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Spatial spectrum of a general family of self-similar arrays

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We describe the structure factors of a fairly general family of self-similar deterministic arrays (fractals). These are constructed recursively by an inflation method which imitates real physical processes where large clusters are grown from smaller entities. An algebraic approach is used to describe the behavior of the structure factors in the limit of arbitrarily large objects. These structure factors exhibit several scaling properties which reflect the self-similarity of the direct space arrays.

I. INTRODUCTION

Structure analysis in condensed matter physics is frequently based on the diffraction of visible light, x rays, or neutrons depending on the length scales which are probed. These scattering experiments provide valuable information about two-body structural properties such as the radial correlation function $g(r)$. In particular, they are well suited to investigate the power-law decay of the correlations in scale-invariant systems in the absence of any characteristic length scale. Therefore, scattering methods appear quite appealing to probe self-similar structures encountered in various experimental situations. Several recent works have dealt with x rays and light-scattering-intensity measurements on various physical systems such as silica gels or aggregates.[1]

In a previous paper,[2] we have described an experimental arrangement which performs optical scattering experiments on planar self-similar arrays (fractals). The diffraction patterns were shown to provide directly spatial Fourier transforms of the real-space objects; applications to triadic Cantor bars and checkerboard fractals were discussed in detail. Similar applications of spatial Fourier transforms to Pascal-Sierpinski gaskets have also been proposed recently by Lakhatia *et al.*[3] In fact, deterministic objects differ strikingly from random objects because their geometrical regularity can lead to highly ordered diffraction patterns with orientational symmetry and persistent long-range correlations; reciprocally these characteristics provide valuable information about the direct space sets.

The purpose of this work is to study the diffraction properties of a fairly general family of deterministic self-similar fractals constructed according to a recursive inflation method which is described in Sec. II; in the same section the corresponding structure factors, $S_n(\mathbf{k})$, are derived analytically. The goal of Sec. III is to analyze the asymptotic behavior of $S_n(\mathbf{k})$ in the limit of infinitely large arrays; we distinguish two different classes of systems according to whether $S_n(\mathbf{k})$ vanishes at large wave vectors, which reveals a lack of spatial correlations, or is composed of Bragg peaks densely filling all the reciprocal space. In Sec. IV, we demonstrate that the diffraction spectrum exhibits a bandlike structure with, in some instances, orientational and translational order; finally $S_n(\mathbf{k})$ is shown to exhibit several noteworthy scaling relations which reflects self-similarity in reciprocal space.

II. DIFFRACTION PATTERNS OF RECURSIVE FRACTALS

A. Construction of recursive fractals

Many deterministic fractals can be viewed as the infinite repetition in space of a unit cell arranged in the sites of a lattice which usually exhibits some particular properties like self-similarity, self-affinity, etc. A general way of constructing such fractals proceeds according to the recursive inflation method which follows; it is illustrated on Fig. 1 in a simple case. In the euclidian space of dimensionality E, one starts at stage $n=0$ with a unit cell, which is sometimes referred to as an initiator,[4] and with a set of N points named generator;[4] each of the sites of the generator is characterized by a vector $\mathbf{u}_l$ where $l=1,\ldots,N$. The first-stage fractal consists of N unit cells arranged at $\mathbf{u}_l$. At stage $n=2$, the generator is scaled up by a factor ξ_1 and N first-stage fractals are arranged at $\xi_1\mathbf{u}_l$. This procedure is continued iteratively to infinity scaling up the generator by $\xi_1\cdots\xi_{n-1}$ at stage n and repeating the previous stage patterns at $\xi_1\cdots\xi_{n-1}\mathbf{u}_l$. For convenience, in the following, we designate α_j the product $\xi_1\cdots\xi_j$ ($\alpha_0=1$). The factors ξ_j may be different, the only requirement being to avoid overlappings of the unit cells. When all the inflation factors ξ_j are chosen constant regardless of j, we have $\alpha_j=\xi^j$ and the resulting set is self-similar with a similarity dimension equal to $D=\ln N/\ln\xi$. At stage n, one gets an arrangement of N^n unit cells situated at vectors $\mathbf{v}$ such as

$$\mathbf{v}=\alpha_0\mathbf{u}_{l_1}+\alpha_1\mathbf{u}_{l_2}+\cdots+\alpha_{n-1}\mathbf{u}_{l_n},$$

where the indices l_i run over 1 to N. It is worth noting that in this inflation process, at each stage the smallest scale remains the same, it is the size of a unit cell, ϵ,

