Connection between microscopic and macroscopic maser theory

L. A. Lugiato,* M. O. Scully,† and H. Walther‡
Max-Planck-Institut für Quantenoptik, D-8046 Garching bei München, West Germany
(Received 29 December 1986)

We show that the steady-state photon-number distribution of the microscopic maser can be recovered using the standard macroscopic quantum laser theory. We introduce an appropriate limit of weak pump and weak photon damping to make the connection between the microscopic and the macroscopic quantum theory of laser operation. Simple analytic formulas describe the main features of the stationary photon-number distribution in the microscopic maser.

I. INTRODUCTION

The progress recently achieved in Rydberg atom spectroscopy and the availability of superconducting microwave cavities of extremely high-quality factors, have led to the experimental realization of a truly microscopic maser.[1] In this system inverted atoms are injected into a single-mode high-Q resonator at such a low rate that at most one atom at a time is present inside the cavity. For resonators of exceedingly high-Q values, it is possible even under such extremely low densities for the rate of stimulated emission into the single-cavity mode to exceed the cavity losses, and the system can be brought above threshold.

In general, two approaches to the quantum theory of such a micromaser have developed. In one approach Filipovicz, Javanainen, and Meystre[2(a)] (FJM) have developed a formulation which emphasizes the microscopic nature of the device. They have then proceeded to derive, for example, the photon statistical distribution for the micromaser and have found, e.g., sub-Poissonian statistics. In other work, Krause, Scully, and Walther[2(b)] (KSW) have applied conventional macroscopic laser theory to the micromaser problem. In particular, they have considered the situation in which the active maser atoms are injected into the maser cavity with an off-diagonal atomic density matrix. In their work KSW have made explicit use of the results of the conventional quantum theory of the laser.[3(a),3(b)] The aim of this paper is to establish the connection between these two different treatments and to show that the standard quantum laser theory is equivalent to that of FJM. We then show that using the standard theory one recovers exactly the stationary distribution calculated by FJM. This may seem at first sight surprising because no stochastic average over injection times (as used by FJM) is involved in the usual quantum laser theory. We clarify this point by showing that in the limit of weak pump and weak photon damping, one obtains the same steady-state distribution as in FJM theory without using any average over injection times. Finally, we derive simple analytic formulas which describe the main features of the stationary photon-number distribution in the microscopic maser.

II. MICROSCOPIC MASER THEORY

As discussed in Ref. 2(a), if we call t_i the times at which the atoms are injected in the cavity, the time evolution of the density matrix ρ of the cavity mode is governed by the map

$$\rho(t_{i+1})=\exp(Lt_p)T(\tau)\rho(t_i) \ , \tag{1}$$

where $t_p=t_{i+1}-t_i$, and τ is the interaction time between each atom and the mode. Here L is the Liouvillian operator which describes the coupling of a single harmonic oscillator to a thermal bath[4] and has the form

$$L\rho=-\frac{\gamma}{2}(n_b+1)(a^\dagger a\rho+\rho a^\dagger a-2a\rho a^\dagger) -\frac{\gamma}{2}n_b(aa^\dagger\rho+\rho aa^\dagger-2a^\dagger\rho a) \ , \tag{2}$$

where n_b is the average number of thermal photons in the cavity, γ is the photon damping rate, and a is the annihilation operator of photons of the cavity mode. In Eq. (1) the gain operator $T(\tau)$ is defined as

$$T(\tau)\rho=\mathrm{Tr}_a\left[\exp\left(\frac{-i}{\hbar}H\tau\right)\rho\exp\left(\frac{i}{\hbar}H\tau\right)\right] \ , \tag{3}$$

where Tr_a indicates partial trace over the Hilbert space of the two-level atom and H is the Jaynes-Cummings Hamiltonian[5]

$$H=\left(\frac{\hbar\omega}{2}\right)S_3+\hbar\omega a^\dagger a+\hbar g(S^+a+S^-a^\dagger) \ . \tag{4}$$

In Eq. (4) ω is the energy difference between the two atomic levels, and $S_3,S^\pm$ are the Pauli spin operators. For the sake of simplicity, in this paper we assume that the frequency of the cavity mode coincides with the atomic transition frequency ω.

