Diffraction of an atomic beam by a phase-fluctuating standing light wave

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We consider an atomic beam at normal incidence with a standing wave of light with phase fluctuations and we evaluate the momentum distribution of the diffracted atoms after passing through the light beam. We consider the independent-increment model or the random-jump model for the phase of the light, and the random-jump model for the light frequency. In each of these three cases we distinguish the situation of independent or identical phases of the two counter-running light waves that compose the standing wave. For identical phases, the momentum distribution seems to be sensitive mainly to the spectrum of the light, whereas for independent phases higher correlation functions are also important.

I. INTRODUCTION

When an atomic beam crosses a standing wave of laser radiation at right angles, the dipole force induces a diffraction pattern in the momentum distribution of the atoms in the laser-beam direction. This pattern may be viewed as resulting from the interaction of the atomic wave packet with the periodic potential that is set up by the grating of nodes and antinodes of the standing wave. Alternatively, it may be regarded as the manifestation of the transfer of photon momentum due to the absorption and stimulated emission of photons in either one of the two counter-running plane waves that compose the standing wave. The first picture focuses on the spatial variation of the beam intensity and on the wave character of the atoms. The second picture seems to derive from the quantized nature of the radiation field and the absorption and emission of discrete momentum packets. In any case, the diffraction of atoms by a standing wave is a microscopic demonstration of the quantum-mechanical atom-field interaction. Recent experiments have demonstrated the feasibility of the resolution of single-photon momentum in the diffracted patterns.[1–5] Theoretical work on diffraction by a standing wave has usually considered two-level atoms in a monochromatic radiation field. These treatments are based on the Schrödinger equation for the wave function of the two states,[6–8] or on the generalized optical Bloch equations for the atomic density matrix.[9–13] In these descriptions the optical coherence of the atoms plays an essential part and the diffraction pattern can extend over quite a few photon momenta, even when the excited-state population remains rather small.[12]

This suggests that the diffraction pattern may depend in a sensitive way on the coherence properties of the radiation field. In the limit of very broadband radiation, the stimulated transitions are expected to become fully uncorrelated so that a rate equation description must become valid. Some work has been devoted to the case of deflection of atoms by a single broadband running wave.[14] Also the transition from monochromatic radiation to broadband radiation for deflection by a single beam has been considered as well as the situation of several fully incoherent beams.[15]

In the present paper we consider the case of diffraction of an atomic beam by a standing wave with arbitrary phase fluctuations. We consider various specific models for the phase fluctuations, in particular the independent-increment model for the phase, and the class of random-jump models either for the phase or for the frequency. These models are known to give explicit results for the intensity correlation function and the spectrum of resonance fluorescence.[16–21] We demonstrate that at least in several particular cases these models allow for an evaluation of the momentum distribution of the atoms as a function of the interaction time. The result depends on the specific model for the phase fluctuations.

In the limit of very large bandwidth the momentum distribution may be described by a random-walk model with steps along the momentum axis of a photon momentum, and with a step rate that is equal to the stimulated transition rate. This result holds only when the phases for the two counter-running waves composing the standing wave are independent, so that not only the phase of the field at any position, but also the location of the nodes of the wave are random variables. In the case of a standing wave that results from placing a mirror in a running wave, this case corresponds to the situation that the distance of the interaction region to the mirror is large compared with the coherence length of the radiation. In the opposite case the phase of the two counter-running waves are equal and the standing wave has a fluctuating phase, but fixed node positions. Also in this case we can evaluate the momentum distribution and the result is essentially different from the case of independent phases. These results illustrate the importance of the details of the coherence properties of a standing wave on the matter-radiation interaction.

II. EVOLUTION EQUATIONS

A. Bloch equations

We consider an atomic beam of two-state atoms, moving with uniform momentum $\mathbf{p}_0$ in the z direction. The

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atomic beam crosses a phase-fluctuating standing wave of radiation with wave vector $\pm\mathbf{k}$ along the x axis and polarization $\boldsymbol{\epsilon}$. We describe this standing wave by the classical electric field

$$\mathbf{E}(\mathbf{r},t)=\tfrac{1}{2}\mathrm{Re}[\boldsymbol{\epsilon}E_0(e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_L t-i\psi(t)}+e^{-i\mathbf{k}\cdot\mathbf{r}-i\omega_L t-i\varphi(t)})]\ , \tag{2.1}$$

where $\psi(t)$ and $\varphi(t)$ are real-valued stochastically fluctuating phases. We have assumed that the bandwidth of the radiation is much smaller than the frequency ω_L, so that fluctuations in the wavelength can be neglected. The spectral distribution for either one of the counterpropagating light beams is given by

$$I_L(\omega)=\frac{1}{\pi}\mathrm{Re}\int_0^\infty d\tau\, e^{i(\omega-\omega_L)\tau}\langle\langle e^{-i[\psi(t+\tau)-\psi(t)]}\rangle\rangle\ , \tag{2.2}$$

where $\langle\langle\ \rangle\rangle$ denotes averaging over the stochastic variables.

During the interaction the atom will acquire momentum in the x direction as a result of the exchange of photon momentum at spontaneous and stimulated transitions. To study the evolution of the momentum distribution of the atoms it is convenient to consider the Wigner representation of the density matrix ρ of a single atom.[22] We introduce the quantity $\rho(\mathbf{r},\mathbf{s})$ which is related to the density matrix ρ

$$\rho(\mathbf{r},\mathbf{s})=\langle\mathbf{r}+\tfrac{1}{2}\mathbf{s}\,|\,\rho\,|\,\mathbf{r}-\tfrac{1}{2}\mathbf{s}\rangle \tag{2.3}$$

in the position representation. The Wigner distribution function is then given by[23]

$$W(\mathbf{r},\mathbf{p})=(2\pi\hbar)^{-3}\int d\mathbf{s}\ \rho(\mathbf{r},\mathbf{s})e^{-i\mathbf{p}\cdot\mathbf{s}/\hbar}\ . \tag{2.4}$$

Both ρ and W are still a matrix with respect to the internal states of the atom. The interaction between an atom and the radiation field is described in the atomic dipole approximation.

We shall assume that the change in position along the x axis during the passage time is negligible compared to the wavelength of the standing wave and that the gain in momentum of the atom in the x direction is small enough to make the Doppler shift $\mathbf{k}\cdot\mathbf{p}/m$ negligible compared to the natural linewidth A. If these conditions are fulfilled we may ignore the Doppler shifts and the free-flow term in the evolution equation for ρ, which results from the kinetic energy operator in the Hamiltonian.[11] Furthermore we assume that the interaction time T is so small that spontaneous transitions are negligible ($AT \ll 1$).

