbf{k}\,|^{\,\alpha-1}\exp(i\mathbf{k}\cdot\mathbf{x}) \times\int_{-\infty}^{\infty}d\mathbf{y}\,f(\mathbf{y})\exp(i\mathbf{k}\cdot\mathbf{y}) \tag{A2}$$

for $\mathrm{Re}\alpha<0$. This is the analog to the Fourier representation (14) of the scalar Weyl fractional derivative. For a more general range for α, one can use the easily verified identity

$$\frac{\partial^{\alpha}}{\partial \mathbf{x}^{\alpha}}f(\mathbf{x})=(-1)^n\left[\frac{\partial}{\partial \mathbf{x}}\cdot\frac{\partial}{\partial \mathbf{x}}\right]^n\frac{\partial^{\alpha-2n}}{\partial \mathbf{x}^{\alpha-2n}}f(\mathbf{x}) \tag{A3}$$

in which n is selected so that $\alpha-2n<0$ and the integral in (A2) exists. This is the analog to Eq. (12).

Some identities which prove useful for the calculations in Sec. III follow immediately:

$$i\boldsymbol{\sigma}\cdot\mathbf{y}\exp[-i\boldsymbol{\sigma}\cdot(\mathbf{u}-\mathbf{y})]=-\frac{\partial}{\partial \mathbf{u}}\cdot\mathbf{y}\exp[-i\boldsymbol{\sigma}\cdot(\mathbf{u}-\mathbf{y})] \ , \tag{A4}$$

$$|\,\boldsymbol{\sigma}\,|^{\,\gamma}\exp[-i\boldsymbol{\sigma}\cdot(\mathbf{u}-\mathbf{y})] =i^{\gamma}\left[\frac{\partial}{\partial \mathbf{u}}\cdot\frac{\partial^{\gamma-1}}{\partial \mathbf{u}^{\gamma-1}}\right]\exp[-i\boldsymbol{\sigma}\cdot(\mathbf{u}-\mathbf{y})] \ . \tag{A5}$$

APPENDIX B

Consider the solution of Eq. (93) in one dimension for simplicity,

$$i^{\gamma}\frac{b}{\beta^{\gamma}}\frac{d^{\gamma-1}}{dx^{\gamma-1}}p(x)+\frac{1}{\beta}K(x)p(x)=j_0 \ . \tag{B1}$$

It is convenient to write $p(x)$ as the product of two functions $\phi(x)$ and $\psi(x)$:

$$p(x)=\phi(x)\psi(x) \ , \tag{B2}$$

whose relationship is defined implicitly by the equation

$$-\frac{b}{\beta^{\gamma}}\phi(x)\frac{d^{\gamma-1}}{dx^{\gamma-1}}\psi(x)=j_0 \tag{B3}$$

so that

$$p(x)=-\frac{\beta^\gamma}{b}\phi(x)\frac{d^{1-\gamma}}{dx^{1-\gamma}}[j_0/\phi(x)] . \tag{B4}$$

These definitions will prove helpful in clarifying how to generalize, from the stable equation, the Kramers relation[1] for the special Gaussian case with a force law derivable from a potential. Specifically, if

$$\mathbf{K}(\mathbf{x})=-\frac{d}{d\mathbf{x}}V(\mathbf{x}) , \tag{B5}$$

then with $\gamma=2$ one can write (93) as

$$\frac{d}{d\mathbf{x}}\left[p(\mathbf{x})\exp\left[\frac{\beta V(\mathbf{x})}{b}\right]\right]=-\frac{\beta^2}{b}j_0\exp\left[\frac{\beta V(\mathbf{x})}{b}\right] . \tag{B6}$$

In one dimension this becomes

$$p(x)=-\frac{\beta^2}{b}\left[\exp-\left[\frac{\beta V(x)}{b}\right]\right] \times\int dx\, j_0\left[\exp\left[\frac{\beta V(x)}{b}\right]\right] , \tag{B7}$$

which is cast in the form of (B4) if written formally as

$$p(x)=-\frac{\beta^2}{b}\left[\exp\left[-\frac{\beta V(x)}{b}\right]\right]\frac{d^{-1}}{dx^{-1}} \times\left[j_0/\exp\left[-\frac{\beta V(x)}{b}\right]\right] . \tag{B8}$$

We shall see that $\phi(x)$ in the more generally stable case is not simply exponential, so that the expression for $p(x)$ for $1<\gamma<2$ will generalize both the order of the derivative and the functional dependence on the potential for the $\gamma=2$ normal case.

