Time evolution of squeezing and antibunching in an optically bistable two-photon medium

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We use a numerical technique to study the time development of squeezing and antibunching resulting from the interaction of coherent light with a two-photon medium. While we find that enhanced squeezing occurs for short time, the squeezing is eventually revoked. We also find that antibunching develops but also is eventually revoked.

In light of the recent observations of squeezed electromagnetic radiation[1,2] there is much interest in finding systems which exhibit a significant amount of squeezing. Some time ago Tombesi and Yuen[3] considered the temporal development of squeezing resulting from the interaction of single-mode coherent light with an optically bistable two-photon medium. As the long-time ($t\rightarrow\infty$) behavior of the system is not experimentally relevant, they developed an approximate solution to the equations of motion in order to study the short-time, transient behavior. The solutions they found displayed enhanced squeezing over the short time. Of course, for much longer times damping effects would be expected to reduce the squeezing. Nevertheless, for a more complete assessment of the potential of the system to produce enhanced squeezing one ought to go beyond the short-time limit. In this paper we do just that by studying the time development directly using a numerical technique. We also study the time development of the other nonclassical effect—photon antibunching. We find that for longer times both squeezing and antibunching appear to be revoked.

The Hamiltonian for our system of interest may be written as

$$H=H_0+H_T+H_A\ , \tag{1}$$

where $H_0=\hbar\omega a^\dagger a$ is the free-field Hamiltonian,

$$H_T=\frac{\hbar}{2}[\Gamma^*(t)a^2+\Gamma(t)a^{\dagger 2}] \tag{2}$$

is the usual two-photon Hamiltonian,[4] and

$$H_A=\frac{\hbar}{2}Ka^{\dagger 2}a^2 \tag{3}$$

is an anharmonic term known to give rise to optical bistability.[5] The two-photon Hamiltonian H_T is, in fact, with a proper choice of $\Gamma(t)$, a prototype for the generation of squeezed states.[4,6] As in Ref. 3 we take $\Gamma(t)=-i\Gamma_0e^{-2i\omega t}$, where Γ_0 is real. On the other hand, the anharmonic term H_A has been shown by Tanas[7] also to produce squeezed light. Thus the combination as in Eq. (1) is expected to give enhanced squeezing, at least for some time intervals.

We work in the interaction picture so that

$$\begin{aligned} H_I&=e^{iH_0t/\hbar}(H_T+H_A)e^{-iH_0t/\hbar}\ ,\\ &=H_{TI}+H_{AI}\ , \end{aligned} \tag{4}$$

where, as can be easily shown,

$$H_{TI}=\frac{i\Gamma_0\hbar}{2}(a^2-a^{\dagger 2})\ , \tag{5a}$$

$$H_{AI}=H_A=\frac{\hbar}{2}Ka^{\dagger 2}a^2\ . \tag{5b}$$

In order to check our numerical computations to see the effect of the combined Hamiltonians of Eq. (4), we consider each part of Eq. (5) separately, as in both cases the dynamics is solvable. The evolution operator when $K=0$ is

$$U_T(t,0)=\exp\left[\frac{\Gamma_0 t}{2}(a^2-a^{\dagger 2})\right]\ , \tag{6}$$

so that

$$\begin{aligned} a(t)&=U_T^\dagger(t,0)a(0)U_T(t,0)\ ,\\ &=a\cosh(\Gamma_0 t)-a^\dagger\sinh(\Gamma_0 t)\ , \end{aligned} \tag{7}$$

where $a(0)=a$. Defining the quadrature operators $X_1=a+a^\dagger$ and $X_2=-i(a-a^\dagger)$ which satisfy

$$[X_1,X_2]=2i\ , \tag{8}$$

we obtain the uncertainty relations

$$(\Delta X_1)^2(\Delta X_2)^2\geq 1\ . \tag{9}$$

Squeezing exists if $(\Delta X_1)^2<1$ or $(\Delta X_2)^2<1$. Assuming an initial coherent state

$$|\alpha\rangle=e^{-N/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle\ , \tag{10}$$

where $N=|\alpha|^2$ is the average photon number, we obtain from Eq. (7)

$$(\Delta X_1)^2 = e^{-2\Gamma_0 t} ,$$
$$(\Delta X_2)^2 = e^{2\Gamma_0 t} . \qquad (11)$$

This is the standard result[4,8] showing squeezing in the X_1 quadrature. Henceforth we shall consider only this quadrature. Also for later purposes we evaluate the time zero correlation function

$$g^{(2)}(0) = \frac{\langle a^\dagger a^\dagger a a \rangle}{\langle a^\dagger a \rangle^2} . \qquad (12)$$

