Velocity selection at large undercooling in a two-dimensional nonlocal model of solidification

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In this paper we consider the two-dimensional symmetric model of dendritic solidification with capillary anisotropy in the limit of large undercooling. We show that capillary anisotropy is necessary to obtain steady-state solutions, and that it allows the system to select a particular value of the growth velocity. The results obtained are in complete agreement with what has been derived in the corresponding limit in the case of the boudary-layer model.

I. INTRODUCTION

Much progress has been made in very recent years towards a satisfactory understanding of the physical mechanisms which control the formation of needle-crystal dendrites of a solid from its undercooled melt. In particular, it has become clear[1-7] that surface tension can play the role of a singular perturbation and that, mathematically, this leads to the existence of a nontrivial solvability condition which must be satisfied for a needle-crystal solution growing at constant velocity V to exist. It is the analysis of such a solvability condition which gives a way to select, out of a continuous family of steady-state solutions, the one presumably corresponding to the actually observed needlelike dendrites. This analysis, explicitly carried out analytically on the geometrical model[3] and the boundary-layer model (BLM) in the limits of small and large undercooling,[8] has suggested that crystalline anisotropy plays an essential role in the selection mechanism. This has been confirmed by the study[2,7] of the fully nonlocal model in the limit of small Péclet number p and also, in a more indirect way, in the case $p\rightarrow\infty$.[1]

In this paper we apply the method developed in Ref. 1 to study the symmetric version of the two-dimensional nonlocal model of solidification[9] with finite anisotropy, in the limit of large undercooling. The existence of a singular perturbation parameter (in our case $\epsilon=1-\Delta$ where Δ is the dimensionless undercooling) and the related solvability condition have been exploited to show how a nonzero anisotropy is required in order for needle-crystal solutions to exist.

The analysis shows that the solvability mechanism is operating in the limit of $p\rightarrow\infty$ in much the same way as in the opposite limit of small undercooling, and therefore confirms the mathematical scenario for velocity selection that has been proposed in the past two years. The results obtained and the explicit form of the dependence of the selected velocity on both ϵ and α (the anisotropy strength) agree completely with what has been derived in the corresponding limit in the case of the BLM.[8] This is what one should expect because it is precisely in the limit of large undercooling, when the range of the diffusion field is much smaller than the typical length scale of the solidification front, that the BLM is supposed to give an accurate description of the growing interface.

II. THE SYMMETRIC MODEL IN THE LIMIT OF LARGE UNDERCOOLING

The starting point is the integro-differential equation for the solidification front $\zeta(x)$ in the two-dimensional symmetric model:[9]

$$\Delta-\frac{d_0}{\rho}\kappa=\frac{p}{2\pi}\int_0^{+\infty}\frac{dt}{t}\int_{-\infty}^{+\infty}dx'\exp\left[-\frac{p}{2t}\{x'^2+[\zeta(x'+x)-\zeta(x)-t]^2\}\right]. \tag{2.1}$$

Crystalline anisotropy enters (2.1) via d_0 which is assumed of the form $d_0=\bar{d}_0A(\theta)$ where $\eta\equiv d\zeta/dx=\tan\theta$. We will consider the case of fourfold symmetry and, to be specific, we will take

$$A(\theta)=1-\alpha\cos4\theta=1-\alpha+\frac{8\alpha\eta^2}{(1+\eta^2)^2}, \tag{2.2}$$

where α represents the strength of the anisotropy.

In (2.1) $\Delta=(T_M-T_\infty)c/L$ is the dimensionless undercooling, $\bar{d}_0=\gamma T_Mc/L^2$ is the capillary length associated with the surface tension γ, $p=\rho V/2D$ is the Péclet number, V is the constant velocity of the moving interface in the laboratory frame, and, finally, ρ is a length scale associated with the curvature $\kappa(x)$ at the tip of the needle crystal ($x=0$) whose precise definition will be given later. Equation (2.1) is valid in the frame moving with velocity V, where the interface $\zeta(x)$ is at rest; lengths and time units are, respectively, ρ and ρ/V.

One of the goals of the analysis presented in this paper is to determine the values of V for which such a steady solution actually exists. We will see shortly that, in the limit $p \to \infty$ ($\Delta \to 1$), there exists a continuous family of solutions, analogous to the Ivantsov parabolas[10] in the case $\bar{d}_0 \to 0$. However, if one starts including corrections in the small parameter $\epsilon = 1 - \Delta$, the family is completely destroyed unless $\alpha > 0$. In this case only a discrete set of solutions survives, thus allowing us to select what is probably the physical mode of growth.

