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Overdamped and amplifying meters in the quantum theory of measurement

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We show that a quantum observable can be measured by coupling it to a meter which in turn interacts with a reservoir. The complete Hamiltonian is chosen so as to allow for an explicit exact solution of the quantum dynamics. In the continuum limit for the bath the solution displays irreversible behavior, two varieties of which, overdamping and amplification, turn out to be of special relevance. We establish the limit in which a suitable pointer variable (i) behaves effectively classically and (ii) acquires, through the measurement, a probability density of its eigenvalues with well-defined peaks each of which corresponds to one discrete eigenvalue of the measured observable. Under a slightly more restrictive condition the reduced density matrix of the object diagonalizes, during the measurement, in the eigenbasis of the measured observable.

I. INTRODUCTION

We shall deal with two important laws of quantum mechanics. The first of them claims the feasibility of preparation experiments in which superpositions of eigenstates of an observable $\hat{\xi}$ are irreversibly turned into mixtures, the probability of finding an eigenvalue ξ after the preparation being the squared modulus of the corresponding probability amplitude before the preparation. The second one states that different eigenvalues ξ of a measured microscopic observable $\hat{\xi}$ leave an effectively classical pointer variable of the measurement device in macroscopically distinct values; reading off these pointer values entails no back reaction on either the pointer or the measured observable $\hat{\xi}$.

There is widespread agreement that it is the task of measurement theory to demonstrate the compatibility of these two laws with the more fundamental ones which concern the additivity and the unitary time evolution of probability amplitudes. The task is a nontrivial one since it requires the reconciliation of the unitary and thus reversible quantum dynamics with effective irreversibility and, moreover, of the quantum-mechanical nature of the observable $\hat{\xi}$ with the classical dynamics of the pointer of the measurement device. To accommodate such seemingly paradoxical modes of behavior measurement devices necessarily involve a great number of degrees of freedom, some of which must be highly excited.

We shall carry out the task of measurement theory for a model consisting of an object (to which the observable $\hat{\xi}$ belongs), a meter (with a single degree of freedom), and a collection of oscillators (serving as a heat bath). The meter and the heat-bath oscillators together constitute the measurement device. We choose the model simple enough for its quantum equations of motion to allow for explicit exact solutions; its structure is sufficiently complex, on the other hand, for the exact solution to display, in certain limits to be established, the behavior described by the two laws stated above.

It is for the sake of simplicity that we take the object observable $\hat{\xi}$ to have a discrete spectrum and allot a single degree of freedom to the meter. A pointer variable $\hat{x}$ is associated with the meter which must be capable of effectively classical motion. It is therefore natural to endow $\hat{x}$ with a continuous spectrum and to accompany it with a canonically conjugate variable $\hat{p}$ such that $[\hat{p},\hat{x}]=\hbar/i$. To have a natural reference state as well as simple free dynamics for the meter we let $\hat{x}$ and $\hat{p}$ be the displacement and the momentum of a harmonic oscillator.

The object-meter interaction should not change the occupation probabilities of eigenstates of $\hat{\xi}$. The corresponding piece of the Hamiltonian, $H_{O\text{-}M}$, must therefore be designed so as to commute with $\hat{\xi}$. This requirement does not make $\hat{\xi}$ a constant of the motion unless $\hat{\xi}$ also commutes with the free Hamiltonian H_O of the object. In order to disentangle the free motion of the object from the effects of the object-meter coupling even if $[H_O,\hat{\xi}]\neq 0$ we assume the latter coupling to be an impulsive one, i.e., to have a δ-function modulation in time, $H_{O\text{-}M}\sim\delta(t)$. Finally, $H_{O\text{-}M}$ must cause a shift of the probability density of having eigenstates of $\hat{x}$ populated and that shift must be different for different eigenstates of $\hat{\xi}$ which may be realized initially. All of these requirements plus that of simplicity are met by

$$H_{O\text{-}M}=\epsilon\hat{\xi}\hat{p}\delta(t)\ , \tag{1.1}$$

where ϵ is a coupling constant.

Even before introducing the heat bath several important conclusions concerning our two laws can be drawn from (1.1). As explained in Sec. II the impulsive object-meter coupling suppresses off-diagonal elements $\langle\xi\,|\,\rho_O\,|\,\xi'\rangle$ of the reduced object density matrix to negligible magnitude provided the initial de Broglie wavelength of the meter is small compared to $|\,\epsilon(\xi-\xi')\,|$. If, moreover, the initial displacement variance $\sigma_{xx}(0)$ of the meter is smaller than $\epsilon^2(\xi-\xi')^2$ but much larger than the squared initial de

Broglie wave length $\sigma_{yy}(0)$ of the meter,

$$\sigma_{yy}(0) \ll \sigma_{xx}(0) < \epsilon^2(\Delta\xi)^2 \ , \tag{1.2}$$

the meter both immediately before and immediately after the impulsive object-meter coupling contains sufficiently many quantum action units $\hbar$ for its reduced density matrix to lend itself to classical interpretation. If (1.2) holds for the smallest spacing of eigenvalues of $\hat{\xi}$ the probability density of eigenvalues x of the pointer displacement $\hat{x}$ will display separated peaks defined by $x_i = \epsilon\xi_i$ (i labeling the discrete set of eigenvalues of $\hat{\xi}$)— and readings of those displacements will not noticeably react back on the meter.

The left part of the inequality (1.2) can be realized by providing the meter with a sufficient degree of excitation while the right part calls for a strong object-meter coupling.

A more complete and satisfactory picture of the measurement process arises when we allow the meter to interact with a heat bath. We adopt, in Sec. III, a well-known exactly solvable model of a bath and its interaction with the meter which turns the meter into a damped oscillator. Such a bath provides a most natural mechanism for imposing thermal equilibrium on the meter before the interaction with the object. For sufficiently high temperatures, then, the left part of the inequality (1.2) holds true and the meter is an effectively classical system. Indeed, even if the meter is assumed in a very nonclassical state (maybe a superposition of macroscopically distinguishable displacement eigenstates) at some time t_0, such that $\sigma_{xx}(t_0) \approx \sigma_{yy}(t_0)$, a high-temperature heat bath will subdue the de Broglie wavelength and enforce $\sigma_{yy} \ll \sigma_{xx}$ a few inverse damping constants later. So benign the potential effect of the bath in fact is that an intrinsically quantum-mechanical state with $\sigma_{yy} \approx \sigma_{xx}$ could even be realized immediately before the object-bath coupling; provided only the bath imparts overdamping with two vastly different time constants $1/\Gamma_-$ and $1/\Gamma_+$ to the meter, effectively classical behavior with $\sigma_{yy} \ll \sigma_{xx}$ is established on the shorter time scale $1/\Gamma_+$ while the information on the initial state of the object is wiped out from the probability density of meter readings only much later, i.e., for $t > 1/\Gamma_-$.

An even more interesting model results if the meter-bath coupling is made so strong as to soften the restoring force of the meter beyond the limit of stability. One of the damping constants of the meter then changes sign, $\Gamma_- \to -\Gamma_-$, and the rate Γ_- becomes an amplification rate. The meter dynamics has an eigenmode growing as $\exp(\Gamma_- t)$ and a decaying one, $\exp(-\Gamma_+ t)$. Once a sufficient amount of amplification has taken place, $\exp(\Gamma_- t) \gg 1$, the decaying mode will no longer be noticeable. As we show in Sec. IV the pointer displacement and momentum will then be locked in a rigid adiabatic equilibrium with one another and jointly grow in time as $\langle \hat{p}^{\,n}(t)\rangle = (M\Gamma_-)^n \langle \hat{x}^{\,n}(t)\rangle \sim e^{n\Gamma_- t}$. This behavior, well known from the theory of linear amplification, corresponds to noiseless deterministic amplification of the pointer displacement. By virtue of the correspondence principle, in every run of the experiment triggered by the object-meter interaction the pointer variable will eventually end up in a classical noiseless trajectory. Quantum effects will be manifest in an ensemble of such trajectories generated by many repeated runs or, equivalently, in a randomness of the effective initial displacement an individual classical trajectory appears to originate from. A condition similar to (1.2) secures the initial noise to be small enough for the random effective initial displacements to strongly cluster around certain discrete values uniquely related to the eigenvalues ξ_i. The later deterministic amplification increases the separation of the corresponding peaks of the probability density of meter readings while none of those peaks is washed out by diffusive effects.