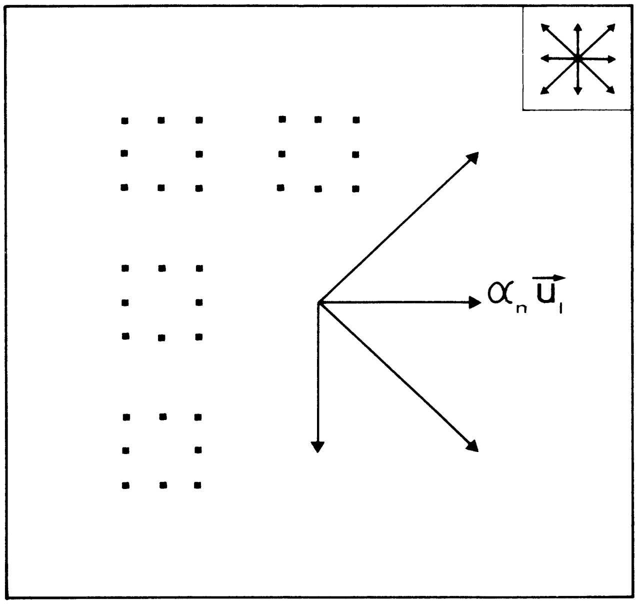

FIG. 1. Construction of a recursive fractal. The generator is drawn in the top right corner of the figure; the unit cell is a square of side ϵ. The first-stage array consists of eight unit cells at $\mathbf{u}_l$. To build the $n+1$-stage array, the generator is scaled up by $\alpha_n=\xi_1 \cdots \xi_n$ and eight n-stage arrays are arranged at $\alpha_n \mathbf{u}_l$. According to the relative values of ϵ and $|\mathbf{u}_l|$ we obtain a set which is connected (the Sierpinski's carpet when $\epsilon=\min(|\mathbf{u}_l|)$ or nonconnected.

while the largest scale L, increases as

$$L(n)\simeq(\alpha_0+\alpha_1+\cdots+\alpha_{n-2}+\alpha_{n-1})\{\sup\}|\mathbf{u}_l-\mathbf{u}_m| .$$

In particular for a self-similar fractal we have $L(n) \simeq \xi^n \epsilon$. One sees immediately that a set is of zero Lebesgue measure if

$$N^{-n}(\alpha_0+\alpha_1+\cdots+\alpha_{n-2}+\alpha_{n-1})^E \to 0 \quad \text{when } n\to+\infty .$$

This inflation method of generating fractals is equivalent to the construction by deflation (curdling construction) usually employed in the mathematical literature.[5] Its advantage is to simulate the way in which real clusters are grown from smaller entities. It applies whenever the n-stage set can be entirely tiled by using the previous stage set. We have named these particular sets recursive fractals. These include many types of well-known fractals such as Cantor sets, Serpinski's figures, some Koch islands, etc. Recursivity of fractals allows direct analytical calculations of their diffraction patterns. In some instances, generalizations to nonrecursive fractals are straightforward; these will be published elsewhere.

B. Diffraction patterns

In a real experiment, one measures the intensity which is scattered in direction $\mathbf{k}$ by a n-stage fractal, $I_n(\mathbf{k})$,

$$I_n(\mathbf{k})=N^n S_n(\mathbf{k})F(\mathbf{k}) .$$

$F(\mathbf{k})$ is the form factor, that is, the intensity scattered by a unit cell, and $N^m F(\mathbf{k})$ would be the intensity diffracted by N^m independent unit cells. Thus, the structure factor, $S_n(\mathbf{k})$, appears as a measure of the spatial correlations between the sites over which the unit cells are distributed. In general, at high frequencies, the form factor smears out the structure of the diffraction spectrum while for low frequencies it can be neglected. In the following we shall focus on the structure factor (or power spectrum), $S_n(\mathbf{k})$,

$$S_n(\mathbf{k})=\frac{1}{N^n}[\hat{\rho}_n(\mathbf{k})\hat{\rho}_n^*(\mathbf{k})] , \tag{2.1}$$

where $\hat{\rho}_n(\mathbf{k})$ is the Fourier transform of the n-stage mass density at site $\mathbf{r}$, $\rho_n(\mathbf{r})$, and $\hat{\rho}_n^*(\mathbf{k})$ its conjugate. It is worth noting that there is no complete correspondence between the structure factor and the direct space sets; indeed, $S_n(\mathbf{k})$ being the square modulus of the Fourier transform of $\rho_n(\mathbf{r})$, it misses all the information about the phases of the diffracted waves.

The structure factor of a recursive fractal can be calculated analytically quite easily. Indeed the j stage mass density at point $\mathbf{r}$ is related to the $j-1$ stage density through the fundamental relation

$$\rho_j(\mathbf{r})=\rho_{j-1}(\mathbf{r})*\sum_{l=1}^{N}\delta(\mathbf{r}-\alpha_{j-1}\mathbf{u}_l) , \tag{2.2}$$

where $*$ denotes a convolution product and δ is the Dirac distribution. The second term of the right-hand side of Eq. (2.2) is a distribution function which will be noted $\Delta_{j-1}(\mathbf{r})$. Finally, $\rho_n(\mathbf{r})$ is expressed as a convolution product of Δ_j distributions,

$$\rho_n(\mathbf{r})=\Delta_0(\mathbf{r})*\Delta_1(\mathbf{r})*\cdots*\Delta_{n-1}(\mathbf{r}) . \tag{2.3}$$