Transform from configuration to Fourier space with the representations

$$\psi(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\,\Psi(k)\exp(-ikx) , \tag{B9}$$

$$\phi(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\,\Phi(k)\exp(-ikx) , \tag{B10}$$

$$K(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\,Q(k)\exp(-ikx) . \tag{B11}$$

The relation (B3) between $\phi(x)$ and $\psi(x)$ becomes

$$-\frac{b}{\beta^\gamma}\int\int_{-\infty}^{\infty}dk\,dl\,(-ik)^{\gamma-1}\Phi(l-k)\Psi(k)\exp(-ilx) =4\pi^2 j_0 , \tag{B12}$$

while Eq. (B1) for $p(x)$ expressed with the decomposition (B2) is

$$\frac{ib}{\beta^\gamma}\int\int_{-\infty}^{\infty}dk\,dl\,l^{\gamma-1}\Phi(l-k)\Psi(k)\exp(-ilx) +\frac{1}{2\pi\beta}\int\int\int_{-\infty}^{\infty}dk\,dl\,d\sigma\,Q(l-\sigma)\Phi(\sigma-k)\Psi(k) \times\exp(-ilx)=4\pi^2 j_0 . \tag{B13}$$

With both integral expressions in (B12) and (B13) equal to $4\pi^2 j_0$ one has

$$\int\int_{-\infty}^{\infty}dk\,dl\left[\left[\frac{ib}{\beta^\gamma}l^{\gamma-1}+\frac{b}{\beta^\gamma}(-ik)^{\gamma-1}\right]\Phi(l-k) +\frac{1}{2\pi\beta}\int_{-\infty}^{\infty}d\sigma\,Q(l-\sigma)\Phi(\sigma-k)\right] \times\Psi(k)\exp(-ilx)=0 . \tag{B14}$$

A linear equation for Φ alone follows if one can set the square bracketed term to zero:

$$\left[\frac{ib}{\beta^\gamma}l^{\gamma-1}+\frac{b}{\beta^\gamma}(-ik)^{\gamma-1}\right]\Phi(l-k) =-\frac{1}{2\pi\beta}\int_{-\infty}^{\infty}d\sigma\,Q(l-\sigma)\Phi(\sigma-k) . \tag{B15}$$

Inverse Fourier transform (B15) for the equivalent configuration-space equation for $\phi(x)$. One finds after applying the operator

$$\int\int_{-\infty}^{\infty}dk\,dl\,\exp(-ikx-ily)$$

and with the use of Weyl fractional derivatives to express the noninteger powers of the transform variables that

$$\left[\frac{i^\gamma b}{\beta^\gamma}\frac{d^{\gamma-1}}{dy^{\gamma-1}}+\frac{b}{\beta^\gamma}\frac{d^{\gamma-1}}{dx^{\gamma-1}}\right]\delta(x+y)\phi(y) =-\frac{1}{\beta}\delta(x+y)K(y)\phi(y) . \tag{B16}$$

Integrate this over all x from $-\infty$ to $+\infty$ to get

$$\frac{i^\gamma b}{\beta^\gamma}\frac{d^{\gamma-1}}{dy^{\gamma-1}}\phi(y)+\frac{b}{\beta^\gamma}\phi(y)\left[\int_{-\infty}^{\infty}dz\frac{d^{\gamma-1}}{dz^{\gamma-1}}\delta(z)\right] =-\frac{1}{\beta}K(y)\phi(y) . \tag{B17}$$