It may be worthwhile to point out that the condition securing effectively classical behavior of the pointer variable $\hat{x}$ (together with its canonically conjugate momentum $\hat{p}$) does not imply that the meter density matrix be diagonal with respect to eigenstates $|x\rangle$ of $\hat{x}$. In fact, $\langle x | \rho_M | x'\rangle = 0$ for $x \neq x'$ is neither necessary nor sufficient for the pair $\hat{x},\hat{p}$ to behave effectively classically. Off-diagonal elements $\langle x | \rho_M | x'\rangle$ contain all the information about the moments $\langle \hat{p}^{\,n}\rangle$. The concept of a diagonal density matrix is too narrow to accommodate effectively classical behavior of a pair of conjugate observables. Only macroscopically distinct states $|x\rangle$ and $|x'\rangle$, i.e., states with a separation $|x - x'|$ by far exceeding a typical de Broglie wavelength, have vanishingly small coherences $\langle x | \rho | x'\rangle$.

It would not be practical to attempt a complete account of the development of the present-day understanding of the measurement process. We think it is appropriate, though, to give reference to those works which have directly influenced our present paper. Our treatment is in the tradition begun by von Neumann's realization that the reduction of a superposition to a mixture is an irreversible phenomenon going along with an increase of the entropy.[1] Moreover, von Neumann first used an object-meter interaction Hamiltonian equivalent to (1.1). We owe to Daneri, Loinger, and Prosperi[2] the idea that apparent irreversibility (apparent on time scales smaller than Poincaré cycles) is quite normal a mode of behavior of many-body systems such as measurement devices, not at all in conflict with formally unitary time evolution. One of us was taught model building by Weidlich[3] along the lines of thought of Ref. 2. More recently, Zurek[4] has expounded the idea that the simplest measurements involve an object, a meter, and a bath; the object-meter interaction serving to correlate different eigenstates of the object observable to be measured with different eigenstates of a suitable pointer variable of the meter; the meter-bath interaction being responsible for the effectively irreversible destruction of coherences between macroscopically distinct eigenstates of the pointer variable. Walls and co-workers[5] have shown, using master-equation techniques similar to the ones used in Ref. 3, that the laws of measurement hold for many model systems, some of which seem realizable in nonlinear optics. Finally, the present paper was preceded by a preliminary account of our work in Ref. 6.

As regards influences from outside measurement theory we should first mention Ullersma's[7] venerable model of a harmonic oscillator turned into a damped one by a heat bath. The strong-coupling limits of overdamping and amplification of that model were investigated in Refs. 8

and 6, respectively. The idea that quantum coherences between macroscopically distinct states have exceedingly short lifetimes even for feebly damped systems has recently been popularized by Leggett and Caldeira[9] [see also Ref. 5(a)]. Finally, we learned about the asymptotic noiselessness of single trajectories in linear amplifiers from the theory of superfluorescence[10] (see also Refs. 11 and 12); its implications for measurements were suggested in Ref. 6 and, independently, by Glauber.[13]

II. OBJECT-METER INTERACTION

We imagine the meter to be a harmonic oscillator with a mass M so large that our quantum description will eventually reveal effectively classical behavior. Denoting the displacement and momentum operators of the meter by $\hat{x}$ and $\hat{p}$ we write the Hamiltonian as

$$H_M = \frac{1}{2M}\hat{p}^2 + \tfrac{1}{2}M\omega^2\hat{x}^2 \; . \tag{2.1}$$

We shall employ the meter displacement $\hat{x}$ as a pointer variable.

We denote the object variable to be measured by $\hat{\xi}$ and assume it to have a discrete spectrum. The object-meter interaction should be designed so as to exert as little a perturbation as possible on the occupation probability of the eigenstates of $\hat{\xi}$. On the other hand, the interaction ought to strongly correlate the observable $\hat{\xi}$ with the pointer variable $\hat{x}$. Both requirements are met by a Hamiltonian first introduced to measurement theory by von Neumann,[1]

$$H_{O\text{-}M} = \epsilon\delta(t)\hat{\xi}\hat{p} \; , \tag{2.2}$$

where ϵ is a coupling constant. The δ-function modulation in (2.2), together with

$$[H_{O\text{-}M}, \hat{\xi}] = 0 \; , \tag{2.3}$$

ensures the object observable $\hat{\xi}$ to be conserved during the interaction process, whatever the unperturbed object Hamiltonian H_O may look like. (Our model will therefore describe what has lately been called a quantum nondemolition measurement[14]). Moreover, since $\hat{p}$ generates translations of the pointer, the Hamiltonian (2.2) associates different pointer shifts with different initial eigenstates of $\hat{\xi}$. The unitary operator

$$U = e^{-i\epsilon\hat{\xi}\hat{p}/\hbar} \tag{2.4}$$

describes the change of state of the combined object-meter system brought about the interaction (2.2). It turns the joint eigenstate $|\xi,x\rangle$ of $\hat{\xi}$ and $\hat{x}$ into

$$U|\xi,x\rangle = |\xi, x+\epsilon\xi\rangle \; . \tag{2.5}$$

We immediately conclude that the expectation values of the object variable $\hat{\xi}$ and the pointer momentum $\hat{p}$ remain unchanged,

$$\begin{aligned} &\langle[\hat{\xi}(0^+)]^n\rangle = \langle[\hat{\xi}(0)]^n\rangle \; , \\ &\langle[\hat{p}(0^+)]^n\rangle = \langle[\hat{p}(0)]^n\rangle, \quad n=1,2,3,\ldots \; . \end{aligned} \tag{2.6}$$

We may assume the object and the meter uncorrelated before the interaction such that the density operator takes the form of a product,

$$\rho_{O\text{-}M}(0) = \rho_O\rho_M \; . \tag{2.7}$$

Right after the interaction the density matrix in the basis $|\xi,x\rangle$ reads

$$\langle\xi,x|\rho_{O\text{-}M}(0^+)|\xi',x'\rangle = \langle\xi|\rho_O|\xi'\rangle\langle x-\epsilon\xi|\rho_M|x'-\epsilon\xi'\rangle \; . \tag{2.8}$$

It is reasonable to require the displacement and the momentum of the meter to vanish in the mean initially. The pointer displacement brought about by the interaction is then characterized by the following mean and variance:

$$\begin{aligned} &\langle\hat{x}(0^+)\rangle = \epsilon\langle\hat{\xi}(0)\rangle \; , \\ &\sigma_{xx}(0^+) = \sigma_{xx}(0) + \epsilon^2\sigma_{\xi\xi}(0) \; . \end{aligned} \tag{2.9}$$

Clearly, for these shifts to give a significant account of the initial behavior of the observable $\hat{\xi}$ the signal-to-noise ratio must be small. If we require pointer readings to resolve the eigenvalues ξ of $\hat{\xi}$ on a scale $\Delta\xi$, the initial mean squared pointer displacement must obey

$$\sigma_{xx}(0) \langle (\epsilon\Delta\xi)^2 \; . \tag{2.10}$$

Beyond the rms displacement $\sqrt{\sigma_{xx}(0)}$ there is another important "length" scale characterizing the initial state of the meter, the de Broglie wavelength, the square of which we shall denote by

$$\sigma_{yy}(0) \equiv \hbar^2/\sigma_{pp}(0) = \sigma_{yy}(0^+) \; . \tag{2.11}$$

Obviously, if and only if

$$\sigma_{yy}(0) << \sigma_{xx}(0) \; , \tag{2.12}$$

the initial meter state contains sufficiently many quantum action units $\hbar$ for the classical use of the term "pointer reading" to become legitimate. If, on the other hand, the inequality (2.12) does not hold initially, the measurement process necessarily involves a second stage. That stage must provide the meter with effectively classical behavior without wiping out the information [(2.8),(2.9)] imparted to the meter by the object-meter interaction. We shall discuss such processes in Secs. III and IV.