Thus, the Fourier transform of the n-stage density, $\hat{\rho}_n(\mathbf{k})$, is

$$\hat{\rho}_n(\mathbf{k})=\prod_{j=0}^{n-1}\hat{\Delta}_j(\mathbf{k}) \quad \text{with} \quad \hat{\Delta}_j(\mathbf{k})=\sum_{l=1}^{N}e^{2\pi i\alpha_j\mathbf{k}\cdot\mathbf{u}_l} . \tag{2.4}$$

Now, expressing the power spectrum of a Δ_j distribution,

$$\hat{\Delta}_j(\mathbf{k})\hat{\Delta}_j^*(\mathbf{k})=N+2\sum_{l=1}^{N}\sum_{m=l+1}^{N}\cos[2\pi\alpha_j\mathbf{k}\cdot(\mathbf{u}_l-\mathbf{u}_m)] , \tag{2.5}$$

and combining (2.1) and (2.4), one finds easily that the structure factor is given by the generalized Riesz product,[6]

$$S_n(k)=\frac{1}{N^n}\prod_{j=0}^{n-1}\left[N+2\sum_{l=1}^{N}\sum_{m=l+1}^{N}\cos[2\pi\alpha_j\mathbf{k}\cdot(\mathbf{u}_l-\mathbf{u}_m)]\right] . \tag{2.6}$$

This expression is quite general; it gives the structure factor of any deterministic recursive fractal. The rest of the paper will be mainly devoted to self-similar objects where the inflation factors ξ_j are taken equal to ξ at each stage of the construction process ($\alpha_j = \xi^j$).

III. ASYMPTOTIC STRUCTURE OF $S_n(\mathbf{k})$ FOR INFINITELY LARGE FRACTALS

A. Conditions of diffraction

In this paragraph, we are concerned with the behavior of structure factors $S_n(\mathbf{k})$ in the limit $n \to +\infty$ for various inflation factors ξ. Similar problems have been extensively studied in the mathematical literature in connection with the classification of the direct space sets in sets of uniqueness and sets of multiplicity for trigonometric series.[7] A list of relevant articles includes works of Kahane and Salem,[8] Salem,[9] and Salem and Zygmund[10] to which we refer the readers for the terminology and some of the results used thereafter. Two different classes of fractals will be distinguished according to whether the spectrum vanishes at large $\mathbf{k}$ when n increases (which reveals destructive interferences between scattered waves), or whether it consists of a set of Bragg peaks of nondecreasing intensity which densely fills the reciprocal space.

To find the conditions required for diffraction, let us consider the quantities $\hat{\Delta}_j(\mathbf{k})\hat{\Delta}_j^*(\mathbf{k})$ defined in formula (2.5) after normalization by their values at $\mathbf{k}=0$ $[\hat{\Delta}_j(0)\hat{\Delta}_j^*(0)=N^2]$. One sees immediately that $0 \le \hat{\Delta}_j(\mathbf{k})\hat{\Delta}_j^*(\mathbf{k})/\hat{\Delta}_j(0)\hat{\Delta}_j^*(0) \le 1$; in particular, it is equal to unity if and only if

$$\cos[2\pi\xi^j\mathbf{k}\cdot(\mathbf{u}_l-\mathbf{u}_m)]=1 \quad \forall(l,m)$$

or in other words if, for each couple of indices (l,m), there exists an integer s_{lm} such that

$$\xi^j\mathbf{k}\cdot(\mathbf{u}_l-\mathbf{u}_m)=s_{lm} \ . \tag{3.1}$$

For an infinite set (in the limit $n \to \infty$) two different situations are encountered. (a) Both the inflation factor ξ and the vectors $\mathbf{u}_l$ of the generator allow condition (3.1) to be satisfied for every $j \ge j_0$, that is, an infinity of factors in the product (2.6) are equal to unity. Then, the structure factor $S_n(\mathbf{k})$ is nonzero at wave vector $\mathbf{k}$. (b) $S_n(\mathbf{k})$ vanishes otherwise in the limit $n \to \infty$.

B. Study of one-dimensional symmetrical Cantor sets

The diffracting condition (3.1) differs strikingly from the well-known Bragg condition which applies to periodic crystals. In particular, it requires several conditions concerning both the inflation factor ξ and the vectors of the generator. Conditions for ξ can be inferred from the study of particular sets. Let us begin with the so-called "symmetrical Cantor sets" which are generated taking a linear unit cell of width ϵ as initiator and two symmetrical sites at $u_l = \pm\epsilon(\xi-1)/2$ as generator [Fig. 2(a)]. To avoid overlappings between the unit cells we have to choose $\xi \ge 2$. The structure factor of this symmetrical