For derivatives of integer n order, the Dirac δ function satisfies

$$\int_{-\infty}^{\infty}dz\,f(z)\frac{d^n}{dz^n}\delta(z)=(-1)^n\frac{d^n}{dz^n}f(z)\,|_{z=0} . \tag{B18}$$

With the same true for noninteger powers

$$\int_{-\infty}^{\infty}dz\frac{d^{\gamma-1}}{dz^{\gamma-1}}\delta(z)=0 \tag{B19}$$

if $\gamma>1$, as has already been assumed.

The equation for $\phi(x)$ is then found to be

$$\frac{i^\gamma b}{\beta^\gamma}\frac{d^{\gamma-1}}{dx^{\gamma-1}}\phi(x)=-\frac{1}{\beta}K(x)\phi(x) . \tag{B20}$$

Hence $\phi(x)$ solves the homogeneous equation for $p(x), j_0=0$ in (B1), for $1<\gamma\le 2$, a result previously known for the special Gaussian case.

One can cast (B20) in stable distribution form, the homogeneous version of (38), by a change of independent variable. Rewrite (B20) as

$$\frac{i^\gamma b}{\beta^{\gamma-1}}\phi(x)=-\frac{d^{1-\gamma}}{dx^{1-\gamma}}K(x)\phi(x)\ . \tag{B21}$$

The generalized chain rule for fractional derivatives[32] is

$$D^\alpha_{g(\rho)}f(\rho)=D^\alpha_{h(\rho)}\left[f(\rho)\frac{dg(\rho)}{d\rho}\left[\frac{dh(\rho)}{d\rho}\right]^{-1}\times 1\left[\frac{h(\rho)-h(\sigma)}{g(\rho)-g(\sigma)}\right]^{1+\alpha}\right]\Bigg|_{\sigma=\rho} \tag{B22}$$

in which $D^\alpha_{g(\rho)}$ is the fractional derivative of order α with respect to $g(\rho)$. With this, transform from x to ρ in (B21):

$$\frac{i^\gamma b}{\beta^{\gamma-1}}\phi(\rho)=-\frac{d^{1-\gamma}}{d\rho^{1-\gamma}}\left[K(\rho)\phi(\rho)\frac{dx}{d\rho}\left[\frac{\rho-\sigma}{x(\rho)-x(\sigma)}\right]^{2-\gamma}\right]\Bigg|_{\sigma=p}\ . \tag{B23}$$

This is of stable form in the new variable ρ,

$$i^\gamma\frac{b}{\beta^{\gamma-1}}\frac{d^{\gamma-1}}{d\rho^{\gamma-1}}\phi(\rho)=\rho\phi(\rho)\ , \tag{B24}$$

if one defines the relationship between x and ρ implicitly by

$$-\frac{dx}{d\rho}\left[\frac{\rho-\sigma}{x(\rho)-x(\sigma)}\right]^{2-\gamma}\Bigg|_{\sigma=\rho}K(\rho)=\rho\ . \tag{B25}$$

Take the $(1-\gamma)$th Riemann-Liouville fractional derivative with respect to ρ of both sides to find

$$-\frac{d^{1-\gamma}}{d\rho^{1-\gamma}}\left[K(\rho)\frac{dx}{d\rho}\left[\frac{\rho-\sigma}{x(\rho)-x(\sigma)}\right]^{2-\gamma}\right]\Bigg|_{\sigma=\rho}=\frac{1}{\Gamma(1+\gamma)}\rho^\gamma \tag{B26}$$

or, equivalently,

$$-\frac{d^{1-\gamma}}{dx^{1-\gamma}}K(x)=\frac{1}{\Gamma(1+\gamma)}\rho^\gamma\ . \tag{B27}$$

(In this last, the fractional derivative is of Riemann-Liouville type with lower limit of integration equal to zero, so that the derivative of ρ on the right-hand side exists as stated.)