In order to illustrate the object-meter correlations in the density matrix (2.8) we now take the meter to be in thermal equilibrium before it interacts with the object. Representing the canonical density operator

$$\rho_M = Z_M^{-1}e^{-\beta H_M} \tag{2.13}$$

in the pointer basis we find the matrix elements

$$\langle x+\tfrac{1}{2}y|\rho_M|x-\tfrac{1}{2}y\rangle = (2\pi\sigma_{xx})^{-1/2}e^{-x^2/2\sigma_{xx}^{\text{th}} - y^2/2\sigma_{yy}^{\text{th}}} \; , \tag{2.14}$$

which take the form of a product of two Gaussians. The quantity σ_{xx}^{th} obviously is the thermal mean-squared pointer reading while σ_{yy}^{th} is the squared thermal de Broglie wave length,

$$\langle \hat{x}^2 \rangle = \sigma_{xx}^{\rm th} = \frac{\hbar}{2M\omega}\coth(\beta\hbar\omega/2) \; ,$$
$$\sigma_{yy}^{\rm th} = \hbar^2/\langle \hat{p}^2 \rangle = \frac{2\hbar}{M\omega}/\coth(\beta\hbar\omega/2) \; . \qquad (2.15)$$

Special interest is due to the high-temperature limit,

$$\sigma_{xx}^{\rm th} = k_B T/M\omega^2 \; ,$$
$$\sigma_{yy}^{\rm th} = \hbar^2/Mk_B T \; , \qquad (2.16)$$

in which, for a macroscopic meter, the rms thermal displacement $(\sigma_{xx}^{\rm th})^{1/2}$ is, even though classical in nature, an exceedingly small length; smaller yet is the thermal de Broglie wavelength $(\sigma_{yy}^{\rm th})^{1/2}$. Indeed, while $(\sigma_{xx}^{\rm th})^{1/2}$ may be smaller than the scale resolvable by macroscopic readings, the thermal de Broglie wavelength $(\sigma_{yy}^{\rm th})^{1/2}$ may appear tiny even when compared to the radius of an atomic nucleus,[15]

$$\sigma_{yy}^{\rm th}/\sigma_{xx}^{\rm th} = (\hbar\omega/k_B T)^2 \ll 1 \; . \qquad (2.17)$$

It is tempting to interpret the disparity of the length scales $(\sigma_{yy}^{\rm th})^{1/2}$ and $(\sigma_{xx}^{\rm th})^{1/2}$ at high temperatures as an "approximate diagonality" of the canonical density matrix (2.14). The element of truth in such an interpretation is the negligibility of off-diagonal elements with a skewness of the order of the length resolution $(\sigma_{xx}^{\rm th})^{1/2}$; there is a danger with that interpretation, though. However small off-diagonal elements with a skewness y of the order of the length resolution $(\sigma_{xx}^{\rm th})^{1/2}$ may be, elements with an arbitrarily small skewness carry all of the information about the pointer momentum, as is obvious from the identity

$$\langle \hat{p}^n \rangle = \int dx \int dy (i\hbar)^n \delta^{(n)}(y) \langle x + \tfrac{1}{2}y \mid \rho_M \mid x - \tfrac{1}{2}y \rangle \; . \qquad (2.18)$$

The inequality (2.17) really means that the uncertainty product $\sigma_{xx}^{\rm th}\sigma_{pp}^{\rm th}$ is larger by far than the quantum limit $\hbar^2$, i.e., that the meter behaves effectively classically at high temperatures.

By inserting the equilibrium density matrix of the meter (2.14) in (2.8) we find the joint object-meter density matrix after the interaction to be

$$\begin{aligned}\langle \xi, x + \tfrac{1}{2}y \mid \rho_{O\text{-}M}(0^+) \mid \xi', x - \tfrac{1}{2}y \rangle &= \langle \xi \mid \rho_O \mid \xi' \rangle (2\pi\sigma_{xx}^{\rm th})^{-1/2} \\ &\quad \times \exp\{ -[x - \epsilon(\xi+\xi')/2]^2/2\sigma_{xx}^{\rm th} \\ &\qquad - [y - \epsilon(\xi-\xi')]^2/2\sigma_{yy}^{\rm th} \} \; . \end{aligned} \qquad (2.19)$$

A remarkable consequence for the reduced density matrix of the object,

$$\langle \xi \mid \rho_O(0^+) \mid \xi' \rangle = \langle \xi \mid \rho_O \mid \xi' \rangle e^{-\epsilon^2(\xi-\xi')^2/2\sigma_{yy}^{\rm th}} \; , \qquad (2.20)$$

now arises. The off-diagonal elements of $\rho_O(0^+)$ are suppressed on the scale $(\sigma_{yy}^{\rm th}/\epsilon)^{1/2}$. Inasmuch as this scale is smaller than the smallest eigenvalue spacing $(\xi - \xi')$ we may conclude that the object-meter interaction prepares the object in a mixture of eigenstates of $\hat{\xi}$,

$$\rho_O = \sum_{\xi,\xi'} \mid \xi \rangle \langle \xi \mid \rho_O \mid \xi' \rangle \langle \xi' \mid$$
$$\to \rho_O(0^+) \approx \sum_{\xi} \mid \xi \rangle \langle \xi \mid \rho_O \mid \xi \rangle \langle \xi \mid \; . \qquad (2.21)$$

This approximate diagonalization is often referred to as the "collapse of the wave function to a mixture."

Similarly important inferences can be drawn from (2.19) for the reduced density matrix of the meter,

$$\langle x + \tfrac{1}{2}y \mid \rho_M(0^+) \mid x - \tfrac{1}{2}y \rangle = e^{-y^2/2\sigma_{yy}^{\rm th}} \sum_{\xi} \langle \xi \mid \rho_0 \mid \xi \rangle (2\pi\sigma_{xx}^{\rm th})^{-1/2} e^{-(x-\epsilon\xi)^2/2\sigma_{xx}^{\rm th}} \qquad (2.22)$$

Like its predecessor before the interaction, (2.14), it has a width in y, $(\sigma_{yy}^{\rm th})^{1/2}$, substantially smaller than the width in x and thus describes an effectively classical ensemble. The probability density for a meter reading x is given as a discrete convolution of the initial probability $\langle \xi \mid \rho_0 \mid \xi \rangle$ for an eigenvalue ξ of the object observable $\hat{\xi}$ with the equilibrium probability density of a meter reading x. The convolution has the effect of smearing out $\langle \xi \mid \rho_0 \mid \xi \rangle$ over an interval proportional to the rms thermal meter reading, $\epsilon^{-1}(\sigma_{xx}^{\rm th})^{1/2}$. An ideal measurement would be one in which the initial probability density $\langle \xi \mid \rho_0 \mid \xi \rangle$ does not vary much on the scale $\epsilon^{-1}(\sigma_{xx}^{\rm th})^{1/2}$ and thus directly gives the density of meter readings as $\langle x \mid \rho_M \mid (0^+) \mid x \rangle = \sum_\xi \langle \xi \mid \rho_O \mid \xi \rangle \delta(x - \epsilon\xi)$. In general, thermal noise will be effective but as long as the inequality (2.10) holds the density $\langle x \mid \rho_M(0^+) \mid x \rangle$ will still display well-separated and well-pronounced peaks.

While our assumption of initial thermal equilibrium for the meter is not unrealistic (see also Sec. III below) and certainly offers technical convenience, it is by no means a necessary one. We could even adopt, instead of an initial mixture of meter states, a highly excited pure state such as an energy eigenstate $\mid n \rangle$ with energy $E_n = \hbar\omega(n + \tfrac{1}{2}) \approx n\hbar\omega$, $n \gg 1$. Like the thermal-equilibrium ensemble at a high temperature such a state is characterized by two vastly different lengths. The one corresponding to the rms thermal displacement $(\sigma_{xx}^{\rm th})^{1/2}$ is the rms of $\hat{x}$ in the state $\mid n \rangle$,

$$(\sigma_{xx}^{(n)})^{1/2} = \left[\frac{\hbar}{M\omega}(n + \tfrac{1}{2}) \right]^{1/2} = (E_n/M\omega^2)^{1/2} \; , \qquad (2.23)$$

while the now appropriate de Broglie wavelength is

$$(\sigma_{yy}^{(n)})^{1/2} = \hbar \langle \hat{p}^2 \rangle^{-1/2} = \left[\frac{\hbar}{M\omega(n + \tfrac{1}{2})} \right]^{1/2} = \frac{\hbar}{\sqrt{ME_n}} \; . \qquad (2.24)$$

Note that $(\sigma_{yy}^{(n)})^{1/2}$ can be interpreted as the mean distance of nodes of the wave function $\langle x \mid n \rangle = \phi_n(x)$. The thermal density matrix in the pointer representation (2.14) is now replaced by

$$\langle x + \tfrac{1}{2}y \mid n \rangle \langle n \mid x - \tfrac{1}{2}y \rangle = \phi_n^*(x + \tfrac{1}{2}y)\phi_n(x - \tfrac{1}{2}y) \; . \qquad (2.25)$$

Instead of a Gaussian falloff we here encounter rapid oscillations in y with $\sigma_{yy}^{(n)}$ as the "wavelength". If probed, as a function of y, on a length scale exceeding $(\sigma_{yy}^{(n)})^{1/2}$, these oscillations tend to cancel and the off-diagonal density matrix elements give zero effect. In this sense the reduced density matrix of the object after the interaction, the analogue of (2.20),

$$\langle \xi | \rho_O(0^+) | \xi' \rangle = \langle \xi | \rho_O | \xi' \rangle \int dx\, \phi_n^*[x - \epsilon(\xi-\xi')/2] \times \phi_n[x + \epsilon(\xi-\xi')/2] , \tag{2.26}$$

is again effectively diagonal and implies an [approximate, with respect to scales exceeding $\epsilon^{-1}(\sigma_{yy}^{(n)})^{1/2}$] collapse of the wave function to a mixture.