The evolution equation for $\rho(\mathbf{r},\mathbf{s},t)$ in the rotating-wave approximation is then given by[11–13]

$$\begin{aligned}
\frac{\partial}{\partial t}\rho_{ee}&=\tfrac{1}{2}i\Omega(\tilde{c}_+\rho_{ge}-\tilde{c}^{\,*}_-\rho_{eg})\ ,\\
\frac{\partial}{\partial t}\rho_{gg}&=\tfrac{1}{2}i\Omega(\tilde{c}^{\,*}_+\rho_{eg}-\tilde{c}_-\rho_{ge})\ ,\\
\frac{\partial}{\partial t}\rho_{eg}&=i\Delta\rho_{eg}+\tfrac{1}{2}i\Omega(\tilde{c}_+\rho_{gg}-\tilde{c}_-\rho_{ee})\ ,\\
\frac{\partial}{\partial t}\rho_{ge}&=-i\Delta\rho_{ge}+\tfrac{1}{2}i\Omega(\tilde{c}^{\,*}_+\rho_{ee}-\tilde{c}^{\,*}_-\rho_{gg})\ ,
\end{aligned} \tag{2.5}$$

where the rapidly oscillating terms $\exp(\pm i\omega_L t)$ are absorbed in the off-diagonal terms ρ_{eg} and ρ_{ge}. Furthermore $\Delta=\omega_L-\omega_0$ is the detuning of the central frequency of the radiation from resonance and

$$\Omega=\boldsymbol{\mu}_{eg}\cdot\boldsymbol{\epsilon}E_0/\hbar \tag{2.6}$$

is the Rabi frequency. The factors $\tilde{c}_\pm$ are defined by

$$\tilde{c}_\pm=\cos[\mathbf{k}\cdot(\mathbf{r}\pm\tfrac{1}{2}\mathbf{s})+\tfrac{1}{2}(\varphi-\psi)]e^{-i(1/2)(\varphi+\psi)}\ , \tag{2.7}$$

with $\mathbf{k}=(k,0,0)$. Equation (2.7) reflects that the positions of the nodes and antinodes are displaced by the fluctuating distances

$$\mathbf{u}=\hat{\mathbf{x}}(\psi-\varphi)/2k\ , \tag{2.8}$$

whereas $(\psi+\varphi)/2$ determines fluctuations in the phase of the standing wave. In (2.8) $\hat{\mathbf{x}}$ denotes the unit vector in the x direction.

For later convenience we introduce the transformed density matrix σ by the definition

$$\sigma=U^\dagger\rho U\ , \tag{2.9}$$

where the stochastic transformation operator is

$$U=\exp\left[-\frac{i}{\hbar}\mathbf{u}\cdot\mathbf{P}+\tfrac{1}{2}i(\psi+\varphi)P_e\right] \tag{2.10}$$

in terms of the momentum operator $\mathbf{P}$ and the projection operator $P_e=|e\rangle\langle e|$ on the excited state. The evolution equation for

$$\sigma(\mathbf{r},\mathbf{s})=\langle\mathbf{r}+\tfrac{1}{2}\mathbf{s}\,|\,\sigma\,|\,\mathbf{r}-\tfrac{1}{2}\mathbf{s}\rangle \tag{2.11}$$

is then

$$\begin{aligned}
\frac{D}{Dt}\sigma_{ee}&=\tfrac{1}{2}i\Omega(c_+\sigma_{ge}-c_-\sigma_{eg})\ ,\\
\frac{D}{Dt}\sigma_{gg}&=\tfrac{1}{2}i\Omega(c_+\sigma_{eg}-c_-\sigma_{ge})\ ,\\
\frac{D}{Dt}\sigma_{eg}&=i[\Delta+\tfrac{1}{2}(\dot{\psi}+\dot{\varphi})]\sigma_{eg}+\tfrac{1}{2}i\Omega(c_+\sigma_{gg}-c_-\sigma_{ee})\ ,\\
\frac{D}{Dt}\sigma_{ge}&=-i[\Delta+\tfrac{1}{2}(\dot{\psi}+\dot{\varphi})]\sigma_{ge}+\tfrac{1}{2}i\Omega(c_+\sigma_{ee}-c_-\sigma_{gg})\ ,
\end{aligned} \tag{2.12}$$

where $\dot{\psi}$ stands for $\partial\psi/\partial t$, etc. Furthermore we have defined

$$\frac{D}{Dt}=\frac{\partial}{\partial t}-\dot{\mathbf{u}}\cdot\frac{\partial}{\partial\mathbf{r}} \tag{2.13}$$

and

$$c_\pm=\cos[\mathbf{k}\cdot(\mathbf{r}\pm\tfrac{1}{2}\mathbf{s})]\ . \tag{2.14}$$

Note that we made a transformation to a coordinate system in which the stochastically fluctuating nodes of the standing wave are at rest. The coupling terms $c_\pm$ are then independent of the stochastic fields at the expense of changing the time derivative in a total derivative (2.13).

The momentum distribution $Z(\mathbf{p},t)$ of the two-state atom at time t, irrespective of the position and internal state is obtained from the density matrix $\rho(\mathbf{r},\mathbf{s},t)$ [or

$\sigma(\mathbf{r},\mathbf{s},t)]$ by taking the trace over the initial states, integrating over the position $\mathbf{r}$, taking the Fourier transform with respect to $\mathbf{s}$ and averaging over the stochastic variables. We may thus write[13]

$$Z(\mathbf{p},t)=\frac{1}{V}(2\pi\hbar)^{-3}\int d\mathbf{r}\,d\mathbf{s}\exp(-i\mathbf{s}\cdot\mathbf{p}/\hbar)\times\mathrm{Tr}\langle\langle\rho(\mathbf{r},\mathbf{s},t)\rangle\rangle\ , \quad (2.15)$$

where $\langle\langle\ \rangle\rangle$ denotes averaging over the stochastic variables and where the integration over $\mathbf{r}$ extends over a large quantization volume V. The same result is obtained when we replace $\rho(\mathbf{r},\mathbf{s},t)$ in (2.15) by $\sigma(\mathbf{r},\mathbf{s},t)$ since the displacement operator does not affect the integral over $\mathbf{r}$. The initial density matrix at zero time is taken to be

$$\rho(\mathbf{r},\mathbf{s},0)\equiv P_g\exp(i\mathbf{s}\cdot\mathbf{p}_0/\hbar)\ , \quad (2.16)$$

where $P_g=|g\rangle\langle g|$ is the projection operator on the ground state and $\mathbf{p}_0=(0,0,p_0)$ is the initial momentum of the atom in the z direction. We are only interested in the momentum distribution in the x direction, which is the propagation direction of the standing wave. Therefore it is sufficient to take the vector $\mathbf{s}$ in this direction. Since the initial momentum is well defined, it follows that the initial position distribution must be a slowly varying function of x, which we have taken to be constant. Furthermore the coupling terms $\tilde{c}_\pm$ (or $c_\pm$) are periodic in x. It is therefore sufficient to replace the $\mathbf{r}$ integration in (2.15) by an integration over x extending over one wavelength $\lambda=(2\pi/k)$ of the standing wave. The momentum distribution for the x direction takes the form

$$Z(p,t)=\frac{1}{2\pi\hbar}\frac{1}{\lambda}\int_0^\lambda dx\int_{-\infty}^{\infty}ds\,e^{isp/\hbar}\mathrm{Tr}\langle\langle\rho(x,s,t)\rangle\rangle\ , \quad (2.17)$$

