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Interacting steady states in thermohaline convection

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We consider the Boussinesq theory for convection in a rectangular box with imposed constant, negative, vertical heat and salt gradients. We analyze the bifurcation of two-dimensional convection steady states near a double instability point defined by a critical value of the thermal Rayleigh number and geometrical aspect ratio. We find that for thermohaline convection the interaction of the two lowest pure-mode steady states does not produce the tertiary bifurcation of periodic solutions because the direction of bifurcation of the modes is always the same, either both supercritical or subcritical. On the other hand, we find that the interaction can produce the secondary bifurcation of mixed-mode steady states, and that jump transitions between multiple, stable, pure-mode steady states are possible. We also find that the mixed-mode steady state is always unstable in the parameter ranges considered.

Thermohaline convection is characterized by the diffusion of two scalar fields: temperature and salt. In ordinary convection the salt is absent so that only the temperature diffuses. The presence of the diffusing salt may substantially alter the qualitative features of the ordinary convection states. For example, for two-dimensional convection in a rectangular box that is heated from below, has "slippery" walls, and whose side walls are insulated, only steady convection states branch from the conduction state as the Rayleigh number varies. Furthermore, these states always branch supercritically. We refer to those bifurcation points of the conduction state as the primary bifurcation points. However, for thermohaline convection, steady states can bifurcate both supercritically and subcritically from the conduction state, depending on the salt Rayleigh number R_S. In addition, subcritical and supercritical bifurcation of time-periodic states from the conduction state can occur for thermohaline convection, but not for ordinary convection.

To study the secondary and cascading bifurcation of solutions, it is customary (see, e.g., Refs. 1–3) to investigate the solutions when two or more primary bifurcation points can coalesce as the system parameters of the problem are varied. By using asymptotic or other methods, it has been shown that, to leading order, the solutions of the bifurcation problem can be expressed as a linear combination of the interacting modes corresponding to the coalescing primary bifurcation points. The amplitudes of these modes, which are time dependent, satisfy a system of nonlinear, ordinary differential equations, called the amplitude equations. Such bifurcation studies for two interacting steady modes for ordinary thermal convection have been presented in Refs. 4–8. These studies obtain new steady convection states which are secondary bifurcations from the primary convection states.

An analysis of a general system of amplitude equations corresponding to the interaction of the steady modes of the *two* lowest primary bifurcation points shows[3] that tertiary bifurcation of time-periodic states occurs only if the bifurcation from the lowest and next lowest primary bifurcation point are supercritical and subcritical, respectively, and if the coefficients in the amplitude equations satisfy certain conditions. Thus, tertiary bifurcation of time-periodic states arising from the interaction of two steady modes cannot occur for ordinary thermal convection because the primary bifurcation is always supercritical. The purpose of this Brief Report is to show that the tertiary bifurcation of time-periodic convection states does not occur for thermohaline convection because the branching from the two lowest primary bifurcation points are in the same direction, i.e., are supercritical or subcritical.

The analysis in Ref. 3 of a general system of amplitude equations corresponding to the interaction of the steady-state modes of the *three* lowest primary bifurcation points shows that it is possible for quarternary bifurcation of time-periodic states to occur when the bifurcations from all three primary bifurcation points are supercritical (as in ordinary convection) and provided that the coefficients in the amplitude equations satisfy special inequalities. Studies of two and of many interacting steady-state modes in Marangoni convection[9] and ordinary thermal convection,[10] respectively, have shown that the nonlinear interaction may give time-periodic convection states. The interaction of a steady and a time-periodic primary convection state also has been previously studied for double-diffusive convection (see, e.g., Ref. 11 and references given therein).

The convection box, which is of depth πd and x-width πL, is heated and salted from below. The solution of the

Boussinesq theory for two-dimensional thermohaline convection depends on the five system parameters: $\delta \equiv d/L$, R_T, R_S, σ, and D. They are, respectively, the aspect ratio of the box, the thermal Rayleigh number, the salt Rayleigh number, the Prandtl number and the Schmidt number; see Ref. 11 for the definitions of these quantities. Since heat diffuses faster than salt, $D<1$. We have assumed that the Dufour and Sorret effects are negligible.

The bifurcation points of two-dimensional steady convection states on the conduction state are given by

$$R_T = R_T^S \equiv \frac{R_S}{D} + \frac{(\delta^2 m^2 + s^2)^3}{\delta^2 m^2}\ , \tag{1}$$

provided that $R_S < D^2(1+\sigma)/(1-D)\sigma$. Here m and s are the wave numbers in the x and z (depth) directions, respectively, of the modes corresponding to (1). Thus, for $R_T > R_T^S$ the conduction state is unstable to a steady mode disturbance with wave numbers (m,s). The minimum on R_T^S over m and s, denoted by R_c, determines the mode which first becomes unstable as R_T is increased. This minimum depends on δ and is always achieved for $s=1$.