The diagonal elements in (2.25), on the other hand, give the initial probability density of a meter reading x. Their spread $(\sigma_{xx}^{(n)})^{1/2}$ tends to deteriorate the information on the initial object probability density $\langle \xi | \rho_0 | \xi \rangle$ retrievable from the final meter probability density [cf. (2.22)]

$$\langle x | \rho_M(0^+) | x \rangle = \sum_{\xi} \langle \xi | \rho_0 | \xi \rangle \, | \phi_n(x - \epsilon\xi) |^2 . \tag{2.27}$$

Again, the condition (2.10) ensures the meter noise does not mask the object signal.

It may be interesting to note that the thermal-equilibrium ensemble and the energy eigenstate yield, in our present context, effectively equivalent initial conditions for the meter if we take the energy eigenvalue and the thermal energy as equal to one another, $E_n = kT$. The usefulness of either initial state for a measurement of the observable $\hat{\xi}$ rests on the disparity of length scales

$$\sqrt{\sigma_{yy}} \ll \sqrt{\sigma_{xx}} < | \epsilon \Delta \xi | . \tag{2.28}$$

The left-hand member of this inequality implies the meter to behave effectively classically initially and leads to the collapse of the wave function of the object. The right-hand member in (2.25) ensures the initial uncertainty of the pointer displacement does not eliminate the signal imparted to the meter by the object.

For an experimenter the $\hat{\xi}\hat{p}$ coupling (2.2) may—in contrast to, say, a $\hat{\xi}\hat{x}$ coupling—not be an easy one to realize. It may therefore be appropriate to point out that our whole discussion would carry over to a model with

$$H_{O\text{-}M} = \tilde{\epsilon}\, \hat{\xi}\, \hat{x}\, \delta(t) \tag{2.29}$$

replacing (2.2). We would have to use the meter momentum $\hat{p}$ as the pointer variable and its eigenstates $|p\rangle$ as the pointer basis. By representing the joint density operator in the basis $|\xi,p\rangle$ we would again find an effective diagonalization ("collapse") with $\langle \hat{p}^2 \rangle = 2ME$ and $\hbar^2/\langle \hat{p}^2 \rangle = \hbar^2/2ME$ replacing σ_{xx} and σ_{yy}, respectively. The inequality (2.28) would read $\langle \hat{p}^2 \rangle^{1/2} \ll \hbar / \langle \hat{x}^2 \rangle^{1/2} \ll \tilde{\epsilon}(\xi - \xi')_{\min}$. Reading the momentum of a highly excited oscillator with a large mass is, of course, no more difficult than reading its displacement.

III. METER-BATH INTERACTION

We now propose to show that a yet larger class of initial conditions is admissible for the meter than the arguments of Sec. II indicate.

Imagine the meter prepared in a pure state $|\phi_M\rangle$ such that the wave function $\phi_M(x)$ is a smooth function of x over its whole extent, i.e., such that the rms displacement and the de Broglie wave length are comparable in magnitude. The pure state in question could be an energy eigenstate $|n\rangle$ with a small quantum number n or a spatially more extended state, possibly even corresponding to a superposition of macroscopically distinguishable pointer states $|x\rangle$. At any rate, we are now considering a very nonclassical situation since no state with

$$\sigma_{yy}(0) \approx \sigma_{xx}(0) \tag{3.1}$$

can assign effectively classical behavior to the meter.

The combined density matrix after the meter-object interaction,

$$\langle x + \tfrac{1}{2}y, \xi | \rho_{O\text{-}M}(0^+) | x - \tfrac{1}{2}y, \xi' \rangle = \langle \xi | \rho_O(0) | \xi' \rangle \phi_M(x + \tfrac{1}{2}y + \epsilon\xi) \phi_M^*(x - \tfrac{1}{2}y + \epsilon\xi') , \tag{3.2}$$

would therefore imply that the object alone is not as well representable by a mixture of eigenstates of $\hat{\xi}$ as in the previously considered case (2.28); the initial states in question are thus not ideally suited for the purpose of preparing eigenstates of $\hat{\xi}$ by our interaction scheme. Worse yet, by reducing (3.2) to the density matrix of the meter alone we still confront a de Broglie wave length comparable to the rms pointer displacement and thus an intrinsically quantum behavior of the meter.

Contrived and difficult to produce in practice as the initial states in question may be, they do not, as we now proceed to explain, preclude the possibility of retrieving information about the state of the object from later pointer readings.

Once out of contact with the object the meter will never, in practice, be an isolated oscillator. It will rather suffer a coupling, however weak, to its environment which we may look upon as a heat bath at some temperature T. For the meter displacement $\hat{x}$ to be an acceptable pointer variable the meter-bath coupling must cause a decay of the length-scale ratio σ_{yy}/σ_{xx} to a magnitude sufficiently small for the meter to approach effectively classical behavior. Moreover, this decay must take place before the excitation imparted to the meter by the object [i.e., the excitation implied by the diagonal elements of (3.2)] is dissipated in the bath.

The dynamics just sketched as desirable can be realized if the meter-bath interaction turns the free oscillations of the meter into an irreversible and, in fact, overdamped motion. Indeed, an overdamped oscillator has two time scales: a short one on which the momentum rushes into an adiabatic equilibrium with the displacement and a large one on which the displacement, dragging along the momentum, creeps towards absolute equilibrium. For a sufficiently high bath temperature the length-scale ratio σ_{yy}/σ_{xx} drops to a very small value on the smaller one of the two time scales mentioned.

For a quantitative investigation we shall employ a well-known exactly solvable model.[7,8] The model consists of a central oscillator (our meter) and N further harmonic oscillators (the heat bath) coupled to the central one by a bilinear coordinate-coordinate coupling. The full Hamiltonian,

$$H=\frac{\hat{p}^2}{2M}+\frac{M\omega^2}{2}\hat{x}^2+\sum_{i=1}^{N}\left[\frac{\hat{p}_i^2}{2m}+\frac{m\omega_i^2}{2}\hat{x}_i^2\right]+\sum_{i=1}^{N}\epsilon_i\sqrt{mM}\,\hat{x}\,\hat{x}_i\ , \tag{3.3}$$

is a quadratic form in all $N+1$ pairs of coordinates and momenta and therefore allows for explicit diagonalization. For H to have a lower bound the coupling constants ϵ_i and the unperturbed frequencies must obey the positivity condition

$$\omega^2-\sum_{i=1}^{N}\epsilon_i^2/\omega_i^2\geq 0\ . \tag{3.4}$$

We shall eventually be interested in the limit where the bath oscillators become infinite in number and even form a continuum.[16] In that limit sums over the bath oscillators take the form of integrals and we need a spectral strength function $\gamma(\omega)$ defined as

$$\gamma(\omega)\Lambda\omega=\sum_{(\omega<\omega_i<\omega+\Delta\omega)}\epsilon_i^2\ . \tag{3.5}$$

The positivity condition (3.4), for instance, then reads

$$\omega^2-\int_0^\infty d\nu\,\gamma(\nu)/\nu^2\geq 0\ . \tag{3.6}$$

To fully specify the model we must make a definite choice for the spectral strength $\gamma(\omega)$. We shall work with[7,8]

$$\gamma(\omega)=\frac{2}{\pi}\frac{\kappa\alpha^2\omega^2}{\alpha^2+\omega^2} \tag{3.7}$$

since in that case all frequency integrals of interest can be expressed in terms of simple known functions. The two parameters κ and α are frequencies by dimension. They are restricted to obey

$$\alpha\kappa\leq\omega^2 \tag{3.8}$$

by the positivity condition (3.6). Obviously, κ can be understood as a measure of the overall strength of the coupling of the central oscillator to the bath. The meaning of α can be inferred from the response function

$$\Theta(t)\frac{i}{\hbar}\left\langle\left[\sum_i\epsilon_i\hat{x}_i(t),\sum_j\epsilon_j\,\hat{x}_j(0)\right]\right\rangle\equiv R_{\text{bath}}(t) \tag{3.9}$$

with the time dependence according to the free bath Hamiltonian. In the continuum limit and with the spectral strength (3.7) we find

$$R_{\text{bath}}(t)=(\kappa\alpha^2/m)e^{-\alpha t}\Theta(t)\ . \tag{3.10}$$

The inverse of α is thus the response time of the bath observable to which the coordinate $\hat{x}$ of the central oscillator is coupled to the Hamiltonian (3.3).