where $\rho(x,s,t)$ must obey (2.5) in which the factor $\mathbf{k}\cdot(\mathbf{r}\pm\frac{1}{2}\mathbf{s})$ in the coupling terms $\tilde{c}_\pm$ can be written as $k(x\pm\frac{1}{2}s)$. Equation (2.17) remains valid when we replace $\rho(x,s,t)$ by $\sigma(x,s,t)$. The initial conditions in both cases are

$$\rho(x,s,0)=\sigma(x,s,0)=P_g\ . \quad (2.18)$$

B. Master equation

In order to model the phase fluctuations of the radiation field, we shall assume that either ψ and φ, or their time derivatives $\dot{\psi}$ and $\dot{\varphi}$ can be described as a Markov process η.[24] Hence the stochastics is completely described by the probability distribution $P(\eta,t)$ and the conditional probability distribution $P(\eta t\mid\eta_0 t_0)$. We shall assume the process to be homogeneous, so that the conditional probability depends only on the time difference $t-t_0$. Then the probability distribution $P(\eta,t)$ obeys the master equation[24]

$$\frac{\partial}{\partial t}P(\eta,t)=-b(\eta)P(\eta,t)+\int d\eta' B(\eta,\eta')P(\eta',t)\ , \quad (2.19)$$

where $B(\eta,\eta')$ is the transition rate from η' to η which obeys the identity

$$\int d\eta\,B(\eta,\eta')=b(\eta')\ . \quad (2.20)$$

The conditional probability $P(\eta t\mid\eta_0 t_0)$ obeys the same master equation (2.19).

III. RANDOM-WALK DESCRIPTION FOR BROADBAND LIMIT

Before we consider the case of arbitrary bandwidth, we turn our attention to the broadband limit, where we may assume that the evolution of the two-state atom can be described by rate equations. The rate α of transitions between the states $|g\rangle$ and $|e\rangle$ by absorption or stimulated emissions is given by the Einstein coefficient for stimulated transitions multiplied by the spectral density of either one of the counterpropagating light beams taken at the atomic transition frequency ω_0.

The atomic momentum p in the x direction is changed by a photon momentum $\hbar k$ at absorption of a photon from the beam with wave vector $\mathbf{k}$, and at stimulated emission of a photon into the beam with wave vector $-\mathbf{k}$. The other two stimulated transitions give a momentum change by $-\hbar k$. This process is illustrated in Fig. 1. When we ignore spontaneous emissions, the evolution of the atomic momentum p may be viewed as a one-dimensional random walk,[24] and the momentum distribution is governed by the master equation

$$\frac{\partial}{\partial t}Z(p,t)=-2\alpha Z(p,t)+\alpha[Z(p+\hbar k,t)+Z(p-\hbar k,t)] \quad (3.1)$$

with

$$\alpha=\tfrac{1}{8}\pi\Omega^2 I_L(\omega_0)\ , \quad (3.2)$$

where the profile of a light spectrum is given in Eq. (2.2).

Taking the Fourier transform

$$\tilde{Z}(s,t)=\int_{-\infty}^{\infty}dp\,e^{isp/\hbar}Z(p,t)\ , \quad (3.3)$$

we obtain from (3.1) and the initial condition $Z(p,0)=\delta(p)$

$$\tilde{Z}(s,t)=e^{-2\alpha t[1-\cos(ks)]}\ . \quad (3.4)$$

Using the expansion in modified Bessel functions[25]

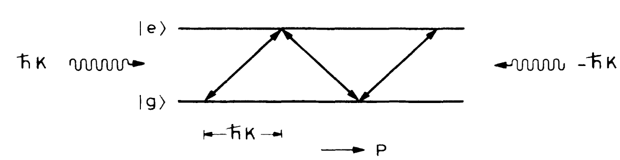

FIG. 1. Scheme of the momentum exchange between a two-level atom and a standing wave of radiation. The solid lines indicate stimulated transitions involving photons from either one of the two running waves, which induce momentum differences by $\pm\hbar k$ between excited and ground-state atoms.

$$e^{z\cos\phi}=\sum_{n=-\infty}^{\infty} I_n(z)e^{in\phi} , \quad (3.5)$$

the inverse Fourier transform can easily be done, yielding

$$Z(p,t)=\sum_{n=-\infty}^{\infty} e^{-2\alpha t}I_n(2\alpha t)\delta(p-n\hbar k) . \quad (3.6)$$

In Fig. 2(a) we plot the momentum distribution (3.6). In subsequent sections we shall compare results for finite bandwidths with this broadband result.

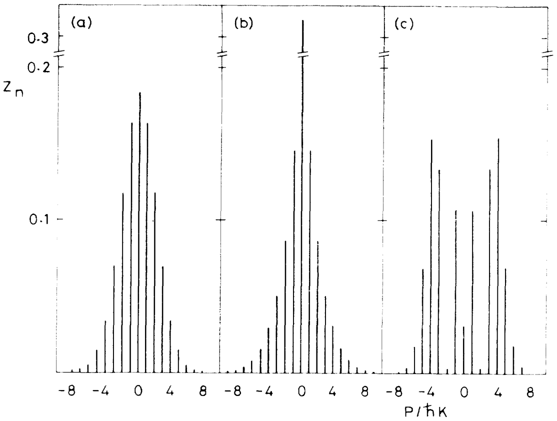

FIG. 2. Momentum distribution of the atoms after diffraction. The distributions are composed of δ peaks at $p=n\hbar k$, and the strengths Z_n are plotted. In (a) we plot the case of independent phases φ and ψ in the broadband limit as described in Eq. (3.6) ($\Delta=0$, $2\alpha t=5$), (b) the case $\varphi=\psi$ in the broadband limit corresponding to Eq. (4.39) and (4.40) ($\Delta=0$, $2\alpha t=5$), and (c) the monochromatic case according to Eq. (4.38) ($\Delta=0$, $\Omega t/2=5$).

IV. THE INDEPENDENT-INCREMENT MODEL FOR THE PHASE

In the independent-increment model the stochastic variable η may take values $-\infty<\eta<\infty$. Then we have to substitute in the master equation (2.19) a transition rate function $B(\eta_2,\eta_1)$ that depends only on the phase increment $\eta_2-\eta_1$.[16] Therefore we can write

$$B(\eta_2,\eta_1)=w(\eta_2-\eta_1) , \quad (4.1)$$

where $w(\eta)$ is a non-negative function. Furthermore we find from (2.20) for the rate of change of η

$$b=\int_{-\infty}^{\infty} w(\beta)d\beta \quad (4.2)$$

which is independent of η.

When the initial condition for the probability distribution is given by

$$P(\eta,0)=\delta(\eta) , \quad (4.3)$$

the master equation (2.19) can easily be solved in Fourier transform and we find

$$P(\eta,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left[-ix\eta-t\int_{-\infty}^{\infty}(1-e^{ix\beta})w(\beta)d\beta\, dx\right] . \quad (4.4)$$

The evolution equation (2.1) for the stochastic function $\sigma(x,s,t)$ has the structure

$$\frac{\partial}{\partial t}\sigma(t)=(L_0+\dot{\varphi}L_-+\dot{\psi}L_+)\sigma(t) , \quad (4.5)$$

where the operators L_0 and $L_\pm$ do not contain $\dot{\varphi}$ or $\dot{\psi}$ but are mutually noncommuting. We assume that φ and ψ are independent stochastic variables, both satisfying the master equation with the same transition rate function (4.1).

