Quantum theory of a nonresonant two-mode laser with coupled transitions

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The master equation for a two-mode laser in a Λ-type three-level atomic system with arbitrary detunings has been derived. The effects of the detunings on the laser operation have been discussed. The asymmetry in threshold conditions for the two modes and the anomalous mutual support between the two modes in certain circumstances have been revealed. The former is due to the two-photon Raman-type resonance and the latter is attributed to the ac-Stark-shift effect.

I. INTRODUCTION

The two-mode laser is a simple and useful example in the study of multimode lasers, in which different modes compete with one another for contributions, resulting from the same excited-level occupation as well as Raman-type two-photon transitions. In the two-mode laser, the two radiation-field modes from stimulated emission of the same atomic level couple to each other and thus one mode affects the operation character of the other to varying degrees.

The two-mode laser in a gain medium consisting of three-level atoms has been investigated recently by some authors[1-4] within the framework of Lamb's quantum theory,[5,6] which is capable of taking into account the quantum nature of the electromagnetic field such as spontaneous emission, photon statistics, and intrinsic linewidth, etc. However, in Refs. 1–4, the detunings between cavity modes and atomic transitions were neglected and subsequently some interesting phenomena resulting from the detunings has been lost. In this paper we investigate the effects of the detunings on the two-mode laser operation in a homogeneously broadened medium composed of three-level atoms with a common upper level, through a generalization of the Scully and Lamb treatment for a single-mode laser.[5,6] Starting from the Schrödinger equation for the atom-field system, the master equation for the reduced density matrix of the two-mode field is obtained. With the aid of the master equation we analyze the threshold conditions and photon statistics by using numerical methods, to see the effects of the detunings on them. Novel phenomena is found that for certain values of the detunings the two modes can support each other rather than compete, because of the presence of the complicated ac-Stark effect.

In Sec. II we derive the master equation for the two-mode laser action. In Sec. III the steady-state properties of this laser are discussed. Section IV gives a summary.

II. MASTER EQUATION

The atomic model for the gain medium is shown in Fig. 1. The atoms have three levels, of which $|a\rangle$ is the upper level, while $|b\rangle$ and $|c\rangle$ are the two lower levels. The transition between $|a\rangle$ and $|b\rangle$ (or $|c\rangle$) is mediated by mode 1 (or 2) with frequency Ω_1 (or Ω_2). The transition between $|b\rangle$ and $|c\rangle$ is forbidden. The energy eigenvalues of levels $|a\rangle$, $|b\rangle$, and $|c\rangle$ are $\hbar\omega_a$, $\hbar\omega_b$, and $\hbar\omega_c$, respectively, and the same mean decay rate γ for all three levels is assumed.[1,2,6]

The Hamiltonian for the atom-field system is[1,2]

$$H=\hbar(H_0+V)\ , \tag{1}$$

$$H_0=\sum_{\alpha=a,b,c}\omega_\alpha A_\alpha^\dagger A_\alpha+\sum_{j=1,2}\Omega_j(a_j^\dagger a_j+\tfrac{1}{2})\ , \tag{2}$$

$$V=g_1a_1A_a^\dagger A_b+g_2a_2A_a^\dagger A_c+\text{H.c.}\ , \tag{3}$$

where the rotating-wave approximation has been made; $a_j^\dagger$ (a_j) is the creation (annihilation) operator for the jth cavity mode; $A_\alpha^\dagger$ (A_α) are those for level $|\alpha\rangle$, and g_j is the atom-field coupling constant.

In the interaction picture, the perturbation V becomes

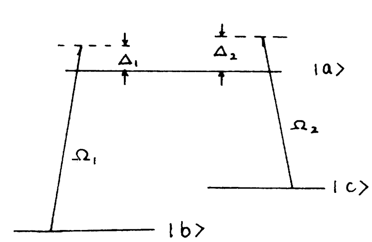

FIG. 1. Three-level atomic system.

$$V^I=g_1 e^{-i\Delta_1 t} a_1 A_a^\dagger A_b + g_2 e^{-i\Delta_2 t} a_2 A_a^\dagger A_c + \text{H.c.} \; , \tag{4}$$

where $\Delta_1=\Omega_1-(\omega_a-\omega_b)$ and $\Delta_2=\Omega_2-(\omega_a-\omega_c)$ are two detunings between free atoms and mode 1 and mode 2. For simplicity, in this paper only a pumping to the upper level $|a\rangle$ with pumping rate R_a is considered. So the state vector at time t_0 may be written as

$$|\psi(t_0)\rangle = \sum_{n_1,n_2} F_{n_1,n_2}(t_0)\,|a,n_1,n_2\rangle \; . \tag{5}$$

At time $t=t_0+\tau$, Eq. (5) develops into

$$\begin{aligned}|\psi(t)\rangle &= |\psi(t_0+\tau)\rangle \\ &= \sum_{n_1,n_2} [a_{n_1,n_2}(t_0+\tau)\,|a,n_1,n_2\rangle \\ &\quad + b_{n_1+1,n_2}(t_0+\tau)\,|b,n_1+1,n_2\rangle \\ &\quad + c_{n_1,n_2+1}(t_0+\tau)\,|c,n_1,n_2+1\rangle] \; .\end{aligned} \tag{6}$$

The development of the state vector obeys the Schrödinger equation,

$$\frac{d}{dt}|\psi(t)\rangle = -iV^I|\psi(t)\rangle \; . \tag{7}$$

Substituting Eq. (6) into Eq. (7), we obtain

$$i\frac{d}{dt}a_{n_1,n_2}(t) = V_1 e^{-i\Delta_1 t} b_{n_1+1,n_2}(t) + V_2 e^{-i\Delta_2 t} c_{n_1,n_2+1}(t) \; , \tag{8a}$$

$$i\frac{d}{dt}b_{n_1+1,n_2}(t) = V_1 e^{i\Delta_1 t} a_{n_1,n_2}(t) \; , \tag{8b}$$

$$i\frac{d}{dt}c_{n_1,n_2+1}(t) = V_2 e^{i\Delta_2 t} a_{n_1,n_2}(t) \; , \tag{8c}$$

where

$$V_j = g_j\sqrt{n_j+1} \quad (j=1,2) \; . \tag{9}$$

Equations (8) are solved (see Appendix A) to obtain

$$a_{n_1,n_2}(t) = a^{(1)}_{n_1,n_2} e^{-i\omega_1(t-t_0)} + a^{(2)}_{n_1,n_2} e^{-i\omega_2(t-t_0)} + a^{(3)}_{n_1,n_2} e^{-i\omega_3(t-t_0)} \; , \tag{10a}$$

$$b_{n_1+1,n_2}(t) = b^{(1)}_{n_1+1,n_2} e^{-i(\omega_1-\Delta_1)(t-t_0)} + b^{(2)}_{n_1+1,n_2} e^{-i(\omega_2-\Delta_1)(t-t_0)} + b^{(3)}_{n_1+1,n_2} e^{-i(\omega_3-\Delta_1)(t-t_0)} \; , \tag{10b}$$

$$c_{n_1,n_2+1}(t) = c^{(1)}_{n_1,n_2+1} e^{-i(\omega_1-\Delta_2)(t-t_0)} + c^{(2)}_{n_1,n_2+1} e^{-i(\omega_2-\Delta_2)(t-t_0)} + c^{(3)}_{n_1,n_2+1} e^{-i(\omega_3-\Delta_2)(t-t_0)} \; , \tag{10c}$$

where ω_1, ω_2, and ω_3 are the three roots of the following algebraic equation:

$$\omega^3-(\Delta_1+\Delta_2)\omega^2+(\Delta_1\Delta_2-V_1^2-V_2^2)\omega+V_1^2\Delta_2+V_2^2\Delta_1=0 \; . \tag{11}$$

Letting the two sides of Eq. (11) be divided by γ^3, we have

$$\mu^3-(\delta_1+\delta_2)\mu^2+(\delta_1\delta_2-\bar{V}_1^2-\bar{V}_2^2)\mu+\bar{V}_1^2\delta_2+\bar{V}_2^2\delta_1=0 \; , \tag{11a}$$

where

$$\mu=\omega/\gamma, \quad \delta_j=\Delta_j/\gamma, \quad \bar{V}_j=\frac{g_j}{\gamma}\sqrt{n_j+1} \quad (j=1,2) \; .$$

The three roots of Eq. (11a) are μ_1, μ_2, and μ_3.