At steady state, we have $\rho(t_{i+1})=\rho(t_i)\equiv\rho_{\mathrm{st}}$. In order to obtain the stationary solution in analytic form, the treatment of 2(a) assumes that the injection time intervals obey a Poisson statistics with average $\bar{t}_p$. It is then found that the steady-state equation takes the form

$$\rho_{\rm st}=\frac{1}{1-L\bar{t}_p}T(\tau)\rho_{\rm st}\ , \tag{5}$$

which leads to the following expression for the photon probability distribution P_n $(n=0,1,2,\dots)$:

$$P_n=P_0\left[\frac{n_b}{n_b+1}\right]^n\prod_{m=1}^{n}\left[1+\frac{1}{n_b\gamma\bar{t}_p}\frac{\sin^2(g\sqrt{m}\,\tau)}{m}\right]\ , \tag{6}$$

for $n\geq 1$, where the probability P_0 is determined by the normalization condition $\sum_{n=0}^{\infty}P_n=1$.

III. STANDARD LASER THEORY

The description of the microscopic maser dynamics can be obtained following the same method formulated in Ref. 3. One considers a coarse-grained time Δt small with respect to the time scale of the evolution and writes the equation for the maser radiation density matrix

$$\dot{\rho}(t)\equiv\frac{\rho(t+\Delta t)-\rho(t)}{\Delta t}=r\delta\rho_\tau(t)+L\rho\ , \tag{7}$$

where the first term describes the gain, and the second the loss. Now, $r=t_p^{-1}$ is the injection rate and $\delta\rho_\tau(t)$ is the change in ρ due to one atom interacting for a time τ. In the case of the microscopic maser, the cutoff on the interaction time is not introduced by spontaneous emission, but directly by time of flight through the interaction region. Hence, one arrives straightforwardly at the following time evolution equation for the elements $\rho_{n,m}$ of the density matrix in the photon number representation $(n,m=0,1,\dots)$:

$$\dot{\rho}_{n,m}=-r\{1-[\cos(g\sqrt{n+1}\tau)][\cos(g\sqrt{m+1}\tau)]\}\rho_{n,m}+r[\sin(g\sqrt{n}\,\tau)][\sin(g\sqrt{m}\,\tau)]\rho_{n-1,m-1}$$
$$-\frac{\gamma}{2}(n_b+1)[(n+m)\rho_{n,m}-2\sqrt{(n+1)(m+1)}\rho_{n+1,m+1}]-\frac{\gamma}{2}n_b[(n+1+m+1)\rho_{n,m}-2\sqrt{nm}\,\rho_{n-1,m-1}]\ . \tag{8}$$

The steady-state photon-number distribution $P_n=\rho_{n,n}$ is then found, by detailed balance, i.e., from the equation

$$\gamma(n_b+1)nP_n=[r\sin^2(g\sqrt{n}\,\tau)+n_b\gamma n]P_{n-1}\ , \tag{9}$$

which gives

$$P_n=P_0\prod_{l=1}^{n}\frac{n_b\gamma+r\sin^2(g\tau\sqrt{l}\,)/l}{\gamma(n_b+1)}\ . \tag{10}$$

Clearly, Eq. (10) is identical to Eq. (6) because $r=t_p^{-1}$.

IV. CONNECTION BETWEEN MICROSCOPIC AND MACROSCOPIC THEORY OF THE LASER

In Sec. III we found that the stationary solution of the microscopic laser theory is exactly recovered in the framework of the standard theory. Next, we explain why this microscopic-to-macroscopic step works, even though in Sec. III we did not introduce the assumption that the injection time intervals obey Poisson statistics.

For a nonstochastic t_p, we consider the following limit: with a smallness parameter ϵ it is assumed that

$$(\gamma t_p)^{1/2}\simeq\epsilon,\quad g\tau\simeq\epsilon\ , \tag{11}$$

with

$$g\tau/\sqrt{\gamma t_p}\simeq\epsilon^0\ . \tag{12}$$

The first condition in (11) means that the decrease in photon number in a time between two successive atomic injections is very small. The second condition says that the rotation of the Bloch vector of the two-level atoms during the interaction time is small, at least when the number of photons in the cavity is small.

