Scaling relation for a growing interface

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A scaling law is derived relating the static and the dynamic scaling exponents for the width of an interface growing through ballistic deposition, and is applied to recent numerical results of Meakin *et al.* [Phys. Rev. A **34**, 5091 (1986)].

Recently, Meakin *et al.*[1] have presented numerical results for the width ξ of the active zone in the single-step model of ballistic deposition. They adopt the scaling relationship

$$\xi \sim L^{\alpha} f(\bar{h}/L^{\gamma}) , \tag{1}$$

where L is the width of the substrate and $\bar{h}$ is the mean height of the deposit above some reference plane. Since particles are added at a constant rate, $\bar{h}$ is proportional to time. The authors give numerical estimates for the scaling exponents α and γ in two and three space dimensions. Furthermore, they propose the scaling relation

$$\alpha+\gamma=2 . \tag{2}$$

In this Comment we derive (2) from the continuum equation of Kardar *et al.*[2] for the interface height $h(\mathbf{x},t)$,

$$\frac{\partial h}{\partial t}=\nu\nabla^2 h+\frac{\lambda}{2}(\nabla h)^2+\eta(\mathbf{x},t) . \tag{3}$$

The Laplacian term on the right-hand side describes relaxation due to a surface tension ν. The nonlinearity arises from the growth process, λ being proportional to the net rate of deposition, and $\eta(\mathbf{x},t)$ being a stochastic force with zero mean and short-range correlations. Detailed arguments for the equivalence of (3) to the single-step model will be presented elsewhere.[3] Our main point here is that the scaling relation (2) seems to be the only available analytic result for a growing interface in three dimensions, and therefore deserves some attention. Also we note that the numerical values for α and γ in three dimensions given in Ref. 1, $\alpha=0.36$ and $\gamma=1.64$, satisfy (2) very well, which strongly supports the accuracy of these values. In contrast, the authors' conjecture that $\alpha=\frac{1}{3}$ and $\gamma=\frac{4}{3}$ is ruled out by the scaling relation.

For the continuum model (3) we study the intermediate scattering function $S(\mathbf{k},t)=\langle\hat{h}(\mathbf{k},t)\hat{h}(-\mathbf{k},0)\rangle$ in the steady state. The scaling form (1) requires the static correlations $S(\mathbf{k},0)$ to scale as $|\mathbf{k}|^{1-d-2\alpha}$ for small $|\mathbf{k}|$. Knowing this, we derive an equation for the full dynamical correlations in the mode coupling approximation,[3,4]

$$\frac{\partial}{\partial t}S(\mathbf{k},t)=-\nu\mathbf{k}^2 S+\frac{\lambda^2}{2}|\mathbf{k}|^{2\alpha+d-1}\int_0^t ds\, S(\mathbf{k},t-s)\int d^{d-1}\mathbf{q}\,[\mathbf{q}\cdot(\mathbf{k}-\mathbf{q})]^2 S(\mathbf{q},s)S(\mathbf{k}-\mathbf{q},s) . \tag{4}$$

We look for scale-invariant solutions,

$$|\mathbf{k}|^{2\alpha+d-1}S(\mathbf{k},t)=g(|\mathbf{k}|^{\gamma}t) , \tag{5}$$

for some scaling function g. We find the dynamic exponent to be determined by $\gamma=2-\alpha$ in accordance with (2). In the scaling limit $|\mathbf{k}|\to 0$, $t\to\infty$ such that $|\mathbf{k}|^{\gamma}t=\text{const}$, the diffusive term in (4) vanishes compared to the nonlinearity as $|\mathbf{k}|^{\alpha}$. In two dimensions the stationary state of (3) is known explicitly[5] and the static correlations are those of a rough equilibrium surface, i.e., $\alpha=\frac{1}{2}$ (Refs. 1 and 2). Therefore, the scaling relation pins down the dynamic exponent at the anomalous value $\gamma=\frac{3}{2}$ (Ref. 4). In three dimensions, however, no such additional information is available.

Finally we note that a rough argument for the scaling relation (2) can be obtained by mapping (3) to a time-dependent diffusion equation, as described in Ref. 2. Then $h(\mathbf{x},t)$ is expressed as the free energy of a directed polymer in d dimensions, subject to a quenched random potential $V(\mathbf{x},t)=(\lambda/2\nu)\eta(\mathbf{x},t)$. The energy of a specific path $\mathbf{x}(t)$ is thus

$$H\{\mathbf{x}(t)\}=\int_0^T dt\left[\frac{\dot{\mathbf{x}}(t)^2}{2}+V[\mathbf{x}(t),t]\right] . \tag{6}$$

On rescaling the d-1 space coordinates as $\mathbf{x}\to b\mathbf{x}$ and the "time" coordinate as $t\to b^{\gamma}t$, we see that H scales as $b^{2-\gamma}$. At low "temperature" $\lambda\to\infty$ we expect the free energy h to scale as the energy H thus arriving at $h\to b^{\alpha}h$ with $\alpha=2-\gamma$. For $d=2$ this is the argument given by Huse, Henley, and Fisher for the scaling of domain walls in a random-exchange Ising model.[5]

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