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Photon-counting distribution in squeezed states

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We calculate the counting (multiplicity) distribution $P_N=\langle N|\rho|N\rangle$ where ρ is (i) the density matrix corresponding to a (pure) squeezed state and (ii) the density matrix of a superposition of squeezed states with thermal light. In the first case, we find that P_N is an oscillating function of N. In the second case, this behavior, characteristic for squeezed states, disappears even for small amounts of noise.

Squeezed states[1] are of great theoretical and possibly practical interest. They contain less than the standard zero-point fluctuations in one quadrature, at the expense of having more fluctuations at the other quadrature. Applications in optical communications have been considered in Ref. 2.

Here we study both pure squeezed states and the more realistic case of mixtures of squeezed states with thermal fields, and we evaluate numerically the multiplicity (counting) distribution $P_N=\langle N|\rho|N\rangle$.

We consider a representation of the SU(1,1) realized with the unitary operators

$$U_2(r,\theta,\lambda)=\exp[-\tfrac{1}{4}re^{-i\theta}(a^\dagger)^2+\tfrac{1}{4}re^{i\theta}a^2]\exp(i\lambda a^\dagger a)\ , \tag{1}$$

$$U_2^\dagger U_2=1,\ r\geqq 0,\ r,\theta,\lambda\in R,\ [a,a^\dagger]=1\ ,$$

and we introduce the squeezed coherent states $|A;r\theta\lambda\rangle$ as

$$|A;r\theta\lambda\rangle=U_2(r,\theta,\lambda)|A\rangle$$
$$=U_2(r,\theta,\lambda)U_1(A)|0\rangle\ , \tag{2}$$

$$U_1(a)=\exp(Aa^\dagger-A^*a)\ . \tag{3}$$

We can prove the relations

$$U_2aU_2^\dagger=\mu a+\nu a^\dagger=b,\ U_2a^\dagger U_2^\dagger=\nu^*a+\mu^*a^\dagger=b^\dagger\ ,$$
$$\mu=e^{-i\lambda}\cosh(\tfrac{1}{2}r),\ \nu=e^{-i(\lambda+\theta)}\sinh(\tfrac{1}{2}r)\ , \tag{4}$$
$$|\mu|^2-|\nu|^2=1\ .$$

The operators $b,b^\dagger$ obey the boson commutation relations $[b,b^\dagger]=1$ and the transformation (4) is a Bogoliubov transformation. From Eq. (4) and the fact that U_2 is unitary we can trivially prove for any function $f(a,a^\dagger)$ the relations

$$U_2f(a,a^\dagger)U_2=f(b,b^\dagger)\to U_2f(a,a^\dagger)=f(b,b^\dagger)U_2\ . \tag{5}$$

Equation (5) implies that $U_2a=bU_2$ and hence the $|A;r\theta\lambda\rangle$ are eigenstates of the destruction operator b

$$b|A;r\theta\lambda\rangle=bU_2|A\rangle=U_2a|A\rangle=A|A;r\theta\lambda\rangle\ . \tag{6}$$

Note also that

$$|A;r\theta\lambda\rangle=U_2\exp(Aa^\dagger-A^*a)|0\rangle$$
$$=\exp(Ab^\dagger-A^*b)U_2|0\rangle$$
$$=\exp(Ab^\dagger-A^*b)|0;r\theta\lambda\rangle\ . \tag{7}$$

From (6) and (7) we see clearly that $|A;r\theta\lambda\rangle$ may be viewed as ordinary coherent states with respect to the operators $b,b^\dagger$.

The authors of Ref. 1 calculated the quantities

$$\langle a^\dagger a\rangle\equiv\langle A;r\theta\lambda|a^\dagger a|A;r\theta\lambda\rangle=|A_1|^2+|\nu|^2\ ,$$
$$g^{(2)}\equiv\frac{\langle(a^\dagger)^2a^2\rangle}{\langle a^\dagger a\rangle^2}$$
$$=1+\frac{|A_1\mu-A_1^*\nu|^2-|A_1|^2+|\nu|^2+2|\nu|^4}{(|A_1|^2+|\nu|^2)^2}\ , \tag{8}$$

$$A_1=\mu^*A-\nu A^*\ .$$

The $g^{(2)}$ can take values less than 1 (antibunching) or between 1 and 2 (bunching) or greater than 2 (enhanced bunching). We consider here the particular case in which $\lambda=\theta=0$ and A is a real positive number. Equation (8) now simplifies into

$$\langle N\rangle=A^2[\cosh(\tfrac{1}{2}r)-\sinh(\tfrac{1}{2}r)]^2+\sinh^2(\tfrac{1}{2}r)\ , \tag{9}$$

$$g^{(2)}=1+\frac{1}{\langle N\rangle}(e^{-r}-1)+\frac{1}{\langle N\rangle^2}(1+\sinh r)\sinh^2(\tfrac{1}{2}r)\ . \tag{10}$$