Let ρ_f and ρ represent the density matrices of the field (including mode 1 and mode 2) and of the total system including the atoms and the field; $\rho_{n_1,n_2;m_1,m_2}$ and $\rho_{\alpha,n_1,n_2;\beta,m_1,m_2}$ are their elements, respectively.

The change of ρ_f is caused by (1) pumping to level $|a\rangle$ and (2) cavity losses,

$$\dot{\rho}_f=\dot{\rho}_f^a+\dot{\rho}_f^l \; . \tag{12}$$

Following the procedure of Refs. 2 and 6, we have

$$\dot{\rho}^a_{n_1,n_2;m_1,m_2}(t_0) = R_a\int_0^\infty d\tau \left[\sum_\beta \rho^a_{\beta,n_1,n_2;\beta,m_1,m_2}(t_0+\tau) - \rho_{n_1,n_2;m_1,m_2}(t_0)\right]\gamma e^{-\gamma\tau} \; . \tag{13}$$

With the aid of Eqs. (A18)–(A24) and after integration, we obtain the following equations for the diagonal elements:

$$\dot{\rho}^{a}_{n_1,n_2;n_1,n_2}(t_0)=-2R_a\left[\frac{(\mu_1-\mu_2)^2}{(\mu_1-\mu_2)^2+1}A^{(1)}_{n_1,n_2}A^{(2)}_{n_1,n_2}+\frac{(\mu_1-\mu_3)^2}{(\mu_1-\mu_3)^2+1}A^{(1)}_{n_1,n_2}A^{(3)}_{n_1,n_2}+\frac{(\mu_2-\mu_3)^2}{(\mu_2-\mu_3)^2+1}A^{(2)}_{n_1,n_2}A^{(3)}_{n_1,n_2}\right]\rho_{n_1,n_2;n_1,n_2}(t_0)$$
$$-2R_a\left[\frac{(\mu_1'-\mu_2')^2}{(\mu_1'-\mu_2')^2+1}B^{(1)}_{n_1,n_2}B^{(2)}_{n_1,n_2}+\frac{(\mu_1'-\mu_3')^2}{(\mu_1'-\mu_3')^2+1}B^{(1)}_{n_1,n_2}B^{(3)}_{n_1,n_2}+\frac{(\mu_2'-\mu_3')^2}{(\mu_2'-\mu_3)^2+1}B^{(2)}_{n_1,n_2}B^{(3)}_{n_1,n_2}\right]\rho_{n_1-1,n_2;n_1-1,n_2}(t_0)$$
$$-2R_a\left[\frac{(\mu_1''-\mu_2'')^2}{(\mu_1''-\mu_2'')^2+1}C^{(1)}_{n_1,n_2}C^{(2)}_{n_1,n_2}+\frac{(\mu_1''-\mu_3'')^2}{(\mu_1''-\mu_3'')^2+1}C^{(1)}_{n_1,n_2}C^{(3)}_{n_1,n_2}+\frac{(\mu_2''-\mu_3'')^2}{(\mu_2''-\mu_3'')^2+1}C^{(2)}_{n_1,n_2}C^{(3)}_{n_1,n_2}\right]\rho_{n_1,n_2-1;n_1,n_2-1}(t_0)\ , \tag{14}$$

where $\rho_{n_1,n_2;n_1,n_2}(t_0)=F_{n_1,n_2}(t_0)F^{*}_{n_1,n_2}(t_0)$ has been used and $A^{(i)}_{n_1,n_2}$, $B^{(i)}_{n_1,n_2}$, $C^{(i)}_{n_1,n_2}$ are defined in Eqs. (A22)–(A24). By using Eqs. (B1) and (B4)–(B8), the change of the diagonal matrix elements of the field caused by the pumping to the level $|a\rangle$ is obtained,

$$\dot{\rho}^{a}_{n_1,n_2;n_1,n_2}(t_0)=-2R_a\frac{g_1^2}{\gamma^2}(n_1+1)F_1(n_1,n_2)p(n_1,n_2)-2R_a\frac{g_2^2}{\gamma^2}(n_2+1)F_2(n_1,n_2)p(n_1,n_2)$$
$$+2R_a\frac{g_1^2}{\gamma^2}n_1F_1(n_1-1,n_2)p(n_1-1,n_2)+2R_a\frac{g_2^2}{\gamma^2}n_2F_2(n_1,n_2-1)p(n_1,n_2-1)\ , \tag{15}$$

where

$$F_1(n_1,n_2)=\frac{(\mu_1-\delta_2)(\mu_2-\delta_2)}{(\mu_1-\mu_3)(\mu_2-\mu_3)[(\mu_1-\mu_2)^2+1]}+\frac{(\mu_1-\delta_2)(\mu_3-\delta_2)}{(\mu_3-\mu_2)(\mu_1-\mu_2)[(\mu_3-\mu_1)^2+1]}+\frac{(\mu_2-\delta_2)(\mu_3-\delta_2)}{(\mu_3-\mu_1)(\mu_2-\mu_1)[(\mu_2-\mu_3)^2+1]}\ , \tag{16}$$

and $F_2(n_1,n_2)$ is the same as $F_1(n_1,n_2)$ with δ_1 and δ_2 interchanged; $F_1(n_1-1,n_2)$ or $F_2(n_1,n_2-1)$ is the same as $F_1(n_1,n_2)$ or $F_2(n_1,n_2)$ with n_1 or n_2 replaced by (n_1-1) or (n_2-1) for all quantities appeared in $F_1(n_1,n_2)$ or $F_2(n_1,n_2)$. Here we must notice that it is through $\bar{V}_j^2=g_j^2/\gamma^2(n_j+1)$ that $F_j(n_1,n_2)$ depends on n_1 and n_2.