For α chosen as real we obtain

$$\begin{aligned}\langle a^\dagger a^\dagger a a \rangle = {} & \alpha^4[\cosh^4(\Gamma_0 t) - 4\cosh^3(\Gamma_0 t) + 6\cosh^2(\Gamma_0 t)\sinh^2(\Gamma_0 t) - 4\cosh(\Gamma_0 t)\sinh^3(\Gamma_0 t) \\ & + \sinh^4(\Gamma_0 t)] + \alpha^2[8\cosh^2(\Gamma_0 t)\sinh(\Gamma_0 t) - 10\cosh(\Gamma_0 t)\sinh^3(\Gamma_0 t) \\ & - 2\cosh^3(\Gamma_0 t)\sinh(\Gamma_0 t) + 4\sinh^4(\Gamma_0 t)] \\ & + \cosh^2(\Gamma_0 t)\sinh^2(\Gamma_0 t) + 2\sinh^4(\Gamma_0 t) ,\end{aligned} \qquad (13a)$$

$$\langle a^\dagger a \rangle^2 = \alpha^2[\cosh^2(\Gamma_0 t) - 2\cosh(\Gamma_0 t)\sinh(\Gamma_0 t) + \sinh^2(\Gamma_0 t)] + \sinh^2(\Gamma_0 t) . \qquad (13b)$$

Photon antibunching occurs if $g^{(2)}(0) < 1$ which is the case here for some $t > 0$ (see below).

For the anharmonic term alone ($\Gamma_0 = 0$) the evolution operator is

$$U_A(t,0) = \exp\left[-\frac{iKt}{2} a^{\dagger 2} a^2\right] , \qquad (14)$$

from which one obtains

$$\begin{aligned} a(t) &= U_A^\dagger(t,0) a(0) U_A(t,0) , \\ &= \exp(-iKt a^\dagger a) a . \end{aligned} \qquad (15)$$

The variance of X_1 for an initial coherent state is

$$\begin{aligned}(\Delta X_1)^2 = {} & 1 + 2N\{1 - \exp[2N(\cos(Kt) - 1)]\} \\ & + 2N \,\mathrm{Re}\{ e^{-iKt} \exp[N(e^{-2iKt} - 1)] \\ & - \exp[2N(e^{-iKt} - 1)]\} ,\end{aligned} \qquad (16)$$

$(\Delta X_1)^2 < 1$ for various values of N and t[7]. However, the statistics of the light are not altered by the anharmonic interaction so that $g^{(2)}(0) = 1$ for all times.

For the combined Hamiltonians the evolution operator is

$$U(t,0) = \exp\left[-\frac{iKt}{2} a^{\dagger 2} a^2 + \frac{\Gamma_0 t}{2}(a^2 - a^{\dagger 2})\right] . \qquad (17)$$

It is not possible to obtain a closed-form expression for $a(t)$. We proceed instead with a direct numerical technique. The expectation value of any function of a and $a^\dagger$, $F(a^\dagger, a)$, is

$$\begin{aligned}\langle F \rangle &= \langle \alpha | U^\dagger(t,0) F(a, a^\dagger) U(t,0) | \alpha \rangle , \\ &= \sum_{n', n''} \langle \alpha | U^\dagger(t,0) | n' \rangle \langle n' | F(a, a^\dagger) | n'' \rangle \\ &\quad \times \langle n'' | U(t,0) | \alpha \rangle ,\end{aligned} \qquad (18)$$

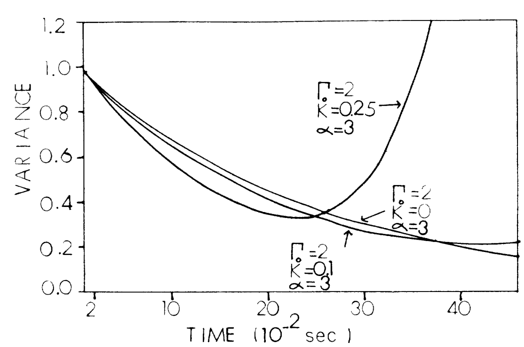

FIG. 1. Time development of the variance $(\Delta X_1)^2$ for $\alpha = 3$, $\Gamma_0 = 2$, and $K = 0$, 0.1, and 0.25.

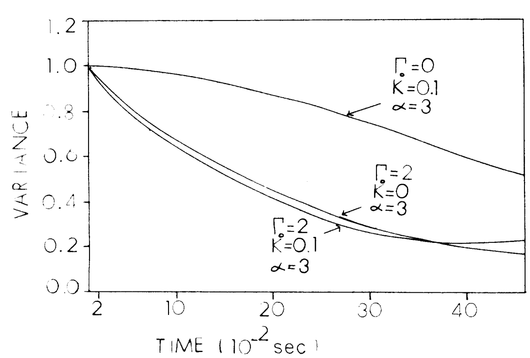

FIG. 2. Time development of the variance $(\Delta X_1)^2$ for the separate case $\Gamma_0 = 0$, $K = 0.1$; $\Gamma_0 = 2$, $K = 0$; and $\Gamma_0 = 2$, $K = 0.1$ and with $\alpha = 3$.

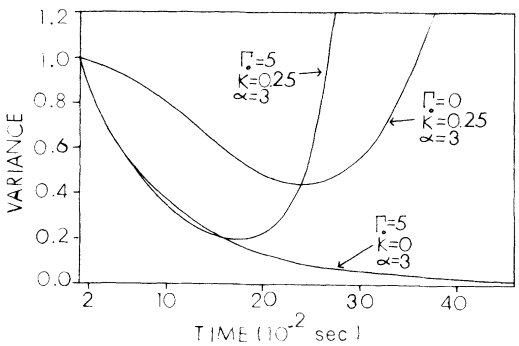

FIG. 3. Same as Fig. 2 but with $\Gamma_0=5$ and $K=0.25$.