The distribution $P_N=|\langle N|A;r\theta\lambda\rangle|^2$ has been calculated in Ref. 1 and simplifies in our case into

$$P_N=|\langle N|A;r\rangle|^2=\frac{1}{N!\cosh\frac{1}{2}r}(\tfrac{1}{2}\tanh\tfrac{1}{2}r)^N\exp(-A^2+A^2\tanh\tfrac{1}{2}r)H_N^2\left(\frac{A}{(\sinh r)^{1/2}}\right)\ . \tag{11}$$

Using (8) we rewrite (11) as

$$P_N(r,\langle N\rangle)=\frac{1}{N!\cosh(\frac{1}{2}r)}[\tfrac{1}{2}\tanh(\tfrac{1}{2}r)]^N H_N^2(Z_1)\cdot e^{Z_2}\ ,$$
$$Z_1=\frac{|\langle N\rangle-\sinh^2(\frac{1}{2}r)|^{1/2}}{[\cosh(\frac{1}{2}r)-\sinh(\frac{1}{2}r)](\sinh r)^{1/2}}\ , \quad (12)$$
$$Z_2=\frac{\langle N\rangle-\sinh^2(\frac{1}{2}r)}{\cosh(\frac{1}{2}r)[\sinh(\frac{1}{2}r)-\cosh(\frac{1}{2}r)]}\ .$$

For fixed r, $\langle N\rangle$ the result is an "oscillating" function of N (see Tables I and II).

In practice it may be difficult to produce *pure* squeezed states. For this reason we now consider a superposition of squeezed states with thermal fields. For ordinary (Glauber) coherent states, this mixture was originally studied by Glauber and by Lachs[3] and is described by the density matrix

$$\rho'=\int\frac{d^2B}{\pi}P_0(B)U_1(B)|A\rangle\langle A|U_1^\dagger(B)\ ,$$
$$P_0(B)=\frac{1}{\langle N_T\rangle}\exp\left[-\frac{|B|^2}{\langle N_T\rangle}\right]\ . \quad (13)$$

Extension of these arguments to squeezed states has been considered in Ref. 4. The mixture of squeezed states with thermal light is described by the density matrix

$$\rho=\int\frac{d^2B}{\pi}P_0(B)U_1(B)|A;r\theta\lambda\rangle\langle A;r\theta\lambda|U_1^\dagger(B)\ ,$$
$$P_0(B)=\frac{1}{\langle N_T\rangle}\exp\left[-\frac{|B|^2}{\langle N_T\rangle}\right]\ . \quad (14)$$

We see clearly that the coherent part $|A\rangle\langle A|$ of (13) has been replaced by the squeezed coherent operator $|A;r\theta\lambda\rangle\langle A;r\theta\lambda|$ in (14). An analytical expression for $\langle N|\rho|M\rangle$ is given in Ref. 4 for the case of real positive A, much larger than r.

$$A\gg\left|\sinh\left(\frac{r}{2}\right)\right|\exp\left(\frac{r}{2}\right),\ \lambda=\theta=0\ . \quad (15)$$

The result is

TABLE I. Pure squeezed states with $r=0.5$, $\langle N_c\rangle=6$.

N	$\langle N\|\rho\|N\rangle$
0	0.599×10^{-3}
1	0.551×10^{-2}
2	0.240×10^{-1}
3	0.658×10^{-1}
4	0.127
5	0.182
6	0.202
7	0.175
8	0.119
9	0.638×10^{-1}
10	0.263×10^{-1}
11	0.796×10^{-2}
12	0.161×10^{-2}
13	0.163×10^{-3}
14	0.761×10^{-6}
15	0.545×10^{-5}
16	0.391×10^{-5}
17	0.810×10^{-6}
18	0.298×10^{-7}
19	0.891×10^{-8}

TABLE II. Pure squeezed states with $r=1$, $\langle N_c\rangle=9$.