Therefore, the master equation of the two-mode laser in a Λ-type three-level atomic system with arbitrary detunings is obtained,

$$\dot{p}(n_1,n_2)=-A_1(n_1+1)F_1(n_1,n_2)p(n_1,n_2)-A_2(n_2+1)F_2(n_1,n_2)p(n_1,n_2)+A_1n_1F_1(n_1-1,n_2)p(n_1-1,n_2)$$
$$+A_2n_2F_2(n_1,n_2-1)p(n_1,n_2-1)+C_1(n_1+1)p(n_1+1,n_2)+C_2(n_2+1)p(n_1,n_2+1)$$
$$-C_1n_1p(n_1,n_2)-C_2n_2p(n_1n_2)\ , \tag{17}$$

where

$$p(n_1,n_2)=\rho_{n_1,n_2;n_1,n_2}(t_0),\quad A_j=2R_ag_j^2/\gamma^2\quad (j=1,2) \tag{18}$$

and the cavity losses have been included in the usual way through,

$$\dot{p}^{\,l}(n_1,n_2)=C_1(n_1+1)p(n_1+1,n_2)+C_2(n_2+1)p(n_1,n_2+1)-C_1n_1p(n_1,n_2)-c_2n_2p(n_1,n_2)\ .$$

The physical meaning of Eq. (17) is very clear. The terms on the right-hand side, as usual, can be interpreted as probability flows which can be expressed by arrows in a two-dimensional probability-flow diagram as done in Refs. 1 and 2.

From the master equation the following equations can be obtained:

$$\dot{p}(n_1)=\sum_{n_2}\dot{p}(n_1,n_2)$$
$$=-A_1(n_1+1)\sum_{n_2}F_1(n_1,n_2)p(n_1,n_2)+A_1n_1\sum_{n_2}F_1(n_1+n_2)p(n_1-1,n_2)+C_1(n_1+1)p(n_1+1)-C_1n_1p(n_1)\ , \tag{19}$$

$$\dot{p}(n_2)=\sum_{n_1}\dot{p}(n_1,n_2)$$
$$=-A_2(n_2+1)\sum_{n_1}F_2(n_1,n_2)p(n_1,n_2)$$
$$+A_2n_2\sum_{n_2}F_2(n_1,n_2-1)p(n_1,n_2-1)$$
$$+C_2(n_2+1)p(n_2+1)-C_2n_1p(n_2)\ . \qquad (20)$$

These two equations can be interpreted by using two one-dimensional probability-flow diagrams as done in Ref. 2.

From Eqs. (19) and (20), the equations for the average photon number, $\langle n_i\rangle=\sum_{n_1,n_2}n_ip(n_1,n_2)$, can be deduced,

$$\langle\dot{n}_1\rangle=A_1\sum_{n_1,n_2}n_1F_1(n_1-1,n_2-1)\times p(n_1-1,n_2-1)-C_1\langle n_1\rangle\ , \qquad (21)$$

$$\langle\dot{n}_2\rangle=A_2\sum_{n_1,n_2}n_2F_2(n_1-1,n_2-1)\times p(n_1-1,n_2-1)-C_2\langle n_2\rangle\ . \qquad (22)$$

If we neglect the correlations,[1] i.e., we assume

$$\langle n_1^{r_1}n_2^{r_2}\rangle=\langle n_1\rangle^{r_1}\langle n_2\rangle^{r_2}\ ,$$

then Eqs. (21) and (22) read

$$\langle\dot{n}_1\rangle=A_1\langle n_1\rangle F_1(\langle n_1-1\rangle,\langle n_2-1\rangle)-C_1\langle n_1\rangle\ , \qquad (23)$$

$$\langle\dot{n}_2\rangle=A_2\langle n_2\rangle F_2(\langle n_1-1,\langle n_2-1\rangle)-C_2\langle n_2\rangle\ , \qquad (24)$$

where $\langle n_i\rangle\gg1$ has been assumed. Expand $F_i(\langle n_1-1\rangle,\langle n_2-1\rangle)$ into Taylor series and only keep the first and the second terms. Then we have

$$F_1(\langle n_1-1\rangle,\langle n_2-1\rangle)=\frac{1}{1+\delta_1^2}+\beta_1'\langle n_1\rangle+\theta_{12}'\langle n_2\rangle\ , \qquad (25)$$

$$F_2(\langle n_1-1\rangle,\langle n_2-1\rangle)=\frac{1}{1+\delta_2^2}+\beta_2'\langle n_2\rangle+\theta_{21}'\langle n_2\rangle\ , \qquad (26)$$

where β_1', β_2', θ_{12}', and θ_{21}' are constants and $F_i(\langle n_1-1\rangle,\langle n_2-1\rangle)\,|_{n_1=n_2=0}=1/(1+\delta_2^2)$, which is easy to prove, has been used. Substituting Eqs. (25) and (26) into Eqs. (21) and (22), we have

$$\dot{I}_1=\left[\frac{A_1}{1+\delta_1^2}-C_1\right]I_1+(\beta_1I_1+\theta_{12}I_2)I_1\ , \qquad (27)$$

$$\dot{I}_2=\left[\frac{A_2}{1+\delta_2^2}-C_2\right]I_2+(\beta_2I_2+\theta_{21}I_1)I_2\ , \qquad (28)$$

where the correspondence $I_i\leftrightarrow\langle n_i\rangle$ has been made and β_1, β_2, θ_{12}, and θ_{21} are constants. The two equations are consistent with the equations deduced from the semiclassical theory.[7] It is apparent that the semiclassical-theory results[7] can be viewed as the results of quantum theory under the conditions of $g_i^2\langle n_i\rangle/\gamma^2\ll1$ and decorrelation approximation.

III. STEADY-STATE OPERATION

In the steady state, $p(n_1)$ and $p(n_2)$ are independent of time, $\dot{p}(n_1)=\dot{p}(n_2)=0$. From Eqs. (19) and (20) and the principle of detailed balance, two equations for the steady-state operation which are equivalent to Eqs. (19) and (20) can been obtained,

$$C_1p(n_1)=A_1\sum_{n_2}F_1(n_1-1,n_2)p(n_1-1,n_2)\ , \qquad (29)$$

$$C_2p(n_2)=A_2\sum_{n_1}F_2(n_1,n_2-1)p(n_1,n_2-1)\ . \qquad (30)$$

These two equations usually cannot be simplified further except for some special cases, and they are the basis of our following discussion. We emphasize effects of the detunings in the discussion.

For the resonant situation, $\delta_1=\delta_2=0$, the three roots of Eq. (11a) become

$$\mu_1=0,\ \ \mu_2=-\mu_3=(\bar{V}_1^2+\bar{V}_2^2)^{1/2}\ ,$$

and then

$$F_1(n_1,n_2)=F_2(n_1,n_2)=\left[1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}(n_2+1)\right]^{-1}\ ,$$

where

$$B_j=4\frac{g_j^2}{\gamma^2}A_j$$

is the self-saturation coefficient for the mode j. Thus Eq. (17) is reduced to the previous result, Eq. (33) of Ref. 2 or Eq. (19) of Ref. 1, and Eqs. (29) and (30) are reduced to

$$C_1p(n_1)=A_1\sum_{n_2}\frac{p(n_1-1,n_2)}{1+\frac{B_1}{A_1}n_1+\frac{B_2}{A_2}(n_2+1)}\ , \qquad (31)$$

$$C_2p(n_2)=A_2\sum_{n_1}\frac{p(n_1,n_2-1)}{1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}n_2}\ . \qquad (32)$$

A. The case of equal detunings

For equal detunings $\Delta_1=\Delta_2=\Delta$, the three roots of Eq. (11) are

$$\omega_1=0,\omega_{2,3}=\tfrac{1}{2}\{\Delta\pm[\Delta^2+4(V_1^2+V_2^2)]^{1/2}\}$$

and then

$$F_1(n_1,n_2)=F_2(n_1,n_2)=\frac{1}{1+\frac{\Delta^2}{\gamma^2}+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}(n_2+1)}\ . \qquad (33)$$