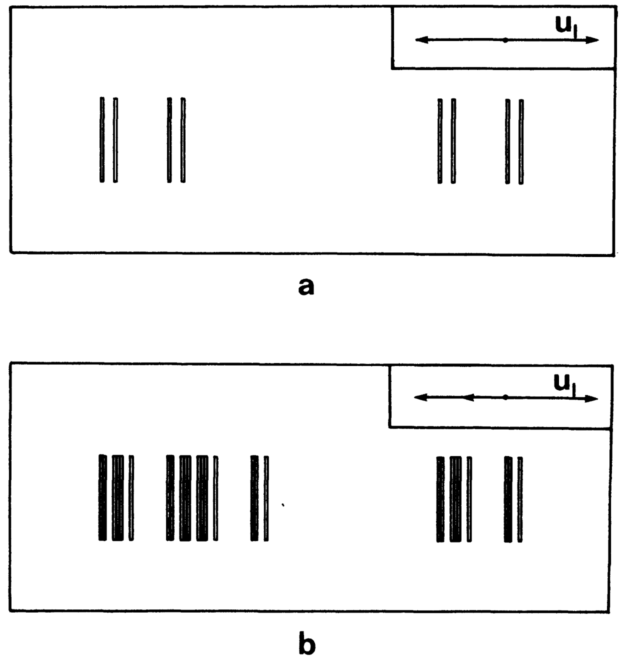

FIG. 2. Cantor sets at stage $n=3$; the unit cells are thin lines of width ϵ; the inflation factor is $\xi=5$; the generators are shown at the top. (a) is a symmetrical Cantor set ($D=\ln 2/\ln 5$), (b) is a general Cantor set ($D=\ln 3/\ln 5$).

Cantor set reduces to the series

$$S_n(k)=2^n\prod_{j=0}^{n-1}\cos^2[\pi\xi^j(\xi-1)k\epsilon] \tag{3.2}$$

and the diffracting condition becomes

$$\exists s \in N, \quad \xi^j(\xi-1)k\epsilon=s \quad \forall j \ . \tag{3.3}$$

First we take $\xi=2$; in the limit $n \to \infty$, the unit cells are distributed on a periodic lattice with ϵ spacing; consequently, $S_n(k)$ exhibits an infinite repetition of Bragg peaks at $k\epsilon=s$ ($s \in N$). This structure reflects the translational invariance of the direct space set. In general, when $\xi>2$, different cases have to be considered according to the properties of ξ.

(i) ξ is an integer. Suppose that there exists integers j_0 and s_0 for which equation (3.3) is satisfied at wave vector k, i.e., $\xi^{j_0}(\xi-1)k\epsilon=s_0$; then ξ being an integer, it holds for any $j>j_0$. Thus although the direct space set is nonperiodic, the structure factor $S_n(k)$ is composed of Bragg peaks of nonvanishing intensity at $k\epsilon=s/(\xi-1)\xi^{n-1}$, $s \in N$. The intensity of each diffraction spot can be calculated directly from (3.2). The brightest peaks correspond to wave vectors of the form shown in Eq. (3.3) where the index is a multiple of ξ. They deduce themselves from one another through similarity transforms of ratios ξ. Several examples will be explained in detail in Sec. IV on Figs. 3 and 4.

(ii) ξ is an irreducible rational number. Then if (3.3) holds for j_0 at wave vector k, in general it is no longer satisfied for $j>j_0$. The structure factor of the corre-

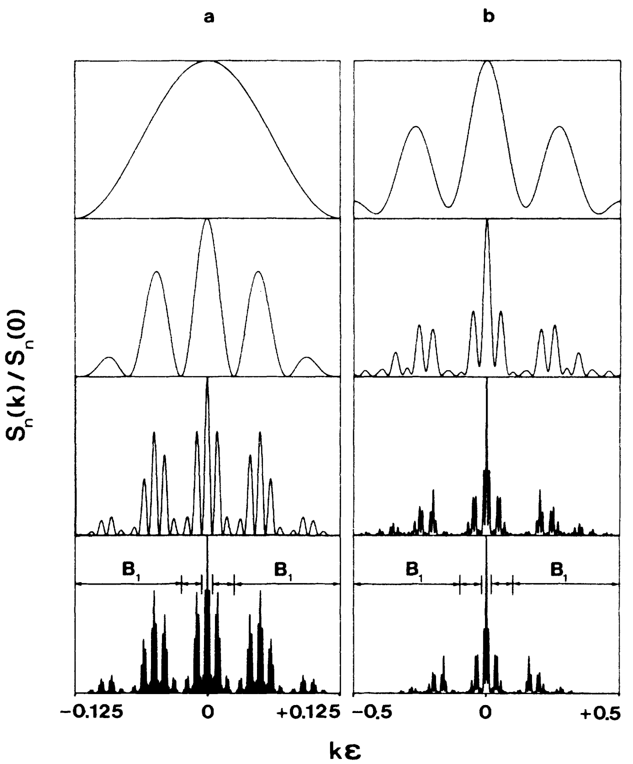

FIG. 3. (a) and (b) are representations of the structure factors $S_n(k)/S_n(0)$ of the Cantor sets shown, respectively, on Figs. 2(a) and 2(b) ($n=1$, 2, 3, and 4 from top to bottom). For clarity, only the two outer edge bands have been represented.

sponding set vanishes when $n \rightarrow +\infty$. This result has already been demonstrated by Bary.[11] Thus, in the asymptotic limit $n \rightarrow \infty$ there is no scattered intensity except in the central peak.