Let us now rewrite Eq. (1) as

$$\rho(t_{i+1})=\{1+[\exp(Lt_p)-1]\}\{1+[T(\tau)-1]\}\rho(t_i)\ . \tag{13}$$

In the limit (11) we have $[\exp(Lt_p)-1]\simeq\gamma t_p$, and $[T(\tau)-1]\sim(g\tau)^2$. Hence we neglect the product $[\exp(Lt_p)-1][T(\tau)-1]$ obtaining

$$\rho(t_{i+1})=(\{1+[\exp(Lt_p)-1]\}+[T(\tau)-1])\rho(t_i)\ . \tag{14}$$

Next, we take into account the definition of the generator of the semigroup $\exp(Lt)$:

$$L=\lim_{t_p\to 0}[\exp(Lt_p)-1]/t_p\ ,$$

which for $\gamma t_p\ll 1$ allows us to reformulate Eq. (14) as

$$\rho(t_{i+1})=\rho(t_i)+Lt_p\rho(t_i)+[T(\tau)-1]\rho(t_i)\ . \tag{15}$$

Since at steady state $\rho(t_{i+1})=\rho(t_i)=\rho_{\rm st}$, we obtain

$$(1-Lt_p)\rho_{\rm st}=T(\tau)\rho_{\rm st}\ , \tag{15$'$}$$

which clearly coincides with Eq. (5). Thus in the limits (11) and (12) the microscopic theory leads to the steady-state distribution (6) even without assuming any stochasticity in the time t_p.

In the transient domain, because $r=t_p^{-1}$, Eq. (15) can be rewritten as

$$\frac{\rho(t_{i+1})-\rho(t_i)}{t_p}=L(t_i)+r[T(\tau)-1]\rho(t_i)\ . \tag{16}$$

On replacing the left-hand side by the time derivative $\dot{\rho}$, Eq. (16) becomes identical to Eq. (8) as one can easily verify.

We emphasize that the weak pump limit assumed in Eq. (11) does not imply at all that Eq. (8) holds only in the threshold region. In fact, as it is shown in Sec. V the maser threshold corresponds to $g\tau=(\gamma t_p)^{1/2}$ and therefore, provided γt_p is small enough, $g\tau$ can be raised many times above threshold without violating conditions (11) and (12). Clearly, Eqs. (11) and (12) identify *sufficient* conditions under which Eq. (1) reduces to Eq. (8), but the agreement between Eq. (8) and Eq. (1) may well persist beyond the domain specified by conditions (11) and (12). As a matter of fact, the direct numerical solution of Eq. (1) leads to a stationary photon-number distribution which is close to Eq. (6) even in situations in which condition (11) is badly violated.[6]

Our demonstration of the connection between the microscopic and the traditional laser theory offers one the possibility of calling on the past store of calculations obtained from the standard laser theory (exponential decay of the off-diagonal elements, Fokker-Planck equation, and generalization thereof, etc.). For example, by considering the diagonal elements of the density matrix $P_n = \rho_{n,n'}$ introduced in Eq. (8),

$$\sin^2(g\tau\sqrt{n}\,) \cong (g\tau)^2 n - \tfrac{1}{3}(g\tau)^4 n^2 \ , \tag{17}$$

which holds in the threshold region, and setting

$$A = r(g\tau)^2, \quad B = \tfrac{1}{3} r(g\tau)^4, \quad C = \gamma, \quad n_b = 0 \ , \tag{18}$$

one obtains the master equation

$$\dot{P}_n = -[A - B(n+1)](n+1)P_n + (A - Bn)nP_{n-1} - C[nP_n - (n+1)P_{n+1}] \tag{19}$$

which is well known in the laser literature.

V. ANALYSIS OF THE STEADY-STATE PHOTON STATISTICS

The maser threshold from Eq. (18) is $A = C$ which, using the fact that $r = 1/t_p$, gives

$$(g\tau)^2 = \gamma t_p \ . \tag{20}$$

This suggests the use of the normalized interaction time

$$\tau_{\text{int}} = g\tau/\sqrt{\gamma t_p} \ , \tag{21}$$

introduced in Ref. (2), which is equal to unity at threshold. A simple analysis of the stationary distribution (10) can be obtained by exploiting a natural continuous approximation. That is, from Eq. (10) we obtain

$$\ln P_n/P_0 = \sum_{l=1}^{n} \ln \left[\frac{n_b\gamma + [r\sin^2(g\tau\sqrt{l}\,)]/l}{\gamma(n_b+1)} \right] \cong \int_0^n dl \ln \left[\frac{n_b\gamma + [r\sin^2(g\tau\sqrt{l}\,)]/l}{\gamma(n_b+1)} \right] \equiv f(n) \ . \tag{22}$$