Let

$$|g_{je}|^2 = |g_j|^2 \frac{\gamma^2}{\gamma^2+\Delta^2} = |g_j|^2 \mathcal{L} ,$$

$$A_{je} = A_j \mathcal{L} , \tag{34}$$

$$B_{je} = B_j \mathcal{L}^2 .$$

Substituting Eq. (34) into Eq. (33) then into Eqs. (29) and (30), we have

$$C_1 p(n_1) = A_{1e} \sum_{n_2} \frac{p(n_1-1,n_2)}{1+\frac{B_{1e}}{A_{1e}}n_1+\frac{B_{2e}}{A_{2e}}(n_2+1)} , \tag{35}$$

$$C_2 p(n_2) = A_{2e} \sum_{n_1} \frac{p(n_1,n_2-1)}{1+\frac{B_{1e}}{A_{1e}}(n_1+1)+\frac{B_{2e}}{A_{2e}}n_2} . \tag{36}$$

Equations (35) and (36) are the same as Eqs. (31) and (32) with the replacement of A_j and B_j by their effective quantities A_{je} and B_{je}. Therefore, all the discussion made in Ref. 1 and 2 is valid under this replacement.

The threshold conditions can formally be expressed as,[2] by using $p(0)=p(1)$,

$$A_1 = C_1 \left[1+\frac{B_2}{A_2}\mathcal{L}H_2(0)\langle n_2\rangle \right] \mathcal{L}^{-1} \quad \text{for mode 1} \tag{37}$$

$$A_2 = C_2 \left[1+\frac{B_1}{A_1}\mathcal{L}H_1(0)\langle n_1\rangle \right] \mathcal{L}^{-1} \quad \text{for mode 2} , \tag{38}$$

where $H_1(0)$ and $H_2(0)$ are constants, and Eqs. (34) have been used.

The threshold is almost $(\gamma^2+\Delta^2)/\gamma^2$ times higher than that at resonance. If $A_1/C_1 = A_2/C_2 = A/C$ and $B_1/A_1 = B_2/A_2 = B/A$, we have

$$\langle n_1\rangle + \langle n_2\rangle = \frac{A}{B}\left[\frac{A}{C} - 1 - \frac{\Delta^2}{\gamma^2} \right] .$$

Near the threshold, the effect of the detunings on $\langle n_1\rangle + \langle n_2\rangle$ is very significant, while as the pumping rate rises it goes down. If the pumping rate is high enough, $A/C \gg 1+\Delta^2/\gamma^2$ and then $\langle n_1\rangle + \langle n_2\rangle = A^2/BC$, the detunings have almost no effect on $\langle n_1\rangle + \langle n_2\rangle$.

The effective gain coefficient A_{je}, which reflects the linear response of the atoms to the light, is $\mathcal{L}^{-1}$ times less than A_j, while the effective self-saturation coefficient B_{je}, which reflects the nonlinear response of the atoms to the light, is $\mathcal{L}^{-2}$ times less than B_j. Near the threshold, the gain coefficient dominates, hence the detunings have great effect. As the pumping rises, the saturation coefficient increases its influence, and then the effect of the detunings coming from the saturation coefficient offsets that coming from the gain coefficient. Consequently, the net effect of the detuning falls down as the pumping rate rises.

B. Unequal detunings $\delta_1 \neq \delta_2$

In the case of $\delta_1 \neq \delta_2$, the three roots of Eq. (11) expressed by Eq. (A5) and $F_j(n_1,n_2)$ cannot be simplified further, nor can Eqs. (29) and (30). However, we can use two H parameters to write Eqs. (29) and (30) formally as[2]

$$C_1 p(n_1) = A_1 F_1(n_1-1, H_2(n_1)\langle n_2\rangle) p(n_1-1) , \tag{39}$$

$$C_2 p(n_2) = A_2 F_2(H_1(n_2)\langle n_1\rangle, n_2) p(n_2-1) , \tag{40}$$

where $H_1(n_2)$ and $H_2(n_1)$ are the two parameters, and depend on n_1 and n_2, respectively. With the aid of these two equations, we can study some properties of the laser operation, such as the relation of the threshold of the mode 1 (or 2) to δ_1,δ_2 and $\langle n_2\rangle$ (or $\langle n_1\rangle$) and photon statistics.

1. Threshold for mode 1

The threshold can be obtained from Eq. (39) by using the condition $p(1)=p(0)$,

$$T_1 = \frac{A_1}{C_1} = 1/F_1(0, H_2(0)\langle n_2\rangle)$$

$$= 1/F_1(0, \langle n_2\rangle_e) \quad \text{for mode 1}$$

where $\langle n_2\rangle_e = \frac{1}{4}(B_2/A_2)H_2(0)\langle n_2\rangle$ and $H_2(0)$ is a constant. Since F_1 is a function of δ_1, δ_2, and $\langle n_2\rangle_e$, the threshold is related to δ_1, δ_2, and $\langle n_2\rangle_e$. We discuss the changes of the threshold with respect to δ_1, δ_2, and $\langle n_2\rangle_e$.

The thresholds for mode 1 versus δ_2 at different values of δ_1 are shown in Fig. 2. The threshold curves are asymmetric when $\delta_1 \neq 0$, and symmetric when $\delta_1 = 0$. There is a maximum at $\delta_1 = \delta_2$, where the influence of mode 2 on mode 1 is strongest because the condition of two-photon Raman-type resonance is satisfied. As $\delta_2 \to \infty$, the influence falls away gradually, so the threshold approaches its minimum.

The threshold for mode 1 versus δ_1 with different values of δ_2 are shown in Fig. 3. It is noticed that there is an extreme maximum value at $\delta_1 = \delta_2$ and the minimum is not at $\delta_1 = 0$. If there were no mode 2, the curve of the threshold for mode 1 versus δ_1 would be hyperboliclike with a minimum at $\delta_1 = 0$ and would have no maximum. Now there exists mode 2; its influence on mode 1 is strongest at $\delta_1 = \delta_2$ because of the two-photon Raman-type resonance, which brings about an extreme maximum or a protrusion on the hyperboliclike curve and shifts the

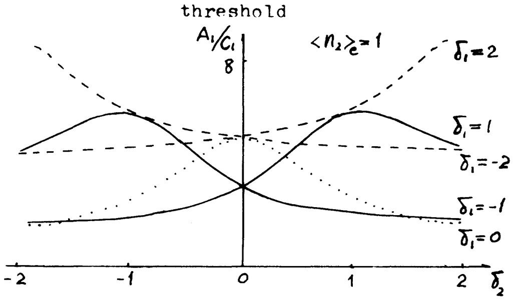

FIG. 2. Threshold for mode 1 vs δ_2 at different values of δ_1.

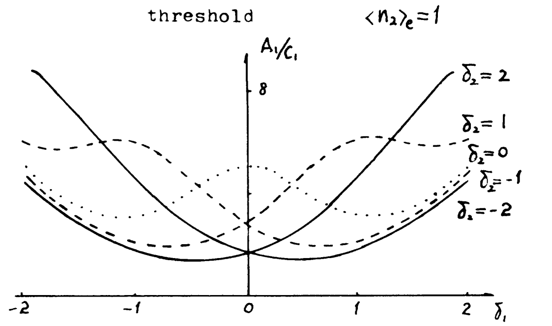

FIG. 3. Threshold for mode 1 vs δ_1 at different values of δ_2.

minimum from $\delta_1=0$ to $\delta_1\neq0$. At minimum threshold, δ_1 and δ_2 have opposite signs.