(iii) ξ is irrational. The behavior of $S_n(k)$ depends on the algebraic or transcendental nature of ξ. Indeed there exists a particular set of algebraic numbers ξ for which ξ^n can be approximated by an integer, that is, $|\xi^n-[\xi^n]|$ converges towards zero as n increases[12] ($[z]$ denotes the nearest integer to z). These numbers are called Pisot-Vijaraghavan (PV) numbers; the PV number class is composed of the algebraic numbers defined by the condition that all their conjugates have their moduli inferior to the unity.[12] (Algebraic numbers are roots of polynomial equations with integer coefficients; the degree of an algebraic number z is the degree of the lowest-order polynomial equation satisfied by z; the roots of this polynomial equation are called the conjugates of z. Note that integers are trivial PV numbers; in this paragraph we are concerned with irrational PV numbers.)[13] Clearly when ξ is a PV number, since ξ^j can be approximated by an integer $[\xi^j]$ from a certain value j_0 of j, the diffracting condition (3.1) is satisfied for any $j \geq j_0$. Reciprocally, it has been shown[9] that if $S_n(k)$ does not vanish, ξ is necessarily a PV number. In conclusion, just as in case (i), $S_n(k)$ is a dense set of Bragg peaks of nonvanishing intensity if and only if ξ belongs to the PV number class.

C. General Cantor sets and higher-dimensional sets

Here we are concerned with the generalization of the preceding results to any self-similar array. Clearly, ξ needs to be a PV number as previously; however, this condition is not sufficient and additional requirements concerning the vectors of the generator have to be satisfied. Let us make conditions (3.1) explicit for arbitrary one-dimensional sets [Fig. 2(b)]. For a given ξ the diffracting condition holds at a wave vector k if and only if the periods of the cosine terms in the expression $\hat{\Delta}_j(k)\hat{\Delta}_j^*(k)$ are commensurate, that is, if their ratios are rational numbers. When ξ is an integer, this is equivalent to choosing the $N(N-1)/2$ quantities $|u_l-u_m|$ to be commensurate. More generally, when ξ is an irrational PV number we consider a special set of numbers, which is called the field of ξ [the field generat-

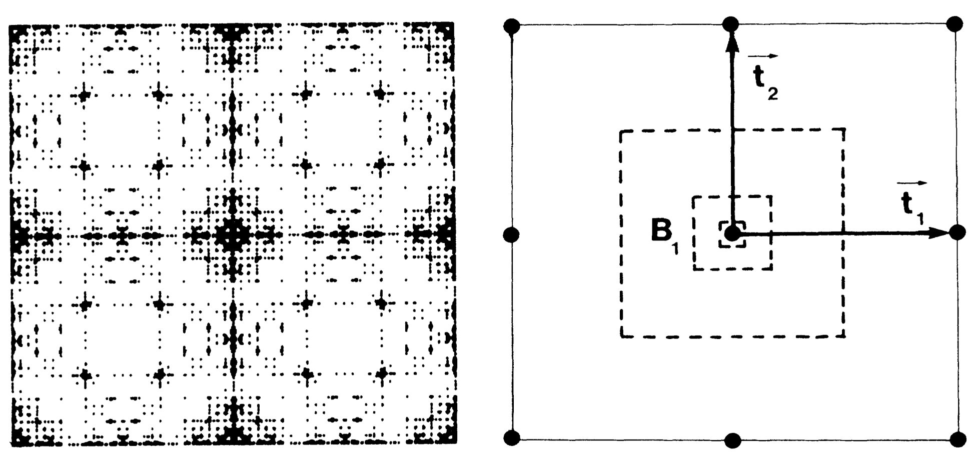

FIG. 4. Representation of the structure factor $S_5(\mathbf{k})/S_5(0)$ for the Sierpinski's carpet of similarity dimension ln8/ln3 (only the peaks of intensity greater than 2×10^{-5} have been drawn). The whole spectrum is deduced through translations from the first Brillouin zone constructed from $(\mathbf{t}_1,\mathbf{t}_2)$. The bandlike structure of the spectrum appears clearly; the boundaries of the two outer edge bands are drawn in dashed line.