We have to evaluate the stochastically averaged quantity $\langle\langle\sigma(t)\rangle\rangle$, with the nonstochastic initial condition (2.18). The evolution equation for the stochastically averaged function can be obtained by the same method as employed in Ref. 17, which we briefly outline here. The formal solution of (4.5) is given by

$$\langle\langle\sigma(t)\rangle\rangle=e^{L_0t}\langle\langle S(t)\rangle\rangle\sigma(0) \quad (4.6)$$

with

$$S(t)=\theta\exp\int_0^t ds[\dot{\varphi}(s)\tilde{L}_-(s)+\dot{\psi}(s)\tilde{L}_+(s)] , \quad (4.7)$$

where θ is the time-ordering operator and

$$\tilde{L}_\pm(t)=e^{-L_0t}L_\pm e^{L_0t} . \quad (4.8)$$

It follows that for infinitimal dt,

$$S(t+dt)=\exp\{[\varphi(t+dt)-\varphi(t)]\tilde{L}_-(t)+[\psi(t+dt)-\psi(t)]\tilde{L}_+(t)\}S(t) . \quad (4.9)$$

For the independent-increment process, the conditional average of the first exponential in (4.9) is independent of the value of $\varphi(t)$ and $\psi(t)$. Hence we obtain after using (4.4) for $t=dt$ and expanding to first order in dt

$$\frac{d}{dt}\langle\langle S(t)\rangle\rangle=-\left[\int_{-\infty}^{\infty}d\beta\, w(\beta)(2-e^{\beta\tilde{L}_+(t)}-e^{\beta\tilde{L}_-(t)})\right]\langle\langle S(t)\rangle\rangle . \quad (4.10)$$

Solving $\langle\langle S(t)\rangle\rangle$ from (4.10) and substituting the result in (4.6), we obtain for the stochastically averaged density matrix

$$\frac{\partial}{\partial t}\langle\langle\sigma(t)\rangle\rangle=(L_0-V)\langle\langle\sigma(t)\rangle\rangle , \quad (4.11)$$

where

$$V=\int_{-\infty}^{\infty}d\beta\, w(\beta)(2-e^{\beta L_+}-e^{\beta L_-}) . \quad (4.12)$$

A. General solution for $Z(p,t)$

Equation (4.11) can be written as

$$\frac{\partial}{\partial t}\langle\langle\sigma(t)\rangle\rangle=[R_0+R_-(s)e^{ikx}+R_+(s)e^{-ikx}-V]\langle\langle\sigma(t)\rangle\rangle\ , \tag{4.13}$$

where V is given by (4.12) with

$$L_\pm\sigma=\tfrac{1}{2}i[P_e,\sigma]\pm\frac{1}{2k}\frac{\partial}{\partial x}\sigma\ . \tag{4.14}$$

The operators R_0 and $R_\pm$ result from an expansion in powers of $\exp(ikx)$ of L_0 as implicitly defined by (4.5) and (2.12).

Since the evolution operator in (4.13) is periodic in x, we can express $\langle\langle\sigma\rangle\rangle$ as a Fourier series according to

$$\langle\langle\sigma(x,s,t)\rangle\rangle=\sum_{n=-\infty}^{\infty}e^{inkx}\sigma_n(s,t)\ . \tag{4.15}$$

Then we obtain the set of coupled differential equations

$$\frac{\partial}{\partial t}\sigma_n(s,t)=(R_0-V_n)\sigma_n(s,t)+R_+(s)\sigma_{n+1}(s,t)+R_-(s)\sigma_{n-1}(s,t) \tag{4.16}$$

with

$$V_n\sigma=2\lambda_n\sigma+(\lambda_{n+1}+\lambda_{n-1}-2\lambda_n)L_e\sigma\ , \tag{4.17}$$

where we have defined

$$\lambda_n=\int_{-\infty}^{\infty}d\beta\, w(\beta)(1-e^{(1/2)i\beta n}) \tag{4.18}$$

and

$$L_e\sigma=[P_e,[P_e,\sigma]]\ . \tag{4.19}$$

In deriving (4.17) we have assumed that $w(\eta)=w(-\eta)$. The initial condition is given by

$$\sigma_n(s,0)=\delta_{0n}P_g\ . \tag{4.20}$$

From (2.17) and (4.15), we obtain for the Fourier-transformed momentum distribution

$$\tilde{Z}(s,t)=\mathrm{Tr}[\sigma_0(s,t)]\ . \tag{4.21}$$

The momentum distribution is thus completely determined by $\sigma_0(s,t)$ which must be solved from (4.16). Taking the Laplace transform

$$\tilde{\sigma}_n(s,v)=\int_0^\infty e^{-vt}\sigma_n(s,t)dt\ , \tag{4.22}$$

we obtain from (4.16) and (4.20)

$$(v-R_0+V_n)\tilde{\sigma}_n(s,v)=R_+(s)\tilde{\sigma}_{n+1}(s,v)+R_-(s)\tilde{\sigma}_{n-1}(s,v)+\delta_{0n}P_g\ . \tag{4.23}$$

This equation admits a formal solution for $\tilde{\sigma}_0(s,v)$ in terms of matrix continued fractions.[26]

We introduce the set of matrices

$$X_n=\tilde{\sigma}_{n+1}\tilde{\sigma}_n^{-1}\ ,\quad n>0\ . \tag{4.24}$$

Multiplying (4.23) on the right by $\tilde{\sigma}_n^{-1}$, we obtain

$$X_{n-1}=(A_n-R_+X_n)^{-1}R_-\ ,\quad n>0 \tag{4.25}$$

where

$$A_n=v-R_0+V_n\ . \tag{4.26}$$

Using (4.25) repeatedly, we obtain a matrix continued fraction for X_0

$$X_0=\cfrac{1}{A_1-R_+\cfrac{1}{A_2-R_+\cfrac{1}{A_3-\cdots}R_-}R_-}R_-\ . \tag{4.27}$$

We shall also need the matrices

$$Y_m=\tilde{\sigma}_{-(m+1)}\tilde{\sigma}_{-m}^{-1},\quad m>0 \tag{4.28}$$

for which we obtain in a similar fashion

$$Y_0=\cfrac{1}{A_1-R_-\cfrac{1}{A_2-R_-\cfrac{1}{A_3-\cdots}R_+}R_+}R_+\ . \tag{4.29}$$

Substituting $\tilde{\sigma}_1=X_0\tilde{\sigma}_0$ and $\tilde{\sigma}_{-1}=Y_0\tilde{\sigma}_0$ in Eq. (4.23) for $n=0$ we obtain

$$\tilde{\sigma}_0=\frac{1}{A_0-R_+X_0-R_-Y_0}P_g\ , \tag{4.30}$$

where we have suppressed the arguments s and v. The momentum distribution is then given by

$$\tilde{Z}(s,v)=\mathrm{Tr}\left[\frac{1}{A_0-R_+X_0-R_-Y_0}P_g\right]\ . \tag{4.31}$$