The threshold for mode 1 versus $\langle n_2\rangle_e$ are shown in Figs. 4(a) and 4(b) at different δ_1 and δ_2. It is very obvious that when δ_1 and δ_2 have the same sign, the threshold for mode 1 usually increases with the increase of $\langle n_2\rangle_e$. For the case of the opposite signs, sometimes it decreases with the increase of $\langle n_2\rangle_e$, such as the case of $\delta_1=2$, $\delta_2=-1$. Such a feature is also very distinct for the change of the photon statistics of mode 1 against $\langle n_2\rangle_e$. The reason for this strange feature is discussed below.

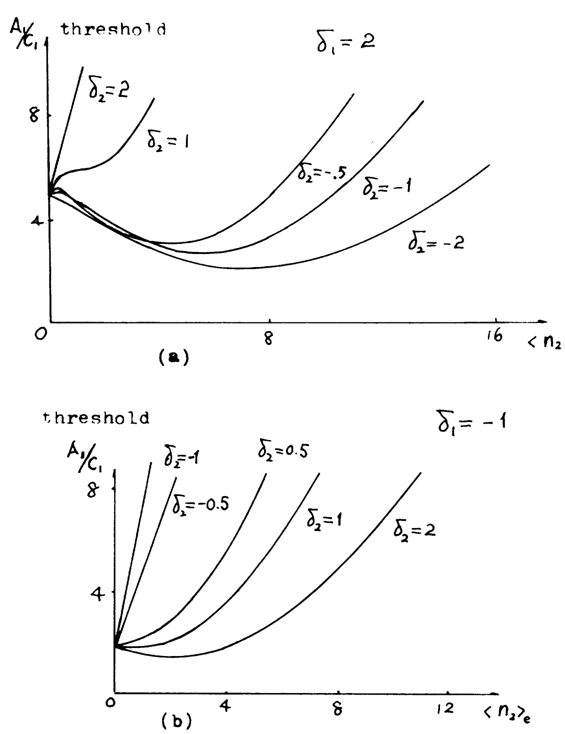

FIG. 4. Threshold for mode 1 vs $\langle n_2\rangle_e$, (a) $\delta_1=2$, (b) $\delta_1=-1$.

2. Photon statistics of mode 1

The photon statistics of mode 1 can approximately be obtained from Eq. (39). The region where $p(n_1)$ has significant value is around $\langle n_1\rangle$ and very small. In this region the change of $H_2(n_1)$ with n_1 is small and can be neglected, so that $H_2(n_1)$ can be treated as a constant, $H_2=H_2(\langle n_2\rangle)$. Apart from this region, $p(n_1)$ is quite small. Therefore, litter error will be brought in, if $H_2(n_1)$ is replaced by the constant H_2 in Eq. (39). For a fixed value of $\langle n_2\rangle_e=\frac{1}{4}(B_2/A_2)H_2(\langle n_1\rangle)\langle n_2\rangle$, all values of $p(n_1)$ can be obtained from Eq. (39).

Figures 5 and 6 present the curve of photon-statistical distribution for mode 1 versus $\langle n_2\rangle_e$ (the change of $\langle n_2\rangle_e$ can be realized by varying C_2). For δ_1 and δ_2 having the same sign (Fig. 5), the peak position of photon-statistical distribution goes rapidly to zero, when $\langle n_2\rangle_e$ increases. For δ_1 and δ_2 having the opposite signs (Fig. 6), the peak position of the photon-statistical distribution first increases and then decreases, when $\langle n_2\rangle_e$ increases. Such a phenomenon, which seems a little strange, can be viewed as a joint effect of ac-Stark shift and competition between mode 1 and mode 2.

According to the perturbation and experiment results,[8] the ac-Stark shift is proportional to the light strength and inversely proportional to the detuning between atom and the light field. When δ_1 (or δ_2) is positive the shift 1 (or 2) of the upper level which is caused by mode 1 (or 2) is negative, while δ_1 (or δ_2) is negative, the shift 1 (or 2) is positive. With $\langle n_2\rangle_e$ (or $\langle n_1\rangle_e$) increasing, the ac-Stark shift 2 (or 1) increases, too. When δ_1 and δ_2 have the same sign, shifts 1 and 2 have the same sign; while δ_1 and δ_2 have opposite signs, shifts 1 and 2 also have opposite ones. If δ_1 and δ_2 have opposite signs, shift 2 reduces the detuning which mode 1 really sees from the atoms in-

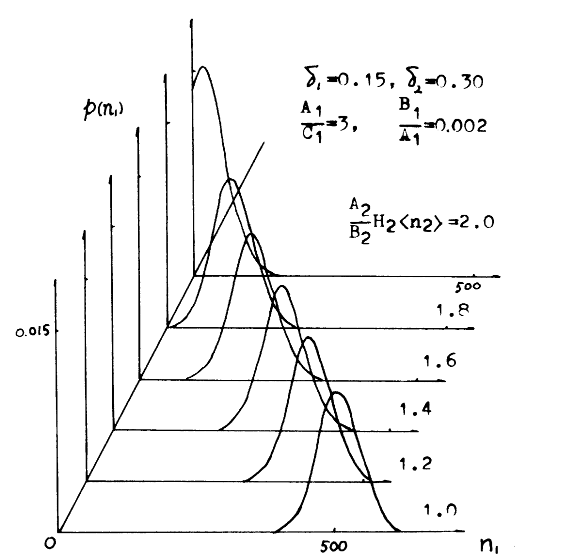

FIG. 5. Photon statistical distribution for mode 1 where the detunings have the same sign.

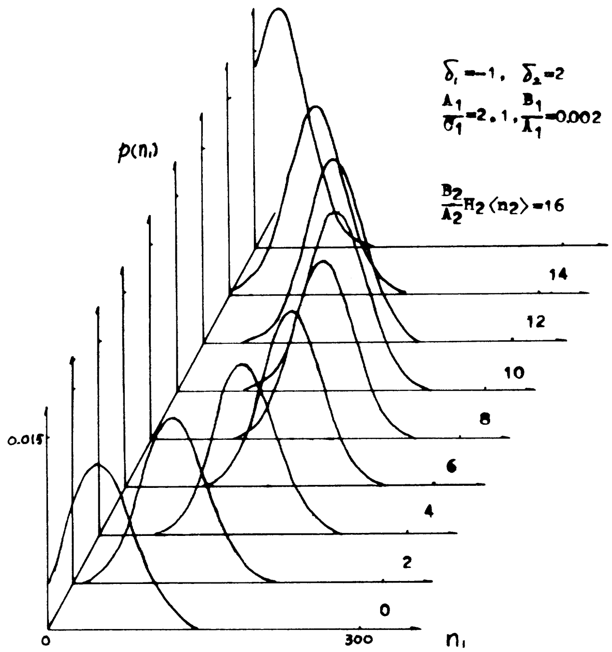

FIG. 6. Photon statistical distribution for mode 1 where the detunings have opposite signs.

teracted with mode 2. If δ_1 and δ_2 have the same sign, shift 2 widens the detuning which mode 1 really sees. Smaller detuning for mode 1 brings about a larger average photon number of mode 1, while larger detuning brings about a smaller average photon number. Besides, an increase of $\langle n_2 \rangle_e$ will reduce the average photon number of mode 1 because of the competition of mode 2 against mode 1 for the population of the upper level.