In pursuing the main goal of this section we shall consider an initial density operator without correlations between the meter and the bath,

$$\rho_{M\text{-}B}(0^+)=\rho_M(0^+)Z_B^{-1}e^{-\beta H_B}\ . \tag{3.11}$$

The canonical operator $Z_B^{-1}\exp(-\beta H_B)$ describes thermal equilibrium in the bath and involves the free bath Hamiltonian ($\epsilon_i=0$). Eventually, we shall specify the meter density operator $\rho_M(0^+)$ as the one produced by our impulsive object-meter interaction.

The initial value problem posed by the Hamiltonian (3.3) and the initial data (3.11) has been solved in Refs. 7 and 8. We therefore need not burden the following discussion with detailed algebra. Due to the harmonicity of the Hamiltonian (3.3) and the ensuring Gaussian nature of the initial density operator of the bath [see (3.11)] the dynamics of the meter can be described in terms of only three functions of time, to be called $A(t)$, $X(t)$, and $Y(t)$. These functions are related to the means and the variances of the meter displacement and momentum,

$$\begin{aligned}
&\langle\hat{x}(t)\rangle=\dot{A}(t)\langle\hat{x}(0^+)\rangle+A(t)\langle\hat{p}(0^+)\rangle/m\ ,\\
&\langle\hat{p}(t)\rangle=M\langle\dot{\hat{x}}(t)\rangle\ ,\\
&\sigma_{xx}(t)=\dot{A}(t)^2\sigma_{xx}(0^+)+2\dot{A}(t)A(t)M^{-1}\sigma_{xp}(0^+)\\
&\qquad+A(t)^2M^{-2}\sigma_{pp}(0^+)+M^{-1}X(t)\ ,\\
&\sigma_{pp}(t)=\ddot{A}(t)^2M^2\sigma_{xx}(0^+)+2\ddot{A}(t)\dot{A}(t)M\sigma_{xp}(0^+)\\
&\qquad+\dot{A}(t)^2\sigma_{pp}(0^+)+MY(t)\ ,\\
&\sigma_{xp}(t)\equiv\tfrac{1}{2}\langle\hat{x}(t)\hat{p}(t)+\hat{p}(t)\hat{x}(t)\rangle-\langle\hat{x}(t)\rangle\langle\hat{p}(t)\rangle\\
&\qquad=\tfrac{1}{2}M\dot{\sigma}_{xx}(t)\ ,
\end{aligned} \tag{3.12}$$

and determine the meter density matrix as

$$\begin{aligned}\langle x+\tfrac{1}{2}y\,|\rho_M(t)|\,x-\tfrac{1}{2}y\rangle=&\int\frac{dk}{2\pi}\int dx'\exp[ikx-i(k\dot{A}-My\ddot{A}/\hbar)x']\exp(-k^2X/2M-y^2YM/2\hbar^2+ky\dot{X}/2\hbar)\\
&\times\langle x'-\hbar kA/2M+\tfrac{1}{2}y\dot{A}\,|\rho_M(0^+)|\,x'+\hbar kA/2M-\tfrac{1}{2}y\dot{A}\rangle\ .\end{aligned} \tag{3.13}$$

We shall reveal the amplitude $A(t)$ to describe an effectively irreversible modification of the oscillation of the meter brought about by the bath. Similarly, the quantities $X(t)$ and $Y(t)$ describe the buildup of thermal equilibrium in the meter. The latter statement is nicely illustrated by the following expressions which relate $X(t)$ and $Y(t)$ to the equilibrium fluctuations of the bath observable $\sum_i\epsilon_i\hat{x}_i$ to which the meter displacement $\hat{x}$ is coupled in the Hamiltonian (3.3):

$$\begin{aligned}X(t)&=\int_0^t dt'\int_0^t dt''A(t')A(t'')mC_{\text{bath}}(t'-t'')\\
Y(t)&=\int_0^t dt'\int_0^t dt''\dot{A}(t')\dot{A}(t'')mC_{\text{bath}}(t'-t'')\ ,\end{aligned} \tag{3.14}$$

$$C_{\rm bath}(t)=\left\langle \left\{ \sum_i \epsilon_i x_i(t), \sum_j \epsilon_j x_j(0) \right\} \right\rangle . \tag{3.15}$$

The ensemble average and the time evolution in the bath correlation function (3.15) are meant with respect to the free bath ($\epsilon_i=0$). The curly bracket in (3.15) denotes the symmetrized product. By straightforward evaluation of the average we find, in the continuum limit,

$$C_{\rm bath}(t)=\frac{1}{m}\int_0^\infty d\omega \frac{\gamma(\omega)}{\omega^2} E(\beta\omega)\cos(\omega t) , \tag{3.16}$$

where

$$E(\beta\omega)=\tfrac{1}{2}\hbar\omega \coth(\beta\hbar\omega/2) \tag{3.17}$$

is the thermal energy of a harmonic oscillator at the temperature T. We are especially interested in the high-temperature limit where $E\to k_B T$ and[17]

$$C_{\rm bath}(t)=\frac{k_B T}{m}\kappa\alpha e^{-\alpha|t|} . \tag{3.18}$$

We now turn to evaluating the amplitude $A(t)$. Obviously, $A(t)$ can be represented as a sum or, in the continuum limit, as an integral over eigenmodes of the Hamiltonian (3.3). By diagonalizing H and again employing the spectral strength (3.7) we find the exact result,

$$A(t)=\frac{\Gamma_+ + \Gamma_-}{(\lambda-\Gamma_+-\Gamma_-)^2}(e^{-\lambda t}-\tfrac{1}{2}e^{-\Gamma_- t}-\tfrac{1}{2}e^{-\Gamma_+ t}) + \frac{\lambda^2-\frac{1}{2}(\Gamma_+^2+\Gamma_-^2)}{(\lambda-\Gamma_+-\Gamma_-)^2}\frac{1}{\Gamma_+-\Gamma_-}(e^{-\Gamma_- t}-e^{-\Gamma_+ t}) , \tag{3.19}$$

where the positive parameters λ and $\Gamma_\pm$ arise as the roots of a cubic equation and obey the Vieta identities

$$\begin{aligned} &\lambda+\Gamma_+ +\Gamma_- =\alpha ,\\ &\lambda(\Gamma_+ +\Gamma_-)+\Gamma_+\Gamma_- =\omega^2 ,\\ &\lambda\Gamma_+\Gamma_- =\alpha(\omega^2-\kappa\alpha) . \end{aligned} \tag{3.20}$$

The two exponentials in $A(t)$ pertaining to the pair of roots of $\Gamma_\pm$ emerge from the harmonic oscillations of the meter with the unperturbed frequency ω as the coupling to the bath is switched on. Similarly, the exponential $e^{-\lambda t}$ is related to the exponential $e^{-\alpha t}$ in the response function (3.9) and the correlation function (3.18) of the free bath. Actually, overdamping arises only for sufficiently strong meter-bath couplings; otherwise, the two real roots $\Gamma_\pm$ become a pair of complex conjugate roots, $\Gamma_\pm \to \Gamma \pm i\Omega$, and the meter displays damped oscillations.[7,8,18]

With the help of (3.19), (3.18), and (3.14) we can now easily obtain explicit rigorous expressions for the thermal amplitudes $X(t)$ and $Y(t)$ at high temperatures. However, these expressions deserve interest only in the limit

$$\alpha \gg \Gamma_\pm \tag{3.21}$$

in which bath correlations decay effectively instantaneously with respect to the time scales $1/\Gamma_\pm$ of the meter relaxation. Indeed, without the separation of time scales expressed in (3.21) the bath would not even deserve its name. We infer from (3.20) that the limit in consideration implies

$$\lambda\approx\alpha, \quad \Gamma_\pm \approx \frac{\kappa}{2} \pm \left[\frac{\kappa^2}{4}-\omega^2+\kappa\alpha\right]^{1/2} . \tag{3.22}$$

With respect to the time scale $1/\Gamma_\pm$ we can therefore drop the fast transient $e^{-\lambda t}$ from (3.19) and have the asymptotic amplitude

$$A_{\rm as}(t)=(\Gamma_+-\Gamma_-)^{-1}(e^{-\Gamma_- t}-e^{-\Gamma_+ t}), \quad t\gg 1/\alpha . \tag{3.23}$$