B. Broadband limit

From (2.2) we obtain for the spectral profile for either one of the counterpropagating light beams, using (4.4) and (4.18)

$$I_L(\omega)=\frac{1}{\pi}\mathrm{Re}\left[\frac{1}{\lambda_2-i(\omega-\omega_L)}\right]\ . \tag{4.32}$$

The profile is thus Lorentzian with half width at half maximum (HWHM) λ_2. In the limit of large bandwidth $\lambda_2\gg\Omega$, Δ the lowest-order contribution in λ_2^{-1} to the momentum distribution is given by

$$\tilde{Z}(s,v)=\mathrm{Tr}\left[\frac{1}{A_0-R_+A_1^{-1}R_1-R_-A_1^{-1}R_1}P_g\right]\ . \tag{4.33}$$

Substituting the expression for the operators we find after a straightforward calculation to lowest order in λ_2^{-1}

$$\tilde{Z}(s,v)=\left[v+\frac{\Omega^2}{4\lambda_2}[1-\cos(ks)]\right]^{-1} . \quad (4.34)$$

This result is exact in the limit $\Omega\rightarrow\infty$, $\lambda_2\rightarrow\infty$, and $\Omega^2/\lambda_2\rightarrow 8\alpha$. In this limit the moment distribution is identical to (3.6), which demonstrates that the intuitive random-walk picture is justified in this limit.

C. Identical phases

In the previous subsections φ and ψ were taken as independent stochastic variables. The standing wave can then be viewed as being composed of two independent running waves. A standard way of producing a standing wave is using a running wave that is reflected by a mirror placed in the beam. The assumption of independent phases φ and ψ is justified when the distance between the interaction region and the mirror is large compared with the coherence length of the field. In the opposite case of a distance that is short compared with the coherence length, we may assume that $\varphi=\psi$. In this case the limit of large bandwidth is no longer described by rate equations for independent running waves. In this case the position of the nodes are not fluctuating and the calculations are simplified. For the evolution equation for $\langle\langle\sigma\rangle\rangle$ we obtain

$$\frac{\partial}{\partial t}\langle\langle\sigma(t)\rangle\rangle=(L_0-\lambda_2 L_e)\langle\langle\sigma(t)\rangle\rangle , \quad (4.35)$$

where L_0 is defined in (4.5) and λ_2 is defined in (4.18) and L_e in (4.19). It is interesting to notice that the evolution equation (4.35) depends only on the parameter λ_2, and not on the higher moments of the transition rate distribution.

After a straightforward calculation analogous to Ref. 13 we obtain for the momentum distribution

$$\tilde{Z}(s,v)=\frac{1}{\lambda}\int_0^\lambda\left[\frac{v[(v+\lambda_2)^2+\Delta^2]+\frac{1}{4}\Omega^2(c_++c_-)^2(v+\lambda_2)}{v^2[(v+\lambda_2)^2+\Delta^2]+\frac{1}{2}\Omega^2(c_+^2+c_-^2)(v+\lambda_2)+\frac{\Omega^4}{16}(c_+^2-c_-^2)^2}\right]dx . \quad (4.36)$$

For excitation on resonance ($\Delta=0$), Eq. (4.36) reduces to the simple form

$$\tilde{Z}(s,v)=\frac{1}{\pi}\int_0^\pi\frac{v+\lambda_2}{v^2+v\lambda_2+\Omega^2\sin^2x\,\sin^2(\frac{1}{2}ks)}dx , \quad (4.37)$$

which for $\lambda_2=0$ gives the well-known result[6]

$$Z(p,t)=\sum_{n=-\infty}^{\infty}J_n^2\left[\frac{\Omega t}{2}\right]\delta(p-n\hbar k) . \quad (4.38)$$

In the broadband limit $\lambda_2\rightarrow\infty$, $\Omega\rightarrow\infty$, and $(\Omega^2/\lambda_2)\rightarrow 8\alpha$, we obtain

$$Z(p,t)=\sum_{n=-\infty}^{\infty}\delta(p-n\hbar k)Z_n(t) \quad (4.39)$$

with

$$Z_n(t)=\sum_{m=|n|}^{\infty}\frac{(-\frac{1}{2}\alpha t)^m}{m!}(-1)^n\binom{2m}{m}\binom{2m}{m-|n|} . \quad (4.40)$$

In Figs. 2(b) and 2(c) we plot the momentum distributions (4.38) for monochromatic radiation, and (4.39) for the broadband limit.

V. THE RANDOM-JUMP PROCESS

A random-jump process is a homogeneous Markov process η where the probability for making a transition from η to η' is independent of the value η before the transition.[27] Therefore the transition rate function determining the master equation (2.19) is given by

$$B(\eta_2,\eta_1)=\gamma h(\eta_2), \quad \gamma>0 \quad (5.1)$$

where $h(\eta)$ is a non-negative function that is normalized according to

$$\int h(\eta)d\eta=1 . \quad (5.2)$$

The rate of change of η is independent of η and we write

$$b(\eta)=\gamma . \quad (5.3)$$

The master equation for the conditional probability now takes the form

$$\frac{\partial}{\partial t}P(\eta t\mid\eta_0 t_0)=-\gamma P(\eta t\mid\eta_0 t_0)+\gamma h(\eta) \quad (5.4)$$

with the obvious solution

$$P(\eta t\mid\eta_0 t_0)=e^{-\gamma(t-t_0)}\delta(\eta-\eta_0)+h(\eta)(1-e^{-\gamma(t-t_0)}) . \quad (5.5)$$

We notice that after a transient time of the order $1/\gamma$, the conditional probability approaches the stationary distribution $h(\eta)$.

The stochastic evolution equation for the density matrix ρ (or σ) can be cast in the form

$$\frac{\partial}{\partial t}\rho(t)=L(\xi(t),\zeta(t))\rho(t) , \quad (5.6)$$

where $\xi(t)$ and $\zeta(t)$ are stochastic variables described by the random-jump process.

It is convenient to consider the marginal average[18,19,27]

$$\Pi(\xi_0,\zeta_0,t)=\langle\langle\delta(\xi(t)-\xi_0)\delta(\zeta(t)-\zeta_0)\rho(t)\rangle\rangle . \quad (5.7)$$

This matrix Π is the conditionally averaged density matrix ρ with the condition that $\xi(t)=\xi_0$ and $\zeta(t)=\zeta_0$, multiplied by $P(\xi_0)P(\zeta_0)$. We shall use the same notation for the case where we consider σ instead of ρ.

When the matrix Π is known, we can obtain $\langle\langle\rho\rangle\rangle$ from the integral

$$\langle\langle\rho(t)\rangle\rangle=\int d\xi_0 d\zeta_0 \Pi(\xi_0,\zeta_0,t) . \tag{5.8}$$

The evolution equation for the matrix Π is given by the Burshtein equation[28,27]

$$\frac{\partial}{\partial t}\Pi(\xi_0,\zeta_0,t)=[L(\xi_0,\zeta_0)-2\gamma]\Pi(\xi_0,\zeta_0,t) +\gamma P(\xi_0)\int \Pi(\xi_0',\zeta_0,t)d\xi_0' +\gamma P(\zeta_0)\int \Pi(\xi_0,\zeta_0',t)d\zeta_0' \tag{5.9}$$

with the initial condition

$$\Pi(\xi_0,\zeta_0,0)=P(\xi_0)P(\zeta_0)P_g . \tag{5.10}$$

The Burshtein equation (5.9) is a combination of the evolution equation (5.6) and the master equation (5.5). In the subsequent sections we shall consider special models for which the Burshtein equation can be solved explicitly.