Therefore, the increase of $\langle n_2 \rangle_e$ has two actions: (1), changing the detuning which mode 1 really sees, and (2), strengthening the competition of mode 2 against mode 1. If δ_1 and δ_2 have the same sign, the two actions all cause the average photon number of mode 1 to decrease, so that the peak position of photon-statistical distribution decreases rapidly to zero, as shown in Fig. 5. If δ_1 and δ_2 have opposite signs, the first action causes the average photon number to increase while the second decreases. The joint effect in some situations (some combination of δ_1 and δ_2) may cause the average photon number of mode 1 to increase as $\langle n_2 \rangle_e$ increases from zero, as shown in Fig. 6. Of course, when $\langle n_2 \rangle_e$ is large enough, the competition of mode 2 against mode 1 will become dominant and the average photon-number of mode 1 will eventually goes down as shown in Fig. 6.

IV. CONCLUSION

We have studied the properties of the two-mode laser in a homogeneously broadened medium composed of three-level atoms with arbitrary detunings, through generalizing the Scully and Lamb quantum theory for a single mode. Novel phenomena appears in the nonresonant operation of this laser which cannot be found in the resonant operation. These new and interesting phenomena are the asymmetry in the threshold condition and the anomalous mutual support between the two modes in certain circumstances. The asymmetry reveals the obvious fact that the two-photon Raman-type resonance makes the influence between the two modes maximum. To explain the latter, we need to take into account the ac-Stark-shift effect of the upper level. At resonance, including two-photon resonance, an increase of one mode's strength is always at the expense of a decrease of the other mode's. Off-resonance, not only the competition resulting from the same population of the common upper level, but also the ac-stark-shift effect, needs to be considered. When the two detunings have opposite signs, within a certain intensity range, as one mode becomes stronger, it causes a stronger ac-Stark-shift effect which reduces the detuning of the other mode, i.e., brings gain to it to overcome the effect of competition.

Although we have obtained analytical formulas, they are so complicated that we can not directly see the above effects from them. Therefore, numerical analysis has been used to show these effects.

APPENDIX A: THE SOLUTION OF EQS. (8)

Let

$$\begin{aligned} a_{n_1,n_2}(t) &= \sum_\omega a_{n_1,n_2}(\omega) e^{-i\omega(t-t_0)} , \\ b_{n_1+1,n_2}(t) &= \sum_\omega b_{n_1+1,n_2}(\omega) e^{-i\omega(t-t_0)} , \\ c_{n_1,n_2+1}(t) &= \sum_\omega c_{n_1,n_2+1}(\omega) e^{-i\omega(t-t_0)} . \end{aligned} \quad \text{(A1)}$$

Substituting Eqs. (A1) into Eqs. (8) in the text, we have

$$\begin{aligned} \sum_\omega \omega a_{n_1,n_2}(\omega) e^{-i\omega(t-t_0)} &= V_1 \sum_\omega b_{n_1+1,n_2}(\omega) e^{-i(\omega+\Delta_2)(t-t_0)} \\ &\quad + V_2 \sum_\omega c_{n_1,n_2+1}(\omega) e^{-i(\omega+\Delta_2)(t-t_0)} , \end{aligned} \quad \text{(A2)}$$

$$\sum_\omega \omega b_{n_1+1,n_2}(\omega) e^{-i\omega(t-t_0)} = V_1 \sum_\omega a_{n_1,n_2}(\omega) e^{-i(\omega-\Delta_1)(t-t_0)} ,$$

$$\sum_\omega \omega c_{n_1,n_2+1}(\omega) e^{-i\omega(t-t_0)} = V_2 \sum_\omega a_{n_1,n_2}(\omega) e^{-i(\omega-\Delta_2)(t-t_0)} ,$$

and then

$$\omega a_{n_1,n_2}(\omega)=V_1 b_{n_1+1,n_2}(\omega-\Delta_1)+V_2 c_{n_1,n_2+1}(\omega-\Delta_2)\ ,$$
$$\omega b_{n_1+1,n_2}(\omega)=V_1 a_{n_1,n_2}(\omega+\Delta_1)\ , \tag{A3}$$
$$\omega c_{n_1,n_2+1}(\omega)=V_2 a_{n_1,n_2}(\omega+\Delta_2)\ .$$

From Eqs. (A3) we find that ω can take three values, which are the three roots of the following equation:

$$\omega^3-(\Delta_1+\Delta_2)\omega^2+(\Delta_1\Delta_2-V_1^2-V_2^2)\omega+\Delta_1 V_2^2+\Delta_2 V_1^2=0\ . \tag{A4}$$

This algebraic equation is similar to Eq. (5) in Ref. 9 and is expected to have three real different roots. They are

$$\omega_1=-\tfrac{1}{3}x_1+x_4\cos x_5\ ,$$
$$\omega_2=-\tfrac{1}{3}x_1+x_4\cos(x_5+\tfrac{2}{3}\pi)\ , \tag{A5}$$
$$\omega_3=-\tfrac{1}{3}x_1+x_4\cos(x_5+\tfrac{4}{3}\pi)\ ,$$

where

$$x_1=-(\Delta_1+\Delta_2)\ ,$$
$$x_2=\Delta_1\Delta_2-V_1^2-V_2^2\ ,$$
$$x_3=\Delta_1 V_2^2+\Delta_2 V_1^2\ , \tag{A6}$$
$$x_4=\tfrac{2}{3}(x_1^2-3x_2)^{1/2}\ ,$$
$$x_5=\tfrac{1}{3}\arccos\left[\frac{9x_1x_2-2x_1^3-27x_3}{2(x_1^2-3x_2)^{2/3}}\right]\ .$$

For some special cases, they can be simplified, for example, the following.

(1) One-photon resonance, $\Delta_1=\Delta_2=0$,

$$\omega_1=0,\quad \omega_{2,3}=\pm(V_1^2+V_2^2)^{1/2}\ ,$$

(2) two-photon resonance, $\Delta_1=\Delta_2=\Delta$,

$$\omega_1=\Delta,\quad \omega_{2,3}=\tfrac{1}{2}\Delta\pm\left[\frac{\Delta^2}{4}+V_1^2+V_2^2\right]^{1/2}\ ,$$

(3) the case of $V_1,V_2\gg|\Delta_1|,|\Delta_2|$, approximately

$$\omega_1=(\Delta_1V_2^2+\Delta_2V_1^2)/(V_1^2+V_2^2)\ ,$$
$$\omega_{2,3}=\tfrac{1}{2}(\Delta_1V_1^2+\Delta_2V_2^2)/[V_1^2+V_2^2\pm(V_1^2+V_2^2)^{1/2}]\ ,$$

(4) the case of $|\Delta_i|\gg V_i$, approximately

$$\omega_1=\Delta_1,\quad \omega_2=\Delta_2,\quad \omega_3=-\left[\frac{V_1^2}{\Delta_1}+\frac{V_2^2}{\Delta_2}\right]\ ,$$

(5) the case of $\Delta_1/\Delta_2=-V_1^2/V_2^2$,

$$\omega_1=0,\quad \omega_{2,3}=\tfrac{1}{2}(\Delta_1+\Delta_2)\pm\left[\frac{(\Delta_1-\Delta_2)^2}{4}+V_1^2+V_2^2\right]^{1/2}\ .$$