The first step is to reduce (2.1), in the limit $p \to \infty$, to a linear, inhomogeneous differential equation of infinite order. We only sketch the derivation which is given in detail in Ref. 1. The right-hand side of (2.1) can be evaluated in the limit $p \to \infty$ by expanding the function

$$g(t,x') = \frac{1}{2t}\{x'^2 + [\zeta(x+x') - \zeta(x) - t]^2\}$$

about its minimum at $t \to 0^+$, $x' = 0$:

$$\exp[-pg(t,x')] = \exp\left[-\frac{p}{2t}[x'^2 + (\eta x' - t)^2]\right] \times \exp\left[-\frac{p}{t}(\eta x' - t)\sum_{n=2}^{\infty}\frac{\eta^{(n-1)}}{n!}x'^n - \frac{p}{2t}\left[\sum_{n=2}^{\infty}\frac{\eta^{(n-1)}}{n!}x'^n\right]^2\right].$$

Here $\eta^{(n)} \equiv d^n\eta/dx^n = d^{n+1}\zeta/dx^{n+1}$.

Now we expand the second exponential on the right-hand side and, following Ref. 1, retain all the terms linear in $\eta^{(n-1)}$ and $\eta^{(1)}\eta^{(n-1)}$ ($n \geq 2$) because they will turn out to give the dominant contribution to the equation after the linearization around the solution at $p \to \infty$ is performed. In this way we get

$$\Delta - \frac{d_0}{\rho}\kappa(x) = 1 + \frac{(\eta^{(1)})^2}{2p^2}b_2 + \sum_{n=2}^{\infty}\frac{\eta^{(n-1)}}{p^{n-1}}a_n + \sum_{n=3}^{\infty}\frac{\eta^{(1)}\eta^{(n-1)}}{p^n}b_n + \cdots , \tag{2.3}$$

where

$$a_n(\eta) = \int_{-\infty}^{+\infty}\frac{dz}{\pi}2^n\frac{z(z+\eta)^{n-1}}{(z^2+1)^{n+1}} ,$$

$$b_n(\eta) = \int_{-\infty}^{+\infty}\frac{dz}{\pi}2^n(n+1)\frac{(z+\eta)^n}{(z^2+1)^{n+2}} \times \left[2(n+2)\frac{z^2}{1+z^2} - 1\right] .$$

If we recall that $d_0/\rho = A(\eta)\nu/p$ with $\nu = \bar{d}_0 V/2D$, $\kappa(x) = -\eta^{(1)}/(1+\eta^2)^{3/2}$, and write $\Delta = 1 - \epsilon$, then (2.3) can be written as

$$-p\epsilon + \nu f(\eta)\eta^{(1)} = \frac{(\eta^{(1)})^2}{2p}b_2 + \sum_{n=2}^{\infty}\frac{\eta^{(n-1)}}{p^{n-2}}a_n + \sum_{n=3}^{\infty}\frac{\eta^{(1)}\eta^{(n-1)}}{p^{n-1}}b_n + \cdots , \tag{2.4}$$

where $f(\eta) \equiv A(\eta)/(1+\eta^2)^{3/2}$.

Notice that, if we write

$$\eta^{(n+1)} = \left[\eta^{(1)}\frac{d}{d\eta}\right]^n \eta^{(1)} , \tag{2.5}$$

(2.4) becomes an equation for $\eta^{(1)}(\eta)$ and that this quantity is closely related to the curvature of the interface.

Equation (2.4) can be solved in the limit $p \to \infty$, for it reduces to

$$\eta_0^{(1)} = -\frac{2p\epsilon}{1 - 2\nu f(\eta)} , \tag{2.6}$$

where we have used the result $a_2 = \frac{1}{2}$. For any fixed value of V and hence of ν, we have a solution at large p provided that $p\epsilon = \text{const}$. This arbitrariness in choosing the constant simply reflects the arbitrariness of the length scale ρ that we are free to choose. Our choice will be such that $2p\epsilon = 1$, i.e., $\rho = D/V\epsilon$; in other words, given a certain velocity V, ρ is closely related to the tip radius of the unperturbed solution because, in these units, $\rho_0(\text{tip}) = -[\eta_0^{(1)}(\eta = 0)]^{-1} = 1 - 2\nu$. Notice that the solution (2.6) only makes sense in the case $\nu(1-\alpha) < \frac{1}{2}$ which, however, is the only one of physical interest because, usually, $\nu = \bar{d}_0 V/2D \ll 1$. Apart from the use of a different length scale, (2.6) precisely corresponds to the modified-Ivantsov parabola which is a solution of the BLM in the limit $\epsilon \to 0$.[8]