Due to (3.12) the corresponding asymptotic meter displacement obeys the equation of motion of an overdamped oscillator,

$$\langle \ddot{\hat{x}}(t)\rangle_{\rm as}+(\Gamma_+ +\Gamma_-)\langle \dot{\hat{x}}(t)\rangle_{\rm as}+\Gamma_+\Gamma_-\langle \hat{x}(t)\rangle_{\rm as}=0 . \tag{3.24}$$

In constructing the asymptotic versions of the thermal amplitudes $X(t)$ and $Y(t)$ from (3.14) we employ (3.23) and observe that with respect to the time scales $1/\Gamma_\pm$ the bath correlation function (3.17) is effectively δ shaped,

$$C_{\rm bath,as}(t)=[2(\Gamma_+ +\Gamma_-)k_B T/m]\delta(t) . \tag{3.25}$$

We thus obtain

$$\begin{aligned} X_{\rm as}(t)&=\frac{k_B T}{\Gamma_+\Gamma_-}\left[1-\frac{\Gamma_+(\Gamma_+ +\Gamma_-)}{(\Gamma_+-\Gamma_-)^2}e^{-2\Gamma_- t} - \frac{\Gamma_-(\Gamma_+ +\Gamma_-)}{(\Gamma_+-\Gamma_-)^2}e^{-2\Gamma_+ t} + \frac{4\Gamma_+\Gamma_-}{(\Gamma_+-\Gamma_-)^2}e^{-(\Gamma_+ +\Gamma_-)t}\right] ,\\ Y_{\rm as}(t)&=k_B T\left[1-\frac{\Gamma_-(\Gamma_+ +\Gamma_-)}{(\Gamma_+-\Gamma_-)^2}e^{-2\Gamma_- t} - \frac{\Gamma_+(\Gamma_+ +\Gamma_-)}{(\Gamma_+-\Gamma_-)^2}e^{-2\Gamma_+ t} + \frac{4\Gamma_+\Gamma_-}{(\Gamma_+-\Gamma_-)^2}e^{-(\Gamma_+ +\Gamma_-)t}\right] . \end{aligned} \tag{3.26}$$

We are now fully equipped to investigate the fate of the density matrix (3.13). As a first observation it may be worth noting that the amplitudes A, X, and Y approach stationary values after a few $1/\Gamma_-$,

$$A(\infty)=0, \quad X(\infty)=k_B T/\Gamma_+\Gamma_-, \quad Y(\infty)=k_B T . \tag{3.27}$$

The x' integral in (3.13) then reduces to the normalization integral of $\rho_M(0^+)$ and wipes out all information about the initial state of the meter. Upon carrying out the remaining Gaussian integral over k we recover the thermal density matrix (2.14) with the rms thermal displacement and the thermal de Broglie wave length given in (2.15), except for the replacement of the unperturbed meter frequency ω by the shifted one, $\sqrt{\Gamma_+\Gamma_-}$. Obviously, this final regime is of no interest for our discussion of the measurement process.

The approach to thermal equilibrium described by (3.23) and (3.26) will be characterized by two vastly different time scales if we assume the meter heavily overdamped,

$$\Gamma_- \ll \Gamma_+ \ . \tag{3.28}$$

In this highly interesting limit, to which the remainder of this section is devoted, $X(t)$ and thus the mean squared pointer displacement thermalize on the long time scale $1/\Gamma_-$. The amplitude $Y(t)$, however, rushes to a $O(\Gamma_-/\Gamma_+)$ neighborhood of its equilibrium value k_BT in a time of the order $1/\Gamma_+$. Accounting now for $\hat{x}(0^+)=\hat{x}(0)+\epsilon\hat{\xi}(0)$, assuming again $\langle\hat{x}(0)\rangle=\langle\hat{p}(0)\rangle=0$, and dropping all transients decaying with the rate Γ_+ we obtain, for $t>1/\Gamma_+$,

$$\begin{aligned}
&\langle\hat{x}(t)\rangle=\epsilon\langle\hat{\xi}(0)\rangle\dot{A}(t)\ ,\\
&\sigma_{xx}(t)=\{\epsilon^2\sigma_{\xi\xi}(0)+\langle\Delta[\hat{x}(0)-\hat{p}(0)/M\Gamma_-]^2\rangle\}\dot{A}(t)^2\\
&\qquad+\sigma_{xx}^{\rm th}\left[1-\frac{\Gamma_+(\Gamma_++\Gamma_-)}{(\Gamma_+-\Gamma_-)^2}e^{-2\Gamma_-t}\right],\\
&\sigma_{pp}(t)=\hbar^2/\sigma_{yy}(t)\\
&\qquad=(M\Gamma_-)^2\{\epsilon^2\sigma_{\xi\xi}(0)\\
&\qquad\quad+\langle\Delta[\hat{x}(0)-\hat{p}(0)/M\Gamma_-]^2\rangle\}\dot{A}(t)^2\\
&\qquad\quad+\sigma_{pp}^{\rm th}\left[1-\frac{\Gamma_-(\Gamma_++\Gamma_-)}{(\Gamma_+-\Gamma_-)^2}e^{-2\Gamma_-t}\right],\\
&\dot{A}(t)=-[\Gamma_-/(\Gamma_+-\Gamma_-)]e^{-\Gamma_-t}\ ,
\end{aligned}\tag{3.29}$$

with

$$\sigma_{xx}^{\rm th}=\sigma_{pp}^{\rm th}/M^2\Gamma_+\Gamma_-=k_BT/M\Gamma_+\Gamma_-\ .$$

The variance $\langle\Delta[\hat{x}(0)-\hat{p}(0)/M\Gamma_-]^2\rangle$ of the initial deviation from adiabatic equilibrium, $\hat{x}-\hat{p}/M\Gamma_-$, here appears as part of the meter noise which the signal received from the object must exceed in order to be detectable. For the displacement variance $\sigma_{xx}(t)$ to be dominated, in the time interval $1/\Gamma_+<t<1/\Gamma_-$, by the signal $\epsilon^2\sigma_{\xi\xi}(0)$ we must require

$$\begin{aligned}
&\epsilon^2\sigma_{\xi\xi}(0)\gg\langle\Delta[\hat{x}(0)-\hat{p}(0)/M\Gamma_-]^2\rangle\ ,\\
&\frac{\Gamma_-}{\Gamma_+}\epsilon^2\sigma_{\xi\xi}(0)\gg\sigma_{xx}^{\rm th}\ ,
\end{aligned}\tag{3.30}$$

the latter condition accounting for the signal reduction during the decay of the fast transient and for $[\Gamma_+(\Gamma_++\Gamma_-)/(\Gamma_+-\Gamma_-)^2]e^{-2\Gamma_-t}=1-O(\Gamma_-/\Gamma_+)$. It is most interesting to see that (3.30) also secures signal predominance in the momentum variance. In fact, effectively classical behavior of our pointer in the time interval under consideration would not be consistent with a signal-dominated displacement and an all-noise momentum. We secure effectively classical behavior of the pointer by requiring $\sigma_{xx}(t)\sigma_{pp}(t)\gg\hbar^2$, or, equivalently,

$$M^2\Gamma_-^2\epsilon^2\sigma_{\xi\xi}(0)\gg\hbar^2\ . \tag{3.31}$$

We should point out that none of the conditions [(3.30) and (3.31)] is in conflict with (3.1), i.e., with an intrinsically quantum-mechanical initial state of the meter. We have thus proven the assertion made in the beginning of this section: the pointer will wind up behaving classically even if it was prepared in a very nonclassical initial state; moreover, if the conditions [(3.30) and (3.31)] are met a set of pointer readings gives a classical statistical account of the initial state of the object.