Thus, the solutions of Eqs. (8) are

$$a_{n_1,n_2}(t)=a^{(1)}_{n_1,n_2}e^{-i\omega_1(t-t_0)}+a^{(2)}_{n_1,n_2}e^{-i\omega_2(t-t_0)}+a^{(3)}_{n_1,n_2}e^{-i\omega_3(t-t_0)}\ ,$$
$$b_{n_1+1,n_2}(t)=b^{(1)}_{n_1+1,n_2}e^{-i(\omega_1-\Delta_1)(t-t_0)}+b^{(2)}_{n_1+1,n_2}e^{-i(\omega_2-\Delta_1)(t-t_0)}+b^{(3)}_{n_1+1,n_2}e^{-i(\omega_3-\Delta_1)(t-t_0)}\ , \tag{A7}$$
$$c_{n_1,n_2+1}(t)=c^{(1)}_{n_1,n_2+1}e^{-i(\omega_1-\Delta_1)(t-t_0)}+c^{(2)}_{n_1,n_2+1}e^{-i(\omega_2-\Delta_2)(t-t_0)}+c^{(3)}_{n_1,n_2+1}e^{-i(\omega_3-\Delta_2)(t-t_0)}\ .$$

According to the initial condition, Eq. (5) in the text,

$$b_{n_1+1,n_2}(t_0)=c_{n_1,n_2+1}(t_0)=0\ ,\qquad a_{n_1,n_2}(t_0)=F_{n_1,n_2}(t_0)\ , \tag{A8}$$

we obtain

$$a^{(1)}_{n_1,n_2}+a^{(2)}_{n_1,n_2}+a^{(3)}_{n_1,n_2}=F_{n_1,n_2}(t_0)\ ,$$
$$b^{(1)}_{n_1+1,n_2}+b^{(2)}_{n_1+1,n_2}+b^{(3)}_{n_1+1,n_2}=0\ , \tag{A9}$$
$$c^{(1)}_{n_1,n_2+1}+c^{(2)}_{n_1,n_2+1}+c^{(3)}_{n_1,n_2+1}=0\ .$$

From Eqs. (A3) and (A7), we have

$$b^{(i)}_{n_1+1,n_2}=\frac{V_1}{\omega_i-\Delta_1}a^{(i)}_{n_1,n_2},\quad i=1,2,3 \tag{A10}$$
$$c^{(i)}_{n_1,n_2+1}=\frac{V_2}{\omega_i-\Delta_2}a^{(i)}_{n_1,n_2},\quad i=1,2,3\ . \tag{A11}$$

Solving the Eqs. (A9)–(A11), we find that

$$a^{(i)}_{n_1,n_2}=F_{n_1,n_2}(t_0)(\omega_{i-1}-\omega_{i+1})\times(\omega_i-\Delta_1)(\omega_i-\Delta_2)/D\ , \tag{A12}$$
$$b^{(i)}_{n_1+1,n_2}=V_1F_{n_1,n_2}(t_0)(\omega_{i-1}-\omega_{i+1})(\omega_i-\Delta_2)/D\ , \tag{A13}$$
$$c^{(i)}_{n_1,n_2+1}=V_2F_{n_1,n_2}(t_0)(\omega_{i-1}-\omega_{i+1})(\omega_i-\Delta_1)/D\ . \tag{A14}$$

where $i=1,2,3$ and $i=0$ is $i=3$ and $i=4$ is $i=1$, and

$$D=(\omega_1-\omega_2)(\omega_2-\omega_3)(\omega_3-\omega_1)\ . \tag{A15}$$

From Eqs. (A13) and (A14), we obtain

$$b^{(i)}_{n_1,n_2}=\frac{V_1'}{B'}F_{n_1-1,n_2}(t_0)(\omega'_{i-1}-\omega'_{i+1})(\omega'_i-\Delta_2)\ , \tag{A16}$$
$$c^{(i)}_{n_1,n_2}=\frac{V_2''}{B''}F_{n_1,n_2-1}(t_0)(\omega''_{i-1}-\omega''_{i+1})(\omega''_i-\Delta_2)\ , \tag{A17}$$

where the quantities with a superscript of one prime (or two primes) are those corresponding with n_1 (or n_2) replaced by n_1-1 (or n_2-1).

Let

$$\delta_j=\Delta_j/\gamma,\quad \bar{V}_j=V_j/\gamma=g_j\sqrt{n_j+1}/\gamma,\quad j=1,2$$
$$\mu_i=\omega_i/\gamma,\quad i=1,2,3 \tag{A18}$$
$$\bar{D}=D/\gamma^3 .$$

Equations (A12), (A16), and (A17) can be rewritten as

$$a^{(i)}_{n_1,n_2}=F_{n_1,n_2}(t_0)(\mu_{i-1}-\mu_{i+1})(\mu_i-\delta_1)(\mu_i-\delta_2)/\bar{D}$$
$$=F_{n_1,n_2}(t_0)A^{(i)}_{n_1,n_2} , \tag{A19}$$

$$b^{(i)}_{n_1,n_2}=\bar{V}_1F_{n_1-1,n_2}(t_0)(\mu'_{i-1}-\mu'_{i+1})(\mu'_1-\delta_2)/\bar{D}'$$
$$=F_{n_1-1,n_2}(t_0)B^{(i)}_{n_1,n_2} , \tag{A20}$$

$$c^{(i)}_{n_1,n_2}=\bar{V}''_2F_{n_1,n_2-1}(t_0)(\mu''_{i-1}-\mu''_{i+1})(\mu''_i-\delta_1)/\bar{D}''$$
$$=F_{n_1,n_2-1}(t_0)C^{(i)}_{n_1,n_2} , \tag{A21}$$

where

$$A^{(i)}_{n_1,n_2}=a^{(i)}_{n_1,n_2}/F_{n_1,n_2}(t_0) , \tag{A22}$$
$$B^{(i)}_{n_1,n_2}=b^{(i)}_{n_1,n_2}/F_{n_1-1,n_2}(t_0) , \tag{A23}$$
$$C^{(i)}_{n_1,n_2}=c^{(i)}_{n_1,n_2}/F_{n_1,n_2-1}(t_0) . \tag{A24}$$

APPENDIX B: SOME MATHEMATICAL DETAILS FOR EQ. (14)

By using Eqs. (A22) and (A19), the expression in the first large parentheses in Eq. (14) in the text equals

$$I=\frac{A^{(1)}_{n_1,n_2}A^{(2)}_{n_1,n_2}(\mu_1-\mu_2)^2}{(\mu_1-\mu_2)^2+1}+\frac{A^{(1)}_{n_1,n_2}A^{(3)}_{n_1,n_2}(\mu_1-\mu_3)^2}{(\mu_1-\mu_3)^2+1}+\frac{A^{(1)}_{n_1,n_2}A^{(3)}_{n_1,n_2}(\mu_2-\mu_3)^2}{(\mu_2-\mu_3)^2+1}$$
$$=\frac{(\mu_1-\delta_1)(\mu_1-\delta_2)(\mu_2-\delta_1)(\mu_2-\delta_2)}{[(\mu_1-\mu_2)^2+1](\mu_2-\mu_3)(\mu_3-\mu_1)}+\frac{(\mu_3-\delta_1)(\mu_3-\delta_2)(\mu_1-\delta_1)(\mu_1-\delta_2)}{[(\mu_3-\mu_1)^2+1](\mu_2-\mu_3)(\mu_1-\mu_2)}+\frac{(\mu_2-\delta_1)(\mu_2-\delta_2)(\mu_3-\delta_1)(\mu_3-\delta_2)}{[(\mu_2-\mu_3)^2+1](\mu_3-\mu_1)(\mu_1-\mu_2)}$$
$$=I_1+I_2+I_3 . \tag{B1}$$