A. Random-jump process for independent phases

In the case of random jumps for the phase, we can identify $\xi(t)=\varphi(t)$ and $\zeta(t)=\psi(t)$ with independent ξ and ζ, both described by the random-jump process. The evolution equation for ρ, as given by (2.5), yields for the operator L in (5.6)

$$L(\varphi(t),\psi(t))=R_0+F_+e^{i\varphi(t)}+F_-e^{-i\varphi(t)} +H_+e^{i\psi(t)}+H_-e^{-i\psi(t)} . \tag{5.11}$$

The operators $F_\pm$ and $H_\pm$ result from an expansion of (2.5) in powers of $e^{i\varphi}$ and $e^{i\psi}$. The operator R_0 is the same as in Eq. (4.13). Because of the particular form of (5.11), we introduce the operators

$$\Xi_{n,m}(t)=\langle\langle e^{in\varphi(t)+im\psi(t)}\rho(t)\rangle\rangle , \tag{5.12}$$

where n and m are integers. This can also be written as

$$\Xi_{n,m}(t)=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}d\varphi_0 d\psi_0 e^{in\varphi_0+im\psi_0}\Pi(\varphi_0,\psi_0,t) . \tag{5.13}$$

Substituting (5.11) in (5.9), multiplying (5.9) with $e^{in\varphi_0+im\psi_0}$, and integrating over φ_0 and ψ_0 yields the set of coupled differential equations

$$\frac{\partial}{\partial t}\Xi_{n,m}(t)=(R_0-2\gamma)\Xi_{n,m}(t)+F_+\Xi_{n+1,m}(t) +F_-\Xi_{n-1,m}(t)+H_+\Xi_{n,m+1}(t) +H_-\Xi_{n,m-1}(t)+\gamma q_n\Xi_{0,m}(t)+\gamma q_m\Xi_{n,0}(t) \tag{5.14}$$

with

$$q_n=\int_{-\pi}^{\pi}d\varphi e^{in\varphi}P(\varphi) \tag{5.15}$$

the Fourier coefficients of P. The initial condition is given by

$$\Xi_{n,m}(0)=q_n q_m P_g . \tag{5.16}$$

We consider now the special case where $\varphi(t)$ and $\psi(t)$ can only take the values $\pm\frac{1}{2}\pi$, with equal probability. The stationary probability distribution is then given by

$$P(\eta)=h(\eta)=\tfrac{1}{2}\delta(\eta-\tfrac{1}{2}\pi)+\tfrac{1}{2}\delta(\eta+\tfrac{1}{2}\pi) . \tag{5.17}$$

Then the phases are described by the random-telegraph process.[24] Using that

$$\Xi_{n+2,m}=\Xi_{n,m+2}=-\Xi_{n,m} , \tag{5.18}$$

the sequence of recurrence relation (5.14) reduces to a set of four equations. Taking the Laplace transform

$$\tilde{\Xi}_{n,m}(v)=\int_0^\infty e^{-vt}\Xi_{n,m}(t)dt , \tag{5.19}$$

we obtain from (5.14), (5.16), and (5.18)

$$\begin{aligned}
&(v-R_0)\tilde{\Xi}_{0,0}(v)=F\tilde{\Xi}_{1,0}(v)+H\tilde{\Xi}_{0,1}(v)+P_g ,\\
&(v+\gamma-R_0)\tilde{\Xi}_{1,0}(v)=-F\tilde{\Xi}_{0,0}(v)+H\tilde{\Xi}_{1,1}(v) ,\\
&(v+\gamma-R_0)\tilde{\Xi}_{0,1}(v)=F\tilde{\Xi}_{1,1}(v)-H\tilde{\Xi}_{0,0}(v) ,\\
&(v+2\gamma-R_0)\tilde{\Xi}_{1,1}(v)=-F\tilde{\Xi}_{0,1}(v)-H\tilde{\Xi}_{1,0}(v) ,
\end{aligned} \tag{5.20}$$

where we have defined

$$F=F_+-F_- \tag{5.21}$$

and

$$H=H_+-H_- . \tag{5.22}$$

Solving $\tilde{\Xi}_{0,0}(v)$ from (5.20) we find for zero detuning ($R_0=0$)

$$\tilde{\Xi}_{0,0}(v)=\cfrac{1}{v+\cfrac{F^2+H^2}{v+\gamma}-\cfrac{FH+HF}{v+\gamma}\cfrac{1}{v+2\gamma+\cfrac{F^2+H^2}{v+\gamma}}\cfrac{FH+HF}{v+\gamma}}P_g . \tag{5.23}$$

After a straightforward calculation we find for the momentum distribution (2.17) in Fourier-Laplace transform in the case of zero detuning ($\Delta=0$)

$$\tilde{Z}(s,v)=\frac{2}{\pi}\int_0^{\pi/2}\left[\frac{v+2\gamma+\Lambda(s,v)}{v^2+2v[\gamma+\Lambda(s,v)]+\Lambda(s,v)[2\gamma+\Lambda(s,v)\sin^2(2x)]}\right]dx \tag{5.24}$$

with

$$\Lambda(s,v)=\frac{\Omega^2}{4(v+\gamma)}[1-\cos(ks)] . \qquad (5.25)$$

The spectral profile for either one of the counterpropagating light waves can be obtained from (2.2) using (5.5) and (5.17). The result is

$$I_L(\omega)=\frac{1}{\pi}\mathrm{Re}\left[\frac{1}{\gamma-i(\omega-\omega_L)}\right] , \qquad (5.26)$$

which is a Lorentz profile with width (HWHM) γ. In the monochromatic limit $\gamma\to 0$, Eq. (5.24) reduces to

$$\tilde{Z}(s,v)=\frac{2}{\pi}\int_0^{\pi/2}\frac{v}{v^2+\Omega^2\sin^2(\frac{1}{2}ks)\sin^2 x}dx , \qquad (5.27)$$

which gives, after inverse Fourier and Laplace transforms, the well-known result (4.38), as it should.[6] In the broadband limit $\gamma\to\infty$, $\Omega\to\infty$, and $(\Omega^2/\gamma)\to 8\alpha$, Eq. (5.24) becomes

$$\tilde{Z}(s,v)=\{v+2\alpha[1-\cos(ks)]\}^{-1} , \qquad (5.28)$$

which shows that the random-walk picture of Sec. III is justified in this limit.