N	$\langle N\|\rho\|N\rangle$
0	0.254×10^{-5}
1	0.475×10^{-4}
2	0.421×10^{-3}
3	0.236×10^{-2}
4	0.935×10^{-2}
5	0.278×10^{-1}
6	0.641×10^{-1}
7	0.117
8	0.171
9	0.199
10	0.183
11	0.130
12	0.679×10^{-1}
13	0.231×10^{-1}
14	0.354×10^{-2}
15	0.237×10^{-5}
16	0.799×10^{-3}
17	0.837×10^{-3}
18	0.281×10^{-3}
19	0.129×10^{-4}

TABLE III. Superposition of squeezed and thermal states; $r=0.5$, $\langle N_c\rangle=6$, $\langle N_T\rangle=0.12$, $\gamma=50$.

N	$\langle N\|\rho\|N\rangle$
0	0.130×10^{-2}
1	0.932×10^{-2}
2	0.326×10^{-1}
3	0.743×10^{-1}
4	0.124
5	0.162
6	0.172
7	0.152
8	0.115
9	0.755×10^{-1}
10	0.434×10^{-1}
11	0.221×10^{-1}
12	0.101×10^{-1}
13	0.413×10^{-2}
14	0.154×10^{-2}
15	0.529×10^{-3}
16	0.169×10^{-3}
17	0.506×10^{-4}
18	0.145×10^{-4}
19	0.407×10^{-5}

$$\langle N|\rho|M\rangle = [\cosh(\tfrac{1}{2}r)]^{-1}\{\langle N_T\rangle[1+\tanh(\tfrac{1}{2}r)]+e^{r}\}^{-1/2}\{\langle N_T\rangle[1-\tanh(\tfrac{1}{2}r)]+e^{-r}\}^{-1/2}$$
$$\times\exp(-\gamma\langle N_T\rangle\{\langle N_T\rangle+e^{-r}[\cosh^2(\tfrac{1}{2}r)][1+\tanh(\tfrac{1}{2}r)]\}^{-1})(N!M!)^{-1/2}\left[\frac{\langle N_T\rangle}{1+\langle N_T\rangle}\right]^{(N+M)/2} H_{N,M}^{(c_{ij})}(R_1,R_2),$$
$$c_{11}=c_{22}=\tfrac{1}{2}(1+\langle N_T\rangle)(\Lambda-K)+\tanh(\tfrac{1}{2}r)(1+\langle N_T\rangle^{-1}) ,$$
$$c_{12}=-\tfrac{1}{2}(1+\langle N_T\rangle)(K+\Lambda) ,$$
$$R_1=R_2=-\left[1-\frac{\tanh(\frac{1}{2}r)}{K\langle N_T\rangle}\right]^{-1} e^{-r/2}\gamma^{1/2}[\cosh(\tfrac{1}{2}r)](1+\langle N_T\rangle)^{-1/2} , \tag{16}$$
$$K=\left[[1+\tanh(\tfrac{1}{2}r)]^{-1}\langle N_T\rangle+e^{-r}\cosh^2\left(\frac{r}{2}\right)\right]^{-1} ,$$
$$\Lambda=\left[[1-\tanh(\tfrac{1}{2}r)]^{-1}\langle N_T\rangle+e^{r}\cosh^2\left(\frac{r}{2}\right)\right]^{-1} ,\quad \gamma=\frac{\langle N_c\rangle}{\langle N_T\rangle} ,$$

where $\langle N_c\rangle$, $\langle N_T\rangle$ are the mean number of coherent and thermal photons.

The $H_{N,M}^{(c_{ij})}(R_1,R_2)$ are Hermitean polynomials of two variables.[5] For a given 2×2 symmetric matrix c_{ij} ($c_{ij}=c_{ji}$) they are defined as

$$H_{N,M}^{(c_{ij})}(R_1,R_2)=(-1)^{N+M}\exp(\tfrac{1}{2}c_{ij}R_iR_j)\frac{\partial^{N+M}}{\partial R_1^N\partial R_2^M}\exp(-\tfrac{1}{2}c_{ij}R_iR_j),\quad i,j=1,2 . \tag{17}$$

The expansion of their generating function is

$$\exp(c_{ij}\alpha_iR_j-\tfrac{1}{2}c_{ij}\alpha_i\alpha_j)=\sum_{N,M}\frac{\alpha_1^N}{N!}\frac{\alpha_2^M}{M!}H_{N,M}^{(c_{ij})}(R_1,R_2),\quad i,j=1,2 . \tag{18}$$

For the numerical evaluation of our Hermitean polynomials we use the relations

$$H_{0,0}=1,\quad H_{1,0}=c_{11}R_1+c_{12}R_2,\quad H_{0,1}=c_{22}R_2+c_{12}R_1 ,$$
$$H_{1,1}=(c_{11}R_1+c_{12}R_2)(c_{22}R_2+c_{12}R_1)-c_{12} , \tag{19}$$
$$H_{N+1,M}=(c_{11}R_1+c_{12}R_2)H_{N,M}-Nc_{11}H_{N-1,M}-Mc_{12}H_{N,M-1} ,$$
$$H_{N,M+1}=(c_{22}R_2+c_{12}R_1)H_{N,M}-Mc_{22}H_{N,M-1}-Nc_{12}H_{N-1,M} .$$

TABLE IV. Superposition of squeezed and thermal states; $r=0.5$, $\langle N_c\rangle=6$, $\langle N_T\rangle=6$, $\gamma=1$.