While it is interesting to see that the overdamped meter can be useful as a measurement dervice even if prepared in a very nonclassical initial state, such states are difficult to produce and thus unlikely to have practical significance. It is, in fact, much more natural to assume the meter in thermal equilibrium with the heat bath before it interacts with the object, i.e., to employ the canonical density matrix defined in (2.13)–(2.16) with $\omega^2=\Gamma_+\Gamma_-$. The impulsive object-meter interaction then yields (2.22) as the meter density matrix $\rho_M(0^+)$ to be used in (3.13). The ensuing Gaussian integrals in (3.13) are easily evaluated. For $t\gg1/\Gamma_+$ and to leading order in the small parameter Γ_-/Γ_+ we obtain the probability density of pointer readings as

$$\langle x\,|\,\rho_M(t)\,|\,x\rangle=\sum_\xi\langle\xi\,|\,\rho_0\,|\,\xi\rangle\frac{1}{\sqrt{2\pi\sigma(t)}}e^{-[x-\epsilon\xi\dot{A}(t)]^2/2\sigma(t)} \tag{3.32}$$

with the width

$$\sigma(t)=\sigma_{xx}^{\rm th}(1-e^{-2\Gamma_-t})\ .$$

Obviously, the density (3.32) allows for a classical interpretation in terms of the Smoluchowski process.[19] We may interpret (3.32) as giving a classical probabilistic account of an ensemble of stochastic pointer trajectories. For times $t\ll1/\Gamma_-$ each such trajectory can be associated with an eigenvalue ξ of the object observable $\hat{\xi}$ provided these eigenvalues are spaced such that $\epsilon\Delta\xi\gg\sigma_{xx}^{\rm th}$ [see (2.28)].

IV. AMPLIFYING METERS

An interesting and not at all unrealistic variant of our model of the meter-bath interaction arises when we relax the positivity condition (3.4) of the Hamiltonian (3.3). Upon increasing $\kappa\alpha/\omega^2$ (keeping $\omega/\alpha\ll1$ fixed) through unity the root Γ_- in (3.20) changes sign and the meter state with $x=0$ loses stability, just as if the potential energy had changed sign to take the form of an inverted parabola [see Eq. (3.24)]. Our meter is thus transformed into an amplifier.

For a damped meter, a signal initially received from the object tends to be dissipated in the bath. In an amplifying meter, on the other hand, such a signal will grow in time. Even if it is so tiny as to be quantum mechanical in nature at early stages it will eventually reach macroscopic magnitude; only then does the term "meter reading" assume its classical meaning which implies no noticeable back reaction on the meter. Of course, noise will also amplify and we must therefore establish the conditions under which a meter reading x reveals significant information about the object observable $\hat{\xi}$.

One problem with an amplifier prepared in an unstable equilibrium deserves immediate discussion: it can fire spontaneously, i.e., even without being fed a signal. Such devices can therefore be useful in practice only if their intrinsic noise is so small compared to the signals to be detected that spontaneous firings are relatively rare events.

If we again adopt the initial condition (3.11), i.e., thermal equilibrium for the free bath and no initial meter-bath correlation the whole analysis of Sec. III remains valid save for the change of sign of $\omega^2-\alpha\kappa$ and of Γ_-. We shall rename the now negative root of (3.20) as $\Gamma_- \to -\Gamma_-$ so that in the following $\Gamma_- > 0$ will be an amplification rate. The amplitudes $A(t)$, $X(t)$, and $Y(t)$ result from (3.23) and (3.26) with $\Gamma_- \to -\Gamma_-$. Especially, for times exceeding $1/\Gamma_-$ at which substantial amplification has already taken place while the attenuated mode $e^{-\Gamma_+ t}$ is no longer noticeable, we have

$$\dot{A}_{\rm as}(t)=\Gamma_- A_{\rm as}(t)=\frac{\Gamma_-}{\Gamma_+ + \Gamma_-}e^{\Gamma_- t}\,,$$
$$Y_{\rm as}(t)=\Gamma_-^2 X_{\rm as}(t)=k_B T\frac{\Gamma_-(\Gamma_+ - \Gamma_-)}{(\Gamma_+ - \Gamma_-)^2}e^{2\Gamma_- t}\,. \tag{4.1}$$

These expressions imply that the means and variances of both the pointer displacement and the pointer momentum grow indefinitely. In contrast to the damped case the de Broglie wavelength therefore does not settle at its thermal equilibrium value $(\sigma_{yy}^{\rm th})^{1/2}$ but suffers an exponential decay to zero. The meter thus tends to behave more and more classically as t grows on the scale $1/\Gamma_-$ set by the amplification rate.

The effectively classical behavior of the meter for $\Gamma_- t \gg 1$ is manifest in the density matrix, too. By using (4.1) in (3.13) we obtain

$$\begin{aligned}\langle x+\tfrac{1}{2}y \,|\, \rho_M^{\rm as}(t) \,|\, x-\tfrac{1}{2}y\rangle &= e^{ixyM\Gamma_-/\hbar - y^2/2\sigma_{yy}^+}[M\Gamma_-/2\pi\hbar \dot{A}_{\rm as}(t)]\\ &\quad\times\int dx'dy' e^{i[x/\dot{A}_{\rm as}(t)-x']y'M\Gamma_-/\hbar - y'^2/2\sigma_{yy}^-}\\ &\quad\times\langle x'+\tfrac{1}{2}y' \,|\, \rho_M(0^+) \,|\, x'-\tfrac{1}{2}y'\rangle \end{aligned}\tag{4.2}$$

with

$$\sigma_{yy}^{\pm}=\sigma_{yy}^{\rm th}\frac{\Gamma_\pm}{\Gamma_+ - \Gamma_-}=\frac{\hbar^2\Gamma_\pm}{Mk_BT(\Gamma_+ - \Gamma_-)}\,. \tag{4.3}$$

To reveal the simple dynamics implied by (4.2) we first consider the ensuing momentum moments. To within relative corrections which vanish as $e^{-\Gamma_- t}$ or as powers of that exponential we find, with the help of (2.18),

$$\langle \hat{p}^{\,m}(t)\rangle=(M\Gamma_-)^m\langle \hat{x}^{\,m}(t)\rangle\,. \tag{4.4}$$

More generally, symmetrized mixed moments obey

$$\langle [\hat{p}^{\,m}(t)\hat{x}^{\,n}(t)]_{\rm sym}\rangle=(M\Gamma_-)^m\langle \hat{x}^{\,m+n}(t)\rangle\,. \tag{4.5}$$

Obviously, Eqs. (4.4) and (4.5) describe a rigid adiabatic equilibrium of the displacement $\hat{x}$ and the momentum $\hat{p}$ of our pointer; they allow us, moreover, to restrict the further discussion of the density matrix (4.2) to its diagonal elements

$$\begin{aligned}\langle x \,|\, \rho_M^{\rm as}(t) \,|\, x\rangle &= \frac{M\Gamma_-}{2\pi\hbar \dot{A}_{\rm as}(t)}\\ &\quad\times\int dx'dy' e^{i[x/\dot{A}_{\rm as}(t)-x']y'M\Gamma_-/\hbar - y'^2/2\sigma_{yy}^-}\\ &\quad\times\langle x'+\tfrac{1}{2}y' \,|\, \rho_M(0^+) \,|\, x'-\tfrac{1}{2}y'\rangle\,. \end{aligned}\tag{4.6}$$

At this point it is obvious that we have entered classical territory. We should note, first of all, that $\langle x \,|\, \rho_M^{\rm as}(t) \,|\, x\rangle$ retains its meaning as the probability density of meter readings x in the classical limit. Moreover, due to the simple exponential growth of the amplitude $\dot{A}_{\rm as}(t)$ the probability density (4.6) has a time dependence such that it obeys

$$\left[\frac{\partial}{\partial t}+\Gamma_-\frac{\partial}{\partial x}x\right]\langle x \,|\, \rho_M^{\rm as}(t) \,|\, x\rangle=0\,. \tag{4.7}$$

The evolution equation (4.7), however, describes deterministic, i.e., noiseless linear amplification of the displacement x. The probability density in question thus drifts along the x axis following deterministic trajectories $x_{\rm as}(t)\sim \dot{A}_{\rm as}(t)\sim e^{\Gamma_- t}$. In fact, we can interpret the quantum average $\langle \hat{x}(t)^n\rangle=\int dx\, x^n\langle x \,|\, \rho_M^{\rm as}(t) \,|\, x\rangle$ with the weight (4.6) as an average over a bundle of such deterministic trajectories,

$$\langle \hat{x}(t)^n\rangle=e^{n\Gamma_- t}\int dx\, x^n\langle x \,|\, \rho_M^{\rm as}(0) \,|\, x\rangle\,, \tag{4.8}$$

the bundle originating from an "initial" cloud of points distributed with the density $\langle x \,|\, \rho_M^{\rm as}(0) \,|\, x\rangle$ obtained from (4.6) by setting $t=0$ in $\dot{A}_{\rm as}(t)$.

The classical noiseless trajectories in question have a physical meaning not only as an ensemble. Rather, in every run of the measurement the pointer will end up in one such trajectory as soon as a sufficient amount of amplification has taken place. In this respect our model provides (as does, in fact, every linear amplifier; see, e.g., Ref. 10) a nice illustration of the correspondence principle. What is quantum mechanical and appears random in a series of runs is the effective initial displacement a given trajectory appears to originate from. It is, of course, this initial randomness which makes it impossible to predict in which classical noiseless trajectory the pointer will wind up in an individual run of the measurement.