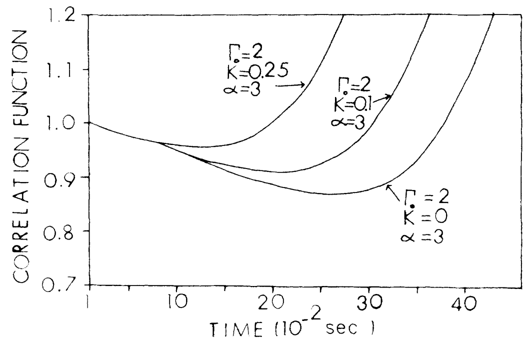

FIG. 4. Time development of the zero time correlation function $g^{(2)}(0)$ for $\Gamma_0=2, K=0$, 0.1, and 0.25 with $\alpha=3$.

which has been expanded in terms of the number states. The evolution operator $U(t,0)=\exp(-iH_I t/\hbar)$ is written via the Trotter product formula as

$$U(t,0)\simeq[U(\epsilon)]^M , \tag{19}$$

where $\epsilon=t/M$. The short-time evolution operator is approximated as[8]

$$U(\epsilon)=\exp(-i\epsilon H_I/\hbar) ,$$

$$\simeq\left[1+\frac{i}{2\hbar}\epsilon H_I\right]^{-1}\left[1-\frac{i}{2\hbar}\epsilon H_I\right] , \tag{20}$$

which has the advantage of being unitary.

In our calculations we retained 45 states in our expansions and set $\epsilon=0.01$ with the number of time steps $M=45$. As a check of the reliability of the method we considered first the separate cases of anharmonic oscillator and parametric amplifiers and compared with the exact results. We find good agreement for α up to about $\alpha=3$ with $K\le 0.25$ and $\Gamma_0\le 5.0$. For higher values of α and K, and longer time, more states are required, exceeding the memory capacity on the PRIME 750 on which our calculations were performed.

In Fig. 1 we show the time development of $(\Delta X_1)^2$ for $\alpha=3.0$ for $\Gamma_0=2$ and $K=0$, 0.1, and 0.25. As can plainly be seen, enhanced squeezing over the $K=0$ case does occur as in Ref. 3. However, we find that the behavior of the squeezing is very sensitive to the anharmonic parameter K. For low values of K a very small amount of enhancement is produced. For larger K greater squeezing is produced but for a shorter time, the squeezing being more rapidly revoked than before. In Fig. 2 we consider the case of the individual interactions $\Gamma_0=0$ and $K=0$ separately and the combination with $\Gamma_0=2$, $K=0.1$ with $\alpha=3$. We note that even while $(\Delta X_1)^2$ is descending for the separate cases, the combined interaction produces an ascending variance over the same time interval. This indicates that the overall behavior of the time development of the variance is not completely dominated by the anharmonic term but is the result of the interference of it and the two-photon nonconserving interaction term.

In the work of Ref. 3 the numerical results are characterized by a parameter $\delta=(K\alpha^2/\Gamma_0)$ and it is shown that enhanced squeezing can occur for $0<\delta<1.2$, where $\delta=0$ is just the case of $K=0$. In our calculations we take δ up to 1.13. However, our choices of Γ_0, K, and α are such that the enhanced squeezing we obtain occurs for shorter time intervals than in Ref. 3, allowing us to see the effect of the anharmonic term at longer time. Thus even though our maximum time is such that $\Gamma_0 t=0.90$ whereas $\Gamma_0 t=1.0$ in Ref. 3, we still see the long-time effects as our choices of Γ_0, K, and α effectively speed up the system. Presumably this would not be possible in the perturbative approach in Ref. 3. To go beyond the $\Gamma_0 t=1.0$ limit we take $\Gamma_0=5$ so that we get $\Gamma_0 t=2.25$. With $K=0.25$ and $\alpha=3$ as before, we see in Fig. 3 the evolution of the variance. Enhanced squeezing is again obtained but is rapidly revoked. (We have checked the exact results with the numerical results for $\Gamma_0=5$ and $K=0$ and found agreement to two decimal places.)

In Fig. 4 we give the results for the zero time correlation function $g^{(2)}(0)$ again for $\Gamma_0=2$, $K=0,0.1$, and $K=0.25$. We find that for $K=0$, $g^{(2)}(0)<1$ over most of the time interval considered, in agreement with the exact result determined from Eq. (13). For $K\neq 0$, we notice that as time goes on the antibunching effect appears to be revoked; the higher the K value the more rapidly it is revoked.

In conclusion, then, we have shown that for a general two-photon medium modeled by the Hamiltonian of Eq. (1) enhanced squeezing can occur but that squeezing in general is eventually revoked. The antibunching effect also occurs but it, too, is eventually revoked. It is possible, of course, that these effects may recur at much longer times but we are unable to study this because of the limitations of the available computing facilities.

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