N	$\langle N\|\rho\|N\rangle$
0	0.586×10^{-1}
1	0.579×10^{-1}
2	0.567×10^{-1}
3	0.551×10^{-1}
4	0.531×10^{-1}
5	0.509×10^{-1}
6	0.486×10^{-1}
7	0.461×10^{-1}
8	0.436×10^{-1}
9	0.411×10^{-1}
10	0.386×10^{-1}
11	0.362×10^{-1}
12	0.338×10^{-1}
13	0.315×10^{-1}
14	0.293×10^{-1}
15	0.272×10^{-1}
16	0.252×10^{-1}
17	0.232×10^{-1}
18	0.214×10^{-1}
19	0.197×10^{-1}

TABLE V. Superposition of squeezed and thermal states; $r=1$, $\langle N_c\rangle=9$, $\langle N_T\rangle=9$, $\gamma=1$.

N	$\langle N\|\rho\|N\rangle$
0	0.385×10^{-1}
1	0.384×10^{-1}
2	0.382×10^{-1}
3	0.377×10^{-1}
4	0.372×10^{-1}
5	0.365×10^{-1}
6	0.357×10^{-1}
7	0.348×10^{-1}
8	0.338×10^{-1}
9	0.328×10^{-1}
10	0.318×10^{-1}
11	0.307×10^{-1}
12	0.296×10^{-1}
13	0.285×10^{-1}
14	0.274×10^{-1}
15	0.263×10^{-1}
16	0.252×10^{-1}
17	0.241×10^{-1}
18	0.230×10^{-1}
19	0.220×10^{-1}

The proof is lengthy but straightforward.

It was proven in Ref. 4 that in the special case of zero squeezing ($r=0$), (16) reduces to the standard Glauber-Lachs[3] formula given in the literature. In the present work we checked this numerically.

We evaluated (16) for various values of r, $\langle N_c \rangle$, γ. Typical examples are presented in Tables III–VI. We did not find any oscillatory behavior of P_N. Note that pure squeezed states in (16) correspond to $\langle N_T \rangle = 0$, i.e., $\gamma \rightarrow \infty$. We checked that for γ up to 50 no oscillatory behavior appears; greater values ($\gamma > 50$) create numerical problems in the sense that very large and very small numbers appear in the calculation. From a practical point of view $\gamma \sim 50$ is already a very small amount of noise, and although we have not explored the region $50 < \gamma < \infty$, we can conclude that even a small amount of noise destroys the oscillatory behavior of P_N.

We have shown in this paper that photon distributions associated with pure squeezed states are oscillatory functions of N. Given the fact that other distributions known in quantum statistics do not present similar behavior, one might be tempted to argue that this could be used as a criterion for detecting in practice squeezed states. For this reason we pursued our calculations into squeezed states with thermal noise, and we found that already very small amounts of noise destroy the oscillatory behavior. Since pure squeezed states might be very difficult to achieve in practice, this finding puts under question the possibility of using the counting distribution as an indicator of squeezed states.

After the completion of our work we became aware of a very recent publication[6] in which the oscillatory behavior of P_N for squeezed states is also found. The effect of noise is not discussed in this paper.

TABLE VI. Superposition of squeezed and thermal states; $r=1$, $\langle N_c \rangle = 9$, $\langle N_T \rangle = 0.18$, $\gamma = 50$.

N	$\langle N \mid \rho \mid N \rangle$
0	0.225×10^{-4}
1	0.276×10^{-3}
2	0.165×10^{-2}
3	0.638×10^{-2}
4	0.181×10^{-1}
5	0.399×10^{-1}
6	0.713×10^{-1}
7	0.106
8	0.134
9	0.146
10	0.139
11	0.117
12	0.872×10^{-1}
13	0.584×10^{-1}
14	0.354×10^{-1}
15	0.196×10^{-1}
16	0.101×10^{-1}
17	0.487×10^{-2}
18	0.226×10^{-2}
19	0.103×10^{-2}

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