Two steady modes $(M,1)$ and $(N,1)$, where $M \neq N$, may become unstable simultaneously at R_c for a critical value $\delta = \delta_c$. To obtain asymptotic solutions of the nonlinear, Boussinesq initial-boundary-value problem for δ near δ_c we define a small parameter ϵ by the relation $\delta = \delta_c(1 \mp \epsilon^2)$ and a slow time τ by $\tau = \epsilon^2 t$. We use a multitime method, as in Ref. 11, to obtain these asymp-

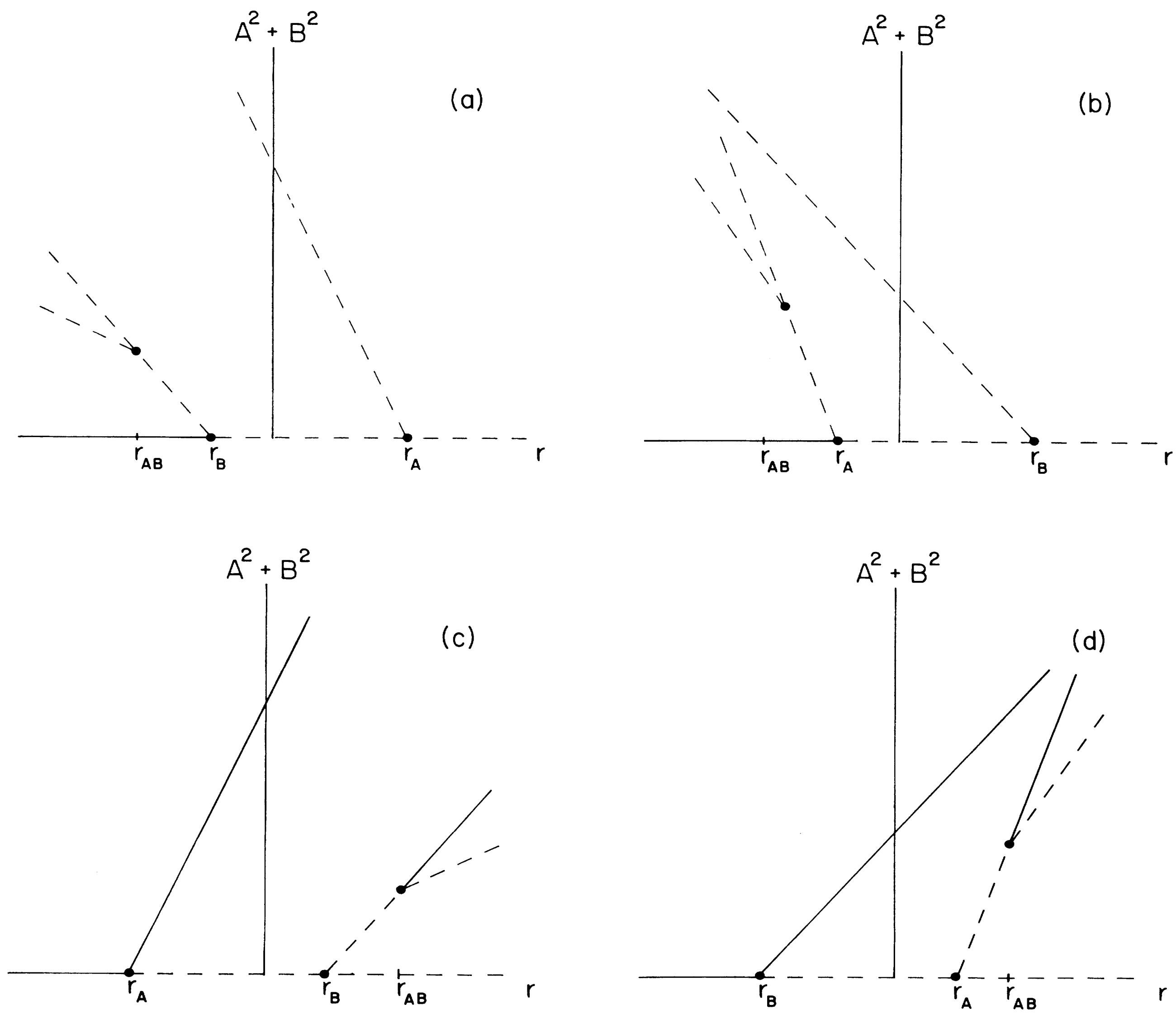

FIG. 1. Schematic bifurcation diagrams (A^2+B^2 vs r) for the steady-state solution branches of (3). (a) $R_S > D^3 R_c$ and $\delta=\delta_c(1-\epsilon^2)$, (b) $R_S > D^3 R_c$ and $\delta=\delta_c(1+\epsilon^2)$, (c) $R_S < D^3 R_c$ and $\delta=\delta_c(1-\epsilon^2)$, (d) $R_S < D^3 R_c$ and $\delta=\delta_c(1+\epsilon^2)$. The pure A states ($A \neq 0, B=0$), pure B states ($B \neq 0$, $A=0$), and mixed states ($A \neq 0, B \neq 0$) bifurcate at r_A, r_B, and r_{AB}, respectively. Dashed lines indicate linearly unstable states. The r axis corresponds to the conduction state.

totic expansions, where R_T is expanded near R_c as

$$R_T = R_c + r\epsilon^2 + O(\epsilon^3) \ . \tag{2}$$

The resulting equations for the amplitudes $A(\tau)$ and $B(\tau)$ of the interacting steady modes $(M,1)$ and $(N,1)$ are given, respectively, by

$$\dot{A} = A[a_1(r) + a_2 A^2 + a_3 B^2] \ , \tag{3a}$$

$$\dot{B} = B[b_1(r) + b_2 B^2 + b_3 A^2] \ . \tag{3b}$$

In (3), r is a free parameter which determines the deviation of R_T from R_c [cf. (2)], and the overdots denote derivatives with respect to τ. For brevity we do not list the coefficients a_j and b_j for $j=1$, 2, and 3, nor the initial conditions for (3). The bifurcation of steady convection states for δ near δ_c and R_T near R_c is thus determined by the time-independent solutions (A,B) of (3), which depend on r. These solutions are either "pure" or "mixed" steady states. The pure states are given by $A=0, B^2=-b_1(r)/b_2$ and $B=0, A^2=-a_1(r)/a_2$.

The mixed steady states are given by

$$A^2 = [a_3 b_1(r) - b_2 a_1(r)]/(a_2 b_2 - a_3 b_3) \ , \tag{4a}$$

$$B^2 = [b_3 a_1(r) - a_2 b_1(r)]/(a_2 b_2 - a_3 b_3) \ . \tag{4b}$$