From Eqs. (A4) and (A18), we have

$$\mu_1\mu_2\mu_3=-(\bar{V}_1^2\delta_2+\bar{V}_2^2\delta_1),\quad \delta_1+\delta_2=\mu_1+\mu_2+\mu_3 ,$$
$$\delta_1\delta_2-\bar{V}_1^2-\bar{V}_2^2=\mu_1\mu_2+\mu_2\mu_3+\mu_1\mu_3 . \tag{B2}$$

$$\therefore(\mu_1-\delta_1)(\mu_1-\delta_2)=\bar{V}_1^2+\bar{V}_2^2+\mu_2\mu_3,\quad (\mu_2-\delta_1)(\mu_2-\delta_2)=\bar{V}_1^2+\bar{V}_2^2+\mu_1\mu_3 .$$

$$\therefore(\mu_1-\delta_1)(\mu_1-\delta_2)(\mu_2-\delta_1)(\mu_2-\delta_2)$$
$$=\mu_1(\bar{V}_1^2+\bar{V}_2^2+\mu_2\mu_3)(\mu_2-\delta_1)(\mu_2-\delta_2)(\mu_1-\mu_2)^{-1}-\mu_2(\bar{V}_1^2+\bar{V}_2^2+\mu_1\mu_3)(\mu_1-\delta_1)(\mu_1-\delta_2)(\mu_1-\mu_2)^{-1}$$
$$=[\bar{V}_1^2\mu_1+\bar{V}_2^2\mu_1-(\bar{V}_1^2\delta_2+\bar{V}_2^2\delta_1)](\mu_2-\delta_1)(\mu_2-\delta_2)(\mu_1-\mu_2)^{-1}$$
$$-[\bar{V}_1^2\mu_2+\bar{V}_2^2\mu_2-(\bar{V}_1^2\delta_2+\bar{V}_2^2\delta_1)](\mu_1-\delta_1)(\mu_1-\delta_2)(\mu_1-\mu_2)^{-1}$$
$$=[\bar{V}_1^2(\mu_1-\delta_2)+\bar{V}_2^2(\mu_1-\delta_1)](\mu_2-\delta_1)(\mu_2-\delta_2)(\mu_1-\mu_2)^{-1}$$
$$-[\bar{V}_1^2(\mu_2-\delta_2)+\bar{V}_2^2(\mu_2-\delta_1)](\mu_1-\delta_1)(\mu_1-\delta_2)(\mu_1-\mu_2)^{-1}$$
$$=\bar{V}_1^2(\mu_1-\delta_2)(\mu_2-\delta_2)(\mu_2-\mu_1)(\mu_1-\mu_2)^{-1}+\bar{V}_2^2(\mu_1-\delta_1)(\mu_2-\delta_1)(\mu_2-\mu_1)(\mu_1-\mu_2)^{-1}$$
$$=-[\bar{V}_1^2(\mu_1-\delta_2)(\mu_2-\delta_2)+\bar{V}_2^2(\mu_1-\delta_1)(\mu_2-\delta_2)] . \tag{B3}$$

By using Eq. (B3) we obtain

$$I_1=\frac{\bar{V}_1^2(\mu_1-\delta_2)(\mu_2-\delta_2)}{(\mu_2-\mu_3)(\mu_1-\mu_3)[(\mu_1-\mu_2)^2+1]}+\frac{\bar{V}_2^2(\mu_1-\delta_1)(\mu_2-\delta_1)}{(\mu_2-\mu_3)(\mu_1-\mu_3)[(\mu_1-\mu_2)^2+1]} . \tag{B4}$$

With the same deduction, we have

$$I_2=\frac{\bar{V}_1^2(\mu_1-\delta_2)(\mu_3-\delta_2)+\bar{V}_2^2(\mu_1-\delta_1)(\mu_3-\delta_1)}{(\mu_3-\mu_2)(\mu_1-\mu_2)[(\mu_3-\mu_1)^2+1]} , \tag{B5}$$

$$I_3=\frac{\bar{V}_1^2(\mu_2-\delta_2)(\mu_3-\delta_2)+\bar{V}_2^2(\mu_2-\delta_1)(\mu_3-\delta_1)}{(\mu_2-\mu_1)(\mu_3-\mu_1)[(\mu_2-\mu_3)^2+1]} . \tag{B6}$$

By using Eqs. (A19), (A23), (A20), and (A24), the expressions in the second and third large parentheses in Eq. (14) are, respectively,

$$J=\frac{B^{(1)}_{n_1,n_2}B^{(2)}_{n_1,n_2}(\mu'_1-\mu'_2)^2}{[(\mu'_1-\mu'_2)^2+1]}+\frac{B^{(1)}_{n_1,n_2}B^{(3)}_{n_1,n_2}(\mu'_1-\mu'_3)^2}{[(\mu'_1-\mu'_3)^2+1]}+\frac{B^{(2)}_{n_1,n_2}B^{(3)}_{n_1,n_2}(\mu'_2-\mu'_3)^2}{[(\mu'_2-\mu'_3)^2+1]}$$
$$=\frac{-(\bar{V}'_1)^2)(\mu'_1-\delta_2)(\mu'_2-\delta_2)}{(\mu'_1-\mu'_3)(\mu'_2-\mu'_3)[(\mu'_1-\mu'_2)^2+1]}+\frac{-(\bar{V}'_1)^2(\mu'_1-\delta_2)(\mu'_3-\delta_2)}{(\mu'_1-\mu'_2)(\mu'_3-\mu'_2)[(\mu'_1-\mu'_3)^2+1]}+\frac{-(\bar{V}'_1)^2(\mu'_2-\delta_2)(\mu'_3-\delta_2)}{(\mu'_2-\mu'_1)(\mu'_3-\mu'_1)[(\mu'_2-\mu'_3)^2+1]}\,, \tag{B7}$$

$$K=\frac{C^{(1)}_{n_1,n_2}C^{(2)}_{n_1,n_2}(\mu''_1-\mu''_2)^2}{(\mu''_1-\mu''_2)^2+1}+\frac{C^{(1)}_{n_1,n_2}C^{(2)}_{n_1,n_2}(\mu''_1-\mu''_3)^2}{(\mu''_1-\mu''_3)^2+1}+\frac{C^{(2)}_{n_1,n_2}C^{(3)}_{n_1,n_2}(\mu''_2-\mu''_3)^2}{(\mu''_2-\mu''_3)^2+1}$$
$$=\frac{-(\bar{V}''_2)^2(\mu''_1-\delta_1)(\mu''_2-\delta_1)}{(\mu''_1-\mu''_3)(\mu''_2-\mu''_3)[(\mu''_1-\mu''_2)^2+1]}+\frac{-(\bar{V}''_2)^2(\mu''_1-\delta_1)(\mu''_3-\delta_1)}{(\mu''_1-\mu''_2)(\mu''_3-\mu''_2)[(\mu''_1-\mu''_3)^2+1]}+\frac{-(\bar{V}''_2)^2(\mu''_2-\delta_1)(\mu''_3-\delta_1)}{(\mu''_2-\mu''_1)(\mu''_3-\mu''_1)[(\mu''_2-\mu''_3)^2+1]}\,. \tag{B8}$$

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