B. Identical phases described by the random-jump process

In Sec. V A we have considered the case of independent stochastic variables $\varphi(t)$. We shall now turn our attention to the opposite case where $\varphi(t)=\psi(t)$. The evolution operator L in (5.11) reduces then to

$$L(\varphi(t))=R_0+(F_++H_+)e^{i\varphi(t)}+(F_-+H_-)e^{-i\varphi(t)} . \qquad (5.29)$$

In analogy to (5.12), we introduce the operators

$$\chi_n(t)=\langle\langle e^{in\varphi(t)}\rho(t)\rangle\rangle \qquad (5.30)$$

which satisfy the set of coupled differential equations

$$\frac{\partial}{\partial t}\chi_n(t)=(R_0-\gamma)\chi_n(t)+(F_++H_+)\chi_{n+1}(t)+(F_-+H_-)\chi_{n-1}(t)+\gamma q_n\chi_0(t) , \qquad (5.31)$$

where q_n is defined by (5.15). The initial condition is given by

$$\chi_n(0)=q_nP_g . \qquad (5.32)$$

We consider again the case where $\varphi(t)$ only takes the value $\pm\frac{1}{2}\pi$ with the probability distribution given by (5.17). In this case the sequence of coupled differential equations (5.31) reduces to a set of two equations which can easily be solved in Laplace transform. The result for zero detuning is

$$\tilde{\chi}_0(v)=\left[v+\frac{1}{v+\gamma}(F+H)^2\right]^{-1}P_g , \qquad (5.33)$$

where F and H are defined in (5.21) and (5.22). In this way we obtain for the momentum distribution (2.17) in the case of zero detuning

$$\tilde{Z}(s,v)=\frac{1}{\pi}\int_0^{\pi}\frac{v+\gamma}{v^2+v\gamma+\Omega^2\sin^2 x\,\sin^2(\frac{1}{2}ks)}dx \qquad (5.34)$$

which is equal to the result (4.37) in the case $\gamma=\lambda_2$. We notice that in the case of identical phases, the momentum distribution is the same for the independent-increment model (Sec. IV C) and for the random-telegraph model for the phase is treated in the present subsection. However, for independent phases, these two models give quite different results.

C. Random-jump process for independent frequencies

When we consider random jumps in the frequency, we identify $\xi(t)=\dot{\varphi}(t)$ and $\zeta(t)=\dot{\psi}(t)$. We take $\dot{\varphi}$ and $\dot{\psi}$ independent and let both be described by the random-jump process. The evolution equation for σ is given by (4.5), which is of the type (5.6) with

$$L(\dot{\varphi}(t),\dot{\psi}(t))=L_0+\dot{\varphi}(t)L_-+\dot{\psi}(t)L_+ . \qquad (5.35)$$

In analogy with Ref. 27, we introduce the operators

$$\Lambda_{n,m}(t)=\langle\langle\dot{\varphi}(t)^n\dot{\psi}(t)^m\sigma(t)\rangle\rangle , \qquad (5.36)$$

where n and m are non-negative integers. As a result of Eq. (5.9) these operators satisfy the set of coupled differential equations

$$\frac{\partial}{\partial t}\Lambda_{n,m}(t)=(L_0-2\gamma)\Lambda_{n,m}(t)+L_-\Lambda_{n+1,m}(t)+L_+\Lambda_{n,m+1}(t)+\gamma[a_n\Lambda_{0,m}(t)+a_m\Lambda_{n,0}(t)] , \qquad (5.37)$$

where we have defined

$$a_n=\int d\eta\,P(\eta)\eta^n \qquad (5.38)$$

as the moments of the stationary distribution. The initial condition for $\Lambda_{n,m}$ is given by

$$\Lambda_{n,m}(0)=a_na_mP_g . \qquad (5.39)$$

As in Sec. V A, we again consider the special case of the random-telegraph process, and we assume that $\dot{\varphi}(t)$ and $\dot{\psi}(t)$ can take only the values $\pm\delta$ $(\delta>0)$ with probability $\frac{1}{2}$.

The spectral profile of the radiation field can now be obtained from (2.2), yielding[21]

$$I_L(\omega)=\frac{1}{\pi}\frac{\delta^2/\gamma}{(\omega-\omega_L)^4/\gamma^2+[1-2(\delta/\gamma)^2](\omega-\omega_L)^2+(\delta^2/\gamma)^2} . \qquad (5.40)$$

In the particular limit $\delta^2 \to \infty$, $\gamma \to \infty$, and $\delta^2/\gamma \to \lambda_L$, the random-telegraph model for the frequency reduces to the Wiener-Levy process[24] for the phase (phase-diffusion model).[21] In this limit (5.40) reduces to the Lorentzian[21]

$$I_L(\omega) = \frac{1}{\pi} \frac{\lambda_L}{(\omega - \omega_L)^2 + \lambda_L^2} . \tag{5.41}$$

Taking the Laplace transform

$$\tilde{\Lambda}_{n,m}(v) = \int_0^\infty e^{-vt} \Lambda_{n,m}(t) dt \tag{5.42}$$

and using the obvious identity

$$\Lambda_{n+2,m} = \Lambda_{n,m+2} = \delta^2 \Lambda_{n,m} \tag{5.43}$$

the sequence of recurrence relations reduces to the four equations

$$\begin{aligned}
&(v - L_0)\tilde{\Lambda}_{0,0}(v) = L_- \tilde{\Lambda}_{1,0}(v) + L_+ \tilde{\Lambda}_{0,1}(v) + P_g ,\\
&(v - L_0 + \gamma)\tilde{\Lambda}_{1,0}(v) = \delta^2 L_- \tilde{\Lambda}_{0,0}(v) + L_+ \tilde{\Lambda}_{1,1}(v) ,\\
&(v - L_0 + \gamma)\tilde{\Lambda}_{0,1}(v) = L_- \tilde{\Lambda}_{1,1}(v) + \delta^2 L_+ \tilde{\Lambda}_{0,0}(v) ,\\
&(v - L_0 + 2\gamma)\tilde{\Lambda}_{1,1}(v) = \delta^2 [L_- \tilde{\Lambda}_{1,0}(v) + L_+ \tilde{\Lambda}_{1,0}(v)] .
\end{aligned} \tag{5.44}$$

From (5.44) we can solve $\tilde{\Lambda}_{0,0}(v) = \langle\langle \tilde{\sigma}(v) \rangle\rangle$, yielding

$$\langle\langle \tilde{\sigma}(s,v) \rangle\rangle = \left[v - L_0(x,s) - \delta^2 Q_+(x,s) - \delta^4 Q_-(x,s) \frac{1}{v - L_0(x,s) + 2\gamma - \delta^2 Q_+(x,s)} Q_-(x,s) \right]^{-1} P_g \tag{5.45}$$

with

$$Q_\pm(x,s) = L_\pm \frac{1}{v - L_0(x,s) + \gamma} L_+ + L_\mp \frac{1}{v - L_0(x,s) + \gamma} L_- . \tag{5.46}$$

The Fourier transform of the distribution is then obtained from (2.17)

$$\tilde{Z}(s,v) = \frac{1}{\pi} \int_0^\pi dx \, \mathrm{Tr} \left[\left[v - L_0(x,s) - \delta^2 Q_+(x,s) - \delta^4 Q_-(x,s) \frac{1}{v - L_0(x,s) + 2\gamma - \delta^2 Q_+(x,s)} Q_-(x,s) \right]^{-1} P_g \right] . \tag{5.47}$$

The expression (5.47) is the central result of this section, and it can serve as a basis for further calculations.