The maxima and minima of the distribution are obtained by setting $f'(n) = 0$. Calling n_0 one such extremum, one easily obtains the equation

$$n_0 = (r/\gamma)\sin^2(g\tau\sqrt{n_0}) \ . \tag{23}$$

All the nonzero solutions of Eq. (23) are solutions of the equation $f'(n_0) = 0$. Now, $n_0 = 0$ is always a solution of Eq. (23), but not of equation $f'(n_0) = 0$ (with the only exception being at threshold). By defining $x = (n_0\gamma/r)^{1/2}$, Eq. (23) can be rewritten as follows:

$$x = \pm\sin(\tau_{\text{int}} x), \quad x > 0 \tag{23'}$$

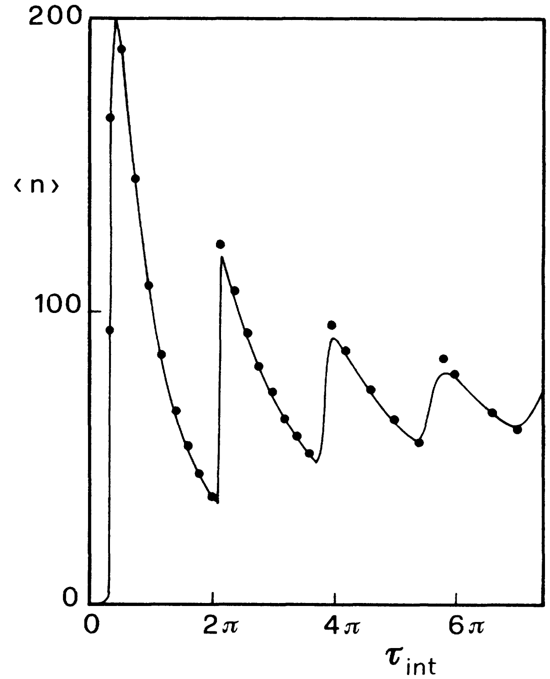

FIG. 1. The curve, which is taken from Ref. 2(a), shows the mean photon number as a function of the normalized interaction time $\tau_{\text{int}} = g\tau/\sqrt{\gamma t_p}$, obtained from the stationary distribution (10) for $r/\gamma = 200$, $n_b = 0.1$. The dots indicate the solution of Eq. (23), selected as indicated in the text.

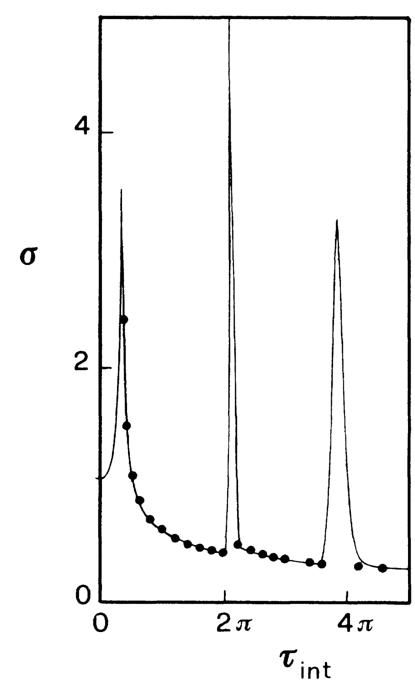

FIG. 2. The curve, taken from Ref. 2(a), shows the normalized mean variance $\sigma = \Delta n/\sqrt{\langle n \rangle}$ obtained from the stationary distribution (10) for the same values of the parameters of Fig. 2. The dots indicate the value obtained from Eq. (26).

which depends only on the parameter τ_{int}, while the parameter r/γ enters only the scaling relation $n_0=(r/\gamma)x$. Clearly, the solutions of Eq. (23′) never exceed unity, and therefore n_0 cannot be larger than r/γ. In particular, for $\tau_{\text{int}}=(2n+1)\pi/2$, $n=0,\pm1,\pm2$, etc., $x=1$ and $n_0=r/\gamma$ is a solution of Eq. (23′) and (23), respectively. On increasing τ_{int}, the zeroes of the function $\sin\tau_{\text{int}}x$ cumulate towards the origin. This implies that Eq. (23′) has one positive solution only for $\tau_{\text{int}}\lesssim 1.5\pi$, whereas for $\tau > 1.5\pi$, Eq. (23′) has more than one positive solution.