If the force law is derivable from a potential (B5), then one finds explicitly for ρ

$$\rho(x)=\left[\Gamma(1+\gamma)\frac{d^{1-\gamma}}{dx^{1-\gamma}}\frac{d}{dx}V(x)\right]^{1/\gamma}\ . \tag{B28}$$

[1]S. Chandrasekhar, Rev. Mod. Phys. **15**, 1 (1943).
[2]J. L. Doob, Ann. Math. **43**, 351 (1942).
[3]P. Lévy, *Calcul des Probabilites* (Gauthier-Villars, Paris, 1925).
[4]B. V. Gnedenko and A. N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables* (Addison-Wesley, Reading, Mass., 1954).
[5]D. R. Holt and E. L. Crow, J. Res. Natl. Bur. Stands., Sec. B **77**, 143 (1973).
[6]See W. Feller, *Introduction to Probability Theory and Its Applications* (Wiley, New York, 1966).
[7]J. L. Doob, Trans. Am. Math. Soc. **42**, 107 (1937).
[8]P. Lévy, *Theorie de l'Addition des Variables Aleatoires* (Gauthier-Villars, Paris, 1937).
[9]B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, San Francisco, 1983).
[10]V. Franchescini, Phys. Fluids **26**, 433 (1983).
[11]R. G. Deissler, Rev. Mod. Phys. **56**, 223 (1984).
[12]V. Seshadri and B. J. West, Proc. Natl. Acad. Sci. U.S.A. **79**, 4501 (1982).
[13]J. Fourier, *The Analytical Theory of Heat* (Dover, New York, 1955).
[14]J. L. Lavoie, T. J. Osler, and R. Trembley, SIAM (Soc. Ind. Appl. Math.) Rev. **18**, 240 (1976).
[15]K. S. Miller, in *Fractional Calculus and Its Applications,* Vol. 457 of *Lecture Notes in Mathematics,* edited by B. Ross (Springer-Verlag, New York, 1975).
[16]B. J. West and V. Seshadri, Physica **113A**, 203 (1982).
[17]G. E. Uhlenbeck and L. S. Ornstein, Phys. Rev. **36**, 823 (1930).
[18]M. V. Smoluchowski, Ann. Phys. (Leipzig) **48**, 1103 (1915).
[19]M. F. Shlesinger, J. Klafter, and B. J. West, Physica **140A**, 212 (1986).
[20]L. F. Richardson, Proc. R. Soc. London, Ser. A **110**, 709 (1926).
[21]H. G. E. Hentschel and I. Procaccia, Phys. Rev. A **29**, 1461 (1984).
[22]F. Peseckis (unpublished).
[23]G. K. Batchelor and A. A. Townsend, Proc. R. Soc. London, Ser. A **190**, 534 (1947).
[24]G. K. Batchelor and A. A. Townsend, Proc. R. Soc. London, Ser. A **199**, 239 (1949).
[25]A. Y. Kuo and S. Corrsin, J. Fluid Mech. **50**, 285 (1971).
[26]J. C. Wyngaard and H. Tennekes, Phys. Fluids **13**, 1962 (1970).
[27]P. C. Martin, E. D. Siggia, and H. A. Rose, Phys. Rev. A **8**, 423 (1973).
[28]See discussion of Kramers-Moyal expansion in H. V. Dop, F. T. M. Nieuwstadt, and J. C. R. Hunt, Phys. Fluids **28**, 1693 (1985).
[29]P. Frenzen (private communication), cited in Ref. 28.
[30]F. Peseckis, NASA/Langley Turbulence Conference, Hampton, Virginia, 1982 (unpublished).
[31]E. W. Montroll and M. F. Shlesinger, in *Studies in Statistical Mechanics,* edited by J. L. Lebowitz and E. N. Montroll (North-Holland, Amsterdam, 1984), Vol. 11.
[32]T. J. Osler, SIAM J. Math. Anal. **1**, 288 (1970).