ed by an algebraic number z is $Q(z)$ where Q denotes the set of rational numbers; it consists of all numbers of the form $\sum_j q_j \xi^j$ where $q_j \in Q$]. Clearly when $|u_l - u_m|$ belongs to the field of ξ, since the coefficients q_j are rational and ξ^j can be approximated by its nearest integer $[\xi^j]$, the periods of the cosine terms in $\hat{\Delta}_j(k)\hat{\Delta}_j^*(k)$ are commensurate provided that j is large enough. Reciprocally it has been demonstrated that ξ belongs to the PV number class and $|u_l - u_m| \in Q(\xi)$ are sufficient conditions to construct a set with a nonvanishing structure factor.[9] For higher-dimensional systems the generalization is straightforward: in an arbitrary basis, all the components of $|\mathbf{u}_l - \mathbf{u}_m|$ have to satisfy the same conditions as just derived for general one-dimensional sets. In conclusion, the asymptotic behaviors of the structure factors of infinitely large recursive fractals depend on the algebraic properties of the inflation factor ξ and of the relative positions of the sites of the generator. In Sec. IV we shall restrict ourselves to the study of nonvanishing structure factors, i.e., when all the conditions detailed in this part hold.

IV. SCALING PROPERTIES OF STRUCTURE FACTORS

A. Reciprocal space construction of an n-stage structure factor

Several scaling relations can be derived for nonvanishing structure factors. Indeed for self-similar arrays, since $\alpha_j = \xi^j$, we obtain from (2.4)

$$\hat{\Delta}_j(\xi\mathbf{k}) = \hat{\Delta}_{j+1}(\mathbf{k}) . \tag{4.1}$$

In addition we have

$$S_{n+1}(\mathbf{k}) = N^{-1} S_n(\mathbf{k}) \hat{\Delta}_n(\mathbf{k}) \hat{\Delta}_n^*(\mathbf{k}) . \tag{4.2}$$

From (4.1), (4.2), and (2.5) we find easily the two following relations:

$$S_{n+1}(\mathbf{k}) = N^{-1} S_n(\xi\mathbf{k}) \hat{\Delta}_0(\mathbf{k}) \hat{\Delta}_0^*(\mathbf{k}) , \tag{4.3}$$

$$S_n(\xi\mathbf{k}) = S_n(\mathbf{k}) \frac{\hat{\Delta}_n(\mathbf{k})\hat{\Delta}_n^*(\mathbf{k})}{\hat{\Delta}_0(\mathbf{k})\hat{\Delta}_0^*(\mathbf{k})} . \tag{4.4}$$

Relations (4.2) and (4.3) describe how structure factors generated at successive stages deduce themselves from one another. For clarity, we begin with the spectra of sets constructed with integer inflation factors ($\xi \in N$). The situation when ξ is an irrational PV number is only slightly different; ξ^j being approximated by its nearest integer $[\xi^j]$ once j is large enough, all the following results remain unchanged at low spatial frequencies. Demonstrations are illustrated on Figs. 3 and 4 in the case of Cantor sets and Sierpinski carpets. Obviously, the orientational symmetries of the diffraction spectra reflect those of the corresponding generators in real space. In addition, one has to notice [see relation (2.6)] that structure factors are invariant over translations of vectors $\mathbf{t}$ such as

$$\exists s_{lm} \in N, \quad \xi^j \mathbf{t}\cdot(\mathbf{u}_l - \mathbf{u}_m) = s_{lm} \quad \forall j \text{ and } \forall (l,m) . \tag{4.5}$$

When ξ belongs to N, it is sufficient that

$$\exists s_{lm} \in N, \quad \mathbf{t}\cdot(\mathbf{u}_l - \mathbf{u}_m) = s_{lm} \quad \forall (l,m) , \tag{4.6}$$

from which the vectors $\mathbf{t}$ can be calculated easily. It is convenient to define primitive vectors for the reciprocal space. In two dimensions a basis $(\mathbf{t}_1, \mathbf{t}_2)$ is composed of two independent vectors joining a bright central peak to its nearest neighbors deduced through the translations just defined (Fig. 4). When $\xi \in N$, it is worth noting that $\mathbf{t}_1$ and $\mathbf{t}_2$ only depend on the generator and are independent of the stage of the construction. Because structure factors are invariant over translations, their whole variations can be inferred from limited domains of the reciprocal space, which correspond to the first Brillouin zones constructed from $(\mathbf{t}_1, \mathbf{t}_2)$.

A first-stage structure factor, $S_1(\mathbf{k})$, is simply the spectrum of the generator; in simple cases it is composed of a single peak, the width of which is in inverse ratio to the largest scale in direct space but in general $S_1(\mathbf{k})$ exhibits secondary maxima (Fig. 3). To describe a second-stage structure factor, we consider separately two different subdomains obtained by dividing the support of $S_1(\mathbf{k})$ into one center region and one edge region which is symmetric about $\mathbf{k}=0$. Over the center region which is obtained by scaling down the support of $S_1(\mathbf{k})$ by a factor ξ, we recover the variation of $S_1(\mathbf{k})$ apart from a reduction in the intensities. At each stage n, a similar division of the previous stage center region generates in turn a new center region and an edge region B_{n-1}. $S_n(\mathbf{k})$ exhibits $n-1$ edge regions the boundaries of which deduce themselves from one another by similarity transforms of ratio $1/\xi$. This reciprocal space division reflects exactly the self-similar growth of the real space set by a factor ξ and gives evidence for the bandlike structure which is observed. In Sec. IV B we shall see that the edge regions play a central role with respect to the scaling properties of $S_n(\mathbf{k})$. Since the structure factor must satisfy simultaneously translational invariance, symmetry about 0 and self-similarity, on the boundary between B_i and B_{i+1}, quantities like $\hat{\Delta}_j(\mathbf{k})\hat{\Delta}_j^*(\mathbf{k})$ with $j=i,n$ must be either minimum or maximum; after derivation