Therefore, the probability distribution has one peak only for $\tau_{\text{int}} < 1.5\pi$. In this situation one cannot in general obtain the curve of the mean photon number $\langle n \rangle$ simply on the basis of Eq. (23). However, for $\tau_{\text{int}} < 1.5\pi$ the nonzero solution of Eq. (23) approximates the mean value very well, as shown by Fig. 1. In this figure, the dots (for $\tau_{\text{int}} > 1.5\pi$) are obtained by selecting the solution of Eq. (23) which is nearest to the mean value. In such a way, one obtains a rather satisfactory approximation. This fact demonstrates that the peak of the stationary distribution, which corresponds to the selected solution of Eq. (23), dominates over the other peaks.

Next, we expand the function $f(n)$ around n_0,

$$f(n)=f(n_0)+\tfrac{1}{2}f''(n_0)(n-n_0)^2+\cdots \tag{24}$$

and accordingly using Eq. (22) we approximate the distribution P_n by a Gaussian

$$P(n)=\eta\exp-\frac{(n-n_0)^2}{2\Delta n^2}\ , \tag{25}$$

with $\Delta n^2=-1/f''(n_0)$. Clearly, this Gaussian approximation cannot hold when the probability distribution has more than one peak of comparable area. By simple calculations, using Eq. (23) one obtains from Eq. (22) the following expression:

$$\sigma=\frac{\Delta n}{\sqrt{n_0}}=\left[\frac{n_b+1}{1\pm g\tau(r/\gamma-n_0)^{1/2}}\right], \tag{26}$$

where the minus sign must be chosen for $2m(\pi/2)<g\tau\sqrt{n_0}<(2m+1)(\pi/2)$, the plus sign for

$$(2m+1)(\pi/2)<g\tau\sqrt{n_0}<(2m+2)(\pi/2),\ m=0,1,\ldots .$$

Figure 2 compares the exact value of σ with that given by Eq. (26) and obtained by selecting the value n_0 as in Fig. 1. Clearly, the agreement is satisfactory. Except for the first one, the narrow peaks in the curve of σ cannot be reproduced by Eq. (26), because they occur in regions of τ_{int} where the distribution P_n has more than one peak of comparable area (i.e., there is no peak which dominates the others). The solid curve in Fig. 2 which is taken from FJM, shows that while in the threshold region the photon-number distribution is broader than a Poisson distribution, well above threshold it is most often sub-Poissonian. In this case, a Poisson distribution can be recovered by introducing an average over the interaction time τ, as shown by FJM.

---

*Permanent address: Dipartimento di Fisica del Politecnico, Torino, Italy.

†Also at Center for Advanced Studies and Department of Physics and Astronomy, University of New Mexico, Albuquerque, NM 87131.

‡Also at Sektion Physik der Universität München, D-8046 Garching, West Germany.

[1]D. Meschede, H. Walther, and G. Muller, Phys. Rev. Lett. **54**, 551 (1985); G. Rempe, H. Walther, and N. Klein, *ibid.* **58**, 353 (1987).

[2](a) P. Filipowicz, J. Javanainen, and P. Meystre, in *Coherence, Cooperation and Fluctuations,* edited by F. Haake, L. M. Narducci, and D. F. Walls (Cambridge University Press, Cambridge, 1986), p. 206; (b) J. Krause, M. O. Scully, and H. Walther, Phys. Rev. A **34**, 2032 (1986).

[3]M. O. Scully and W. E. Lamb, Jr. (a) Phys. Rev. Lett. **16**, 853 (1966); (b) Phys. Rev. **159**, 208 (1967); (c) M. Sargent III, M. O. Scully, and W. E. Lamb, Jr., *Laser Physics* (Addison-Wesley, London, 1974).

[4]H. Haken, *Handbuch der Physik,* edited by L. Genzel (Springer-Verlag, Berlin, 1970), Vol. XXV/2c, W. H. Louisell, *Quantum Statistical Properties of Radiation* (Wiley, New York, 1973).

[5]E. T. Jaynes and F. W. Cummings, Proc. IEEE **51**, 89 (1963).

[6]P. Meystre (private communication).