The bifurcation diagrams for these steady states are plotted in Fig. 1 for both $R_S < D^3 R_c$ and $R_S > D^3 R_c$ with $\delta = \delta_c(1-\epsilon^2)$ and $\delta = \delta_c(1+\epsilon^2)$. The value $R_S = D^3 R_c$ is where the pure states change their direction of bifurcation.[11] In the figure, each point on a bifurcation branch represents the two solutions with reflection symmetry, i.e., (A,B) and $(-A,-B)$. The linearized stability of the conduction steady states is also shown.

We observe in the figure that the pure A and B states, which bifurcate from r_A and r_B, respectively, have the same direction of bifurcation, i.e., they either both bifurcate supercritically or subcritically. Thus, there can be no bifurcation of time-periodic states on the mixed steady-state branches.[3] The secondary bifurcation point r_{AB} may lie on either the pure A or B state, depending on whether $\delta < \delta_c$ or $\delta > \delta_c$. The steady states are all unstable for $R_S > D^3 R_c$, as shown in Fig. 1. Thus, if the initial conditions for (3) lie near these states then the system must jump to a large amplitude state not described by our local analysis. In Figs. 1(c) and 1(d), which are for $R_S < D^3 R_c$, the first pure state to bifurcate from the conduction state is stable but the second is unstable. However, the second pure state becomes stable for $r > r_{AB}$. This stable part of the branch is not experimentally accessible from the conduction state, i.e., by slowly increasing R_T from R_c. It might be reached for $r > R_{AB}$, however, by either applying a perturbation to the system or by choosing initial conditions which are sufficiently close to this state. Thus, jump transitions between multiple, stable, pure-mode steady states can be produced by perturbations, and multiple steady states of the system can be observed by varying the initial conditions. In all four cases considered the mixed states (4) are unstable. In summary, we find that tertiary bifurcation of time-periodic states resulting from the interaction of two primary steady states of convection is not possible. By contrast, in rotating thermal convection, where the pure steady states can bifurcate with opposite directions, their interaction can result in tertiary bifurcation periodic convection states.[12,13]

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