$$\exists s_{lm} \in N, \quad 2\xi^j \mathbf{k}(\mathbf{u}_l - \mathbf{u}_m) = s_{lm}, \quad j = i,n . \tag{4.7}$$

The boundaries of the edge bands are determined directly from this relation.

B. Scaling relations

Let us first be interested in the intensity transmitted at $\mathbf{k}=0$; the unit cell being supposed of characteristic length scale ϵ we have

$$I_n(0) = N^{2n} |F(0)| \simeq N^{2n} \epsilon^E .$$

For an n-stage self-similar fractal of size L, we know that $L \simeq \xi^n \epsilon$; so we find easily that $I_n(0)$ can be expressed as a function of the largest and smallest scales, L and ϵ,

$$I_n(0) \simeq L^{2D} \epsilon^{2(E-D)} . \tag{4.8}$$

A direct space determination of the squared mass according to Minkowski's procedure would lead to the same expression; we simply recover the fact that $I_n(0)$ measures the square of the total mass embedded in the object. In view of this, one can determine D by comparing the intensity transmitted for different-stage fractals or more simply by measuring the intensity transmitted through variable size portions of an object. This last procedure can be extended directly to random fractals.

In the following we concentrate on scaling properties when $\mathbf{k}$ is different from 0. Relation (4.3) relates two successive stage structure factors; at low spatial frequencies, i.e., for $\mathbf{k}$ such that $|\mathbf{k}|\epsilon \ll 1$, we have $\hat{\Delta}_0(\mathbf{k})\hat{\Delta}_0^*(\mathbf{k}) \simeq N^2$ and finally (4.3) leads to

$$S_{n+1}(\mathbf{k}) \simeq N S_n(\xi\mathbf{k}) \simeq \xi^D S_n(\xi\mathbf{k}) , \tag{4.9}$$

where D is the similarity dimension. Now we consider the ratio $S_n(\xi\mathbf{k})/S_n(\mathbf{k})$ which expresses how structure factors scale for different spatial frequencies. For $|\mathbf{k}|\epsilon \ll 1$, from relations (4.4) and (4.9), we deduce $S_n(\xi\mathbf{k})/S_n(\mathbf{k}) \simeq N^{-2}\hat{\Delta}_n(\mathbf{k})\hat{\Delta}_n^*(\mathbf{k})$. In order to break up the fine structure of $S_n(\mathbf{k})$ we now calculate the mean value of the ratio $S_n(\xi\mathbf{k})/S_n(\mathbf{k})$ over the bands B_i defined previously,

$$\left\langle \frac{S_n(\xi\mathbf{k})}{S_n(\mathbf{k})} \right\rangle = \frac{1}{N^2 B_i} \int_{B_i} \left[N + 2\sum_{l=1}^{N} \sum_{m=l+1}^{N} \cos[2\pi\xi^n \mathbf{k}(\mathbf{u}_l - \mathbf{u}_m)] \right] d^2\mathbf{k} .$$

By using relation (4.7) which ensures that $2\xi^n\mathbf{k}(\mathbf{u}_l-\mathbf{u}_m)=s_{lm}$ $\forall(l,m)$ on the boundaries of B_i, it is easily found that the second term of the integral is zero. Finally we have the following result:

$$\left\langle \frac{S_n(\xi\mathbf{k})}{S_n(\mathbf{k})} \right\rangle = \xi^{-D} , \tag{4.10}$$

which is analogous to the relation expected for random self-similar arrays without any geometrical regularity.

There also exists a scaling relation between $\langle S_n(\xi\mathbf{k})\rangle$ and $\langle S_n(\mathbf{k})\rangle$. Indeed from (4.4), assuming that $\hat{\Delta}_0(\mathbf{k})\hat{\Delta}_0^*(\mathbf{k}) \simeq N^2$ (since $|\mathbf{k}|\epsilon \ll 1$) and replacing $\hat{\Delta}_n(\mathbf{k})\hat{\Delta}_n^*(\mathbf{k})$ by its value, we find

$$\langle S_n(\xi\mathbf{k})\rangle = \xi^{-D}\langle S_n(\mathbf{k})\rangle + \frac{2}{N^2 B_i}\int_{B_i} S_n(\mathbf{k}) \sum_{l=1}^{N} \sum_{m=l+1}^{N} \cos[2\pi\xi^n\mathbf{k}(\mathbf{u}_l-\mathbf{u}_m)] d^2k .$$