Our remaining task is the discussion of the effective initial noise. By inserting the meter density matrix from (2.8) in (4.6) we have

$$\langle x \,|\, \rho_M^{\rm as}(0) \,|\, x\rangle=\sum_\xi\langle \xi \,|\, \rho_0 \,|\, \xi\rangle\frac{M\Gamma_-}{2\pi\hbar \dot{A}_{\rm as}(0)}\int dx'dy' e^{i[x/\dot{A}_{\rm as}(0)-x']y'M\Gamma_-/\hbar - y'^2/2\sigma_{yy}^-}\langle x'-\epsilon\xi+\tfrac{1}{2}y' \,|\, \rho_M(0) \,|\, x'-\epsilon\xi-\tfrac{1}{2}y'\rangle\,. \tag{4.9}$$

We want a spacing $\Delta\xi$ of eigenvalues ξ of the object observable $\hat{\xi}$ to be resolved by later pointer readings. It is therefore necessary that $\epsilon\Delta\xi$ be larger than the width of the weight of x with which the probability $\langle \xi | \rho_0 | \xi \rangle$ is convoluted in (4.9). The width in question gets contributions from both the initial density operator $\rho_M(0)$ and the Gaussian integral kernel which accounts for the early-stage transients of the pointer preceding the pure adiabatic amplification regime. The initial meter noise can be roughly characterized by the rms displacement $\sqrt{\sigma_{xx}(0)}$ and a de Broglie wavelength $\sqrt{\sigma_{yy}(0)} = \hbar/\sqrt{\sigma_{pp}(0)}$. The noise increment due to the early-stage transients must be expressible in terms of the lengths $\sqrt{\hbar/M\Gamma_-}$ and $(\sigma_{yy}^{-})^{1/2}$ and the dimensionless ratio Γ_+/Γ_-. Assuming, for simplicity, $\Gamma_+ \approx \Gamma_- = \Gamma$ the lengths accessible are $l_z = (k_B T/MT^2)^{1/2}(\hbar\Gamma/k_B T)^z$ with an arbitrary exponent z. Among these, $l_0 = (\sigma_{xx}^{\text{th}})^{1/2}$ and $l_1 = (\sigma_{yy}^{\text{th}})^{1/2}$ have a special physical significance. On top of our basic assumption $\sigma_{yy}^{\text{th}} \ll \sigma_{xx}^{\text{th}}$ we must therefore stipulate

$$\epsilon\Delta\xi > (\sigma_{xx}^{\text{th}})^{1/2}, \quad \sqrt{\sigma_{xx}(0)}, \quad \sqrt{\sigma_{yy}(0)} \tag{4.10}$$

to provide the probability density $\langle x | \rho_M^{\text{as}}(0) | x \rangle$ with separated peaks at the displacements $x_i \approx \epsilon\xi_i \dot{A}_{\text{as}}(0)$ corresponding to the eigenvalues of $\hat{\xi}$. The effective initial pointer displacement will then not be likely to take on values in between the peaks of the density $\langle x | \rho_M^{\text{as}}(0) | x \rangle$. The later noiseless dynamics will transport the pointer to large displacements again not likely to lie in between the now much more widely separated positions $x_i(t) = \epsilon\xi_i A_i(t)$.

It may be worth pointing out that we do not have to request the initial de Broglie wavelength $\sqrt{\sigma_{yy}(0)}$ to be small compared to the initial rms displacement $\sqrt{\sigma_{xx}(0)}$ for the amplified pointer displacement to be "uniquely" (up to unlikely fakes) related to initial object eigenvalues ξ_i. We would like to recall from Sec. II, however, that the inequality

$$\sqrt{\sigma_{yy}(0)} \ll \sqrt{\sigma_{xx}(0)} \tag{4.11}$$

has the additional virtue of making the meter a good preparation device as well in that it secures strongly suppressed coherences $\langle \xi_i | \rho_0(0^+) | \xi_j \rangle$, $i \neq j$, after the impulsive object meter interaction.

The qualitative dimensional analysis of (4.9) can be replaced by a more specific one for concrete choices of the initial meter density operator $\rho_M(0)$. Special interest may be due to initial coherent states, squeezed states, and thermal equilibrium ensembles. We refrain from giving detailed formulas for any of these cases here since the general resolvability condition (4.10) is the only essential requirement an initial meter state must fulfill to qualify the meter as a measurement device.

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[1]J. von Neumann *Mathematische Grundlagen der Quantenmechanik* (Springer, Berlin, 1932).

[2]A. Daneri, A. Loinger, and G. M. Prosperi, Nucl. Phys. **33**, 297 (1962).

[3](a) W. Weidlich, Z. Phys. **205**, 199 (1967). (b) F. Haake and W. Weidlich, Z. Phys. **213**, 451 (1968).

[4]W. H. Zurek, Phys. Rev. D **24**, 1516 (1981); **26**, 1862 (1982).

[5](a) D. F. Walls and G. J. Milburn, Phys. Rev. A **31**, 2403 (1985); (b) C. M. Savage and D. F. Walls, Phys. Rev. A **32**, 2316 (1985); **32**, 3487 (1985); (c) R. F. O'Connell, C. M. Savage, and D. F. Walls, Proceedings of the New York Academy of Sciences, edited by D. Greenburger (1987); (d) D. F. Walls, M. Collet, and G. J. Milburn, Phys. Rev. D **32**, 3208 (1985).

[6]F. Haake and D. F. Walls, *Quantum Optics IV*, Vol. 12 of *Springer Proceedings in Physics*, edited by G. D. Harvey and D. F. Walls (Springer, Berlin, 1986).

[7]P. Ullersma, Physica (Utrecht) **32**, 27 (1966); **32**, 56 (1966); **32**, 74 (1966); **32**, 90 (1966).

[8]F. Haake and R. Reibold, Phys. Rev. A **32**, 2462 (1985).

[9]A. O. Caldeira and A. J. Leggett, Phys. Rev. A **31**, 1059 (1985).

[10]F. Haake, H. King, G. Schröder, J. W. Haus, and R. Glauber, Phys. Rev. A **20**, 2017 (1979); **23**, 1322 (1981).

[11]R. J. Glauber and T. von Foerster, Phys. Rev. A **3**, 1484 (1971).

[12]C. M. Caves, Phys. Rev. D **26**, 1817 (1982).

[13](a) R. J. Glauber, *Group Theoretical Methods in Theoretical Physics*, Proceedings of the International Conference, Zvenigorod, USSR, 1982 (Marwood Academic, New York, 1985); (b) R. J. Glauber, in *Frontiers in Quantum Optics*, edited by E. R. Pike and S. Sarkar, Vol. X of *Malvern Physics Series*, (Adam Hilger, Bristol, 1986).

[14]V. B. Braginski, Y. I. Vorontsov, and K. S. Thorne, Science **209**, 547 (1980).

[15]For $M = 1$ g, $\omega = 2\pi$ s^{-1}, $T = 300$ K we have $(\sigma_{xx}^{\text{th}})^{1/2} \approx 10^{-10}$ m and $(\sigma_{yy}^{\text{th}})^{1/2} \approx 10^{-24}$ m.

[16]In the continuum limit Poincaré recurrence times are infinite.

[17]We should note, in passing, that the correlation function $C_{\text{bath}}(t)$ and the response function $R_{\text{bath}}(t)$ as given by (3.18) and (3.10) obey the classical fluctuation dissipation theorem $R_{\text{bath}}(t) = -\Theta(t)(1/k_B T)\dot{C}_{\text{bath}}(t)$.

[18]We ought to note that the manifestly irreversible behavior of $A(t)$ is due to the continuum limit for the bath. For a finite number N of bath oscillators the amplitude $A(t)$ would display quasiperiodic behavior; however, for $N \gg 1$ and the discrete set of coupling constants ϵ_i still bunched as required in (3.7) the difference between the quasiperiodic version of $A(t)$ and (3.19) would almost never in time be noticeable; Poincaré recurrences of $A(t)$ to a narrow neighborhood of the initial value $A(0) = 1$, for instance, have a mean separation in the time growing exponentially with N.

[19]G. E. Uhlenbeck and L. S. Ornstein, Phys. Rev. **36**, 823 (1930).

[20](a) E. P. Wigner, Phys. Rev. **40**, 749 (1932); (b) J. E. Moyal, Proc. Chambridge Philos. Soc. **45**, 99 (1949).