In the limit $\delta^2 \to \infty$, $\gamma \to \infty$, and $\delta^2/\gamma \to \lambda_2$, we have seen that the random-jump process reduces to the Wiener-Levy process for the phase, and the spectral profile is given by the Lorentzian (5.41). In the same limit (5.45) gives

$$\langle\langle \tilde{\sigma}(s,v) \rangle\rangle = \frac{1}{v - L_0(x,s) - \lambda_L (L_+^2 + L_-^2)} P_g . \tag{5.48}$$

This shows that $\langle\langle \sigma \rangle\rangle$ obeys the evolution equation (4.13) with V given by

$$V\sigma = \tfrac{1}{2}\lambda_L L_e \sigma - \tfrac{1}{2}\lambda_L \frac{1}{k^2} \frac{\partial^2}{\partial x^2} \sigma , \tag{5.49}$$

with L_e given by (4.19). Equation (5.48) can also be obtained by taking the Gaussian limit in the independent-increment model for the phase. This reflects that the independent-increment model reduces to the Wiener-Levy process in the Gaussian limit.[17] The momentum distribution in this case is given by (4.31) where the operator V_n is now given by

$$V_n \sigma = \tfrac{1}{2}\lambda_L L_e \sigma + \tfrac{1}{2}\lambda_L n^2 \sigma . \tag{5.50}$$

The broadband limit is given by (4.34) where we have to replace λ_2 by λ_L, showing that also in this case the random-walk picture of Sec. III is justified.

D. Identical frequencies described by random-jump process

Finally we consider the random-jump process for the frequency with $\dot{\varphi}(t) = \dot{\psi}(t)$. We shall again consider the random-telegraph model, so that $\dot{\varphi}(t)$ can only take the values $\pm\delta$. In analogy to Sec. V B we introduce

$$\Phi_n(t) = \langle\langle \dot{\varphi}(t)^n \sigma(t) \rangle\rangle , \tag{5.51}$$

where n are non-negative integers. For the random telegraph only two independent operators Φ_n arise. The evolution equation can be obtained from (5.9) in a similar way as in Sec. V B. We can solve $\Phi_0(t) = \langle\langle \sigma(t) \rangle\rangle$ from these equations in Laplace transform. Substituting the result in (2.17) we obtain for the momentum distribution in Fourier transform

$$\tilde{Z}(s,v) = \frac{1}{\lambda} \int_0^\lambda dx \, \mathrm{Tr} \left[\frac{1}{v - L_0(x,s) - \delta^2 (L_+ + L_-) \frac{1}{v + \gamma - L_0(x,s)} (L_+ + L_-)} P_g \right] . \tag{5.52}$$

After a straightforward calculation we find for (5.52) in the case of zero detuning

$$\tilde{Z}(s,v) = \frac{1}{\pi} \int_0^\pi dx \frac{v + \Theta(x,s,v)}{v^2 + v\Theta(x,s,v) + \Omega^2 \sin^2 x \, \sin^2(\tfrac{1}{2}ks)} \tag{5.53}$$

with

$$\Theta(x,s,v)=\delta^2\frac{v+\gamma}{(v+\gamma)^2+\Omega^2\cos^2x\,\cos^2(\frac{1}{2}ks)}\ . \tag{5.54}$$

From this general result we can now recover several special cases. (i) In the monochromatic limit ($\delta\to0$) (5.53) gives the well-known result (4.38).[6] (ii) In the limit $\delta^2\to\infty$, $\gamma\to\infty$, and $\delta^2/\gamma\to\lambda_L$ (5.53) reduces to (4.37) with $\lambda_2=\lambda_L$. (iii) In the case $\gamma=0$ the spectral profile (5.40) reduces to

$$I_L(\omega)=P(\omega-\omega_L)\ . \tag{5.55}$$

The spectrum thus consists of two lines with frequencies $\omega_L\pm\delta$. From (5.53) we obtain ($\Delta=0$)

$$\tilde{Z}(s,v)=\frac{1}{\pi}\int_0^{\pi}dx\frac{v[v^2+\delta^2+\Omega^2\cos^2x\,\cos^2(\frac{1}{2}ks)]}{v^4+v^2\{\delta^2+\Omega^2[\cos^2x\,\cos^2(\frac{1}{2}ks)+\sin^2x\,\sin^2(\frac{1}{2}ks)]\}+\frac{1}{16}\Omega^4\sin^2(2x)\sin^2(ks)}\ . \tag{5.56}$$

The same result can be obtained by using a monochromatic wave which is detuned by δ from resonance.[13] This result is a reflection of the fact that the momentum distribution is independent of the sign of the detuning from resonance. In the particular limit $\delta\gg\Omega$, we obtain from (5.56) the well-known result[9]

$$Z(p,t)=\sum_{n=-\infty}^{\infty}J_n^2\left[\frac{\Omega^2t}{8\delta}\right]\delta(p-2n\hbar k)\ . \tag{5.57}$$

VI. CONCLUSIONS

We have derived explicit expressions for the momentum distribution of an atomic beam after diffraction at normal incidence by a standing light wave with phase fluctuations. These expressions describe the continuous transition from the fully coherent case of monochromatic light, causing Rabi oscillations, to the completely incoherent situation of broadband radiation inducing stimulated transitions between level populations.

We consider three distinct models for the phase fluctuations. In Sec. IV we treat the independent-increment model, where the phase is assumed to make jumps at random instants to another value with a probability distribution that depends only on the difference of the phases after and before the jump. When the phases of the two counter-running waves are independent, we obtain Eq. (4.31) as a formal result for the momemtum distribution in Fourier-Laplace transform. In the broadband limit this reduces to Eq. (4.34) or, equivalently, Eq. (3.6). When the two phases are identical, the general formal result is given in Eq. (4.37), which gives Eq. (4.39) and (4.40) in the broadband limit. Section V is devoted to random-jump models, where the phase or the frequency executes jumps at random instants with a probability distribution for the value after the jump that is independent of the value before the jump. As a special explicit case we treat the random-telegraph model, where the phase or the frequency can attain only two different values. For two independent phases, we arrive at the formal results (5.24) for phase jumps, and Eq. (5.47) for frequency jumps. A comparison of (4.31) and (5.24) demonstrates that the independent-increment model and the random-jump model for the phases give quite different results for the momentum distribution, even though the spectrum of the radiation field is the same in these two cases. Only in the broadband limit are these results identical. When the phases of the two counter-running waves are identical, we obtain Eqs. (5.34) and (5.53) for phase jumps and frequency jumps, respectively. Now a comparison of (5.34) and (4.37) shows that the increment model and the jump model for the phase produce the same momentum distribution. This suggests that for identical phases the momentum distribution is mainly determined by the spectrum of the radiation field, whereas for two independent phases higher correlation functions are also important. In general, these results indicate the importance of the detailed properties of laser fluctuations on the characteristics of the momentum distribution of an atomic beam diffracted by a standing light wave.

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