Let us call δS_n the right-hand side integral. In the range $0 \ll |\mathbf{k}|\epsilon \ll 1$, $S_n(\mathbf{k})$ is a set of symmetrical Bragg peaks centered at wave vectors $\mathbf{k}$ given by $\mathbf{k}(\mathbf{u}_l-\mathbf{u}_m)=s/\xi^{n-1}$ $(s \in N)$ and nonvanishing between $\mathbf{k}-\delta\mathbf{k}$ and $\mathbf{k}+\delta\mathbf{k}$ $(|\delta\mathbf{k}| \ll 0)$. They can be approximated by step functions of width $2\delta\mathbf{k}$ centered at wave vectors $\mathbf{k}$ (a step function is constant between $\mathbf{k}-\delta\mathbf{k}$ and $\mathbf{k}+\delta\mathbf{k}$, zero outside). In addition the cosine terms in δS_n are maximum in $\mathbf{k}$ and exhibit whole numbers of periods between $\mathbf{k}-\delta\mathbf{k}$ and $\mathbf{k}+\delta\mathbf{k}$. Thus we have $\delta S_n \simeq 0$ and

$$\langle S_n(\xi\mathbf{k})\rangle = \xi^{-D}\langle S_n(\mathbf{k})\rangle . \tag{4.11}$$

This expression is a direct consequence of the self-similarity of the direct space sets. On Fig. 5, we have plotted $\langle S_n(\mathbf{k})\rangle$ versus k for a Sierpinski's carpet (k is the mean radius of the edge bands); clearly relation (4.11) holds in a large range of wave vectors except at low frequencies where a significant deviation due to finite size effects is observed.

V. SUMMARY AND CONCLUDING REMARKS

We have presented solutions for the diffraction spectrum of a general family of deterministic self-similar arrays which are generated by means of a recursive inflation method. This construction imitates the way according to which real clusters are grown from smaller entities. The only limitation is that the n-stage sets are entirely tiled without overlappings using the sets generated at stage $n-1$. Generalizations to objects constructed with two iterative rules or involving overlappings are straightforward.

The behavior of the structure factors in the asymptotic limit of arbitrarily large sets depends on the algebraic properties of both the inflation factor and the relative positions of the sites of the generator. Two classes of systems have been distinguished according as their structure factors vanish or are composed of Bragg peaks densely filling the reciprocal space. All the arrays used in the literature to investigate physical phenomena on fractals belong to the second class characterized by

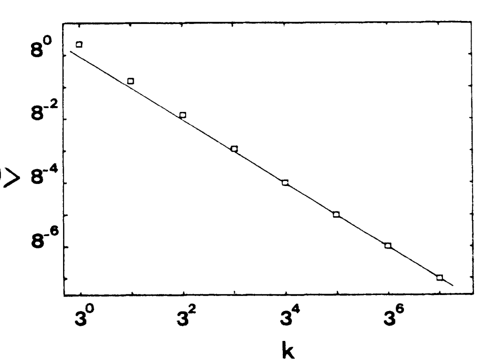

FIG. 5. $\log_8$-$\log_3$ plot showing the variation of $\langle S_{10}(k)\rangle$ for the Sierpinski's carpet of similarity $D=\ln 8/\ln 3$. At large wave vectors the points fall perfectly on a line of slope $-D$ showing that the scaling relation (4.11) holds perfectly; deviations are observed at small wave vectors because of finite-size effects.

strong constructive spatial correlations. In view of this it seems useful to raise the question of the relevance of this classification with respect to other physical properties of deterministic fractals.

We have studied in detail the case of nonvanishing structure factors; because of the geometrical regularity of the corresponding direct space sets, these are highly ordered diffraction patterns with orientational symmetry and translational invariance. Self-similarity is responsible for the existence of a bandlike structure in the diffraction patterns and leads to several scaling relations. In the same way it would be of interest in further studies to know whether this particular spatial spectrum has special implications in the physical properties of the arrays.

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[1]For a review see J. K. Kjems, and T. Freltoft, in *Scaling Phenomena in Disordered Systems,* Vol. 133 of *NATO Advanced Study Institute, Series B: Physics,* edited by R. Pynn and A. Skjeltorp (Plenum, New York, 1985); J. Texeira, in *Growth and Form,* edited by H. E. Stanley and N. Ostrowsky (Martinus Nijhoff, The Hague, 1985).

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[7]A set of points E within an interval I is a set of uniqueness if there exists no identically zero trigonometric series converging towards zero everywhere outside I. On the opposite, E is said a set of multiplicity. For additonal information see Ref. 8 (Chap. 5).

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[13]The gold number $(\sqrt{5}+1)/2$ is a well-known quadratic PV number; it is defined by the second degree quadratic equation $\tau^2-\tau-1=0$; its conjugate is $(1-\sqrt{5})/2$ thus of modulus inferior to the unity.