Now we linearize (2.4) around the modified-Ivantsov solution in this way departing from the analysis of Ref. 1 where linearization was performed around the zero surface tension solution obtained by setting $\nu = 0$ in (2.6). In this way we will be able to investigate the limit $\epsilon \to 0$, ν fixed, which will turn out to be physically relevant for, as we shall see, the value of ν (and hence of V) selected by the crystalline anisotropy remains finite when $\epsilon \to 0$. It is perhaps worth stressing that this limiting procedure is quite different from the one used in Ref. 1 where, instead, the case ϵ fixed, $\nu \to 0$, was investigated.

We write $\eta^{(1)} = \eta_0^{(1)} - h$, use (2.5), and keep only terms at most linear in h. The resulting equation for $h(\eta)$ reads

$$(L_0 - 2\epsilon L_1 - \nu f)h = \tfrac{3}{2}\epsilon(\eta_0^{(1)})^2 , \tag{2.7}$$

where

$$L_0 = \sum_{n=0}^{\infty} A_n\left[-2\epsilon\frac{d}{d\eta}\right]^n , \tag{2.8}$$

$$L_1 = \sum_{n=0}^{\infty} B_n\left[-2\epsilon\frac{d}{d\eta}\right]^n , \tag{2.9}$$

$$A_n(\eta) = (-\eta_0^{(1)})^n a_{n+2} , \tag{2.10}$$

$$B_n(\eta) = -\left[\frac{n(n+1)}{2} + 1\right](-\eta_0^{(1)})^n\frac{d\eta_0^{(1)}}{d\eta}a_{n+3} + (-\eta_0^{(1)})^{n+1}b_{n+2} , \tag{2.11}$$

and we have used the result $b_2 = \frac{3}{2}$.

III. WKB APPROXIMATION IN THE LIMIT $\epsilon \to 0$

In order to solve (2.7) we now proceed in a way which is the direct generalization of the WKB approximation to solve a second-order differential equation of the same structure. The method used is the same as in Ref. 1. First we consider the homogeneous equation and, in the limit of small ϵ, write the homogeneous solution $H(\eta)$ in the form

$$H=\exp\left[\frac{S(\eta,\epsilon)}{\epsilon}\right]=\exp\left[\frac{1}{\epsilon}(S_0+\epsilon S_1+\cdots)\right] ;$$

in this way we obtain the following expressions for S_0' and S_1' (in what follows the prime superscript will always refer to differentiation with respec to η):

$$L_0(S_0',\eta)=\sum_{n=0}^{\infty} A_n(-2S_0')^n=\nu f(\eta) , \tag{3.1}$$

$$S_1'\frac{\partial L_0}{\partial S_0'}+\frac{S_0''}{2}\frac{\partial^2 L_0}{\partial S_0'^2}-2L_1=0 , \tag{3.2}$$

$$L_1(S_0',\eta)=\sum_{n=0}^{\infty} B_n(-2S_0')^n . \tag{3.3}$$

By using (2.11) we explicitly get

$$L_0(X,\eta)=\frac{1}{R+(1+X\eta)R^{1/2}}=\nu f(\eta) , \tag{3.4}$$

where $R=1+2X\eta-X^2$, $X=-2S_0'\eta_0^{(1)}$, and $R^{1/2}$ is defined so that $\mathrm{Re}(R^{1/2})>0$ when $\mathrm{Re}R>0$. The expression for L_1 is

$$L_1=-\tfrac{3}{2}\eta_0^{(1)}R^{-5/2}-\frac{2\nu}{X}(\eta_0^{(1)})^2\frac{df}{d\eta}[L_0(X,\eta)-\tfrac{1}{2}]$$
$$-X\nu(\eta_0^{(1)})^2\frac{df}{d\eta}\frac{\partial^2 L_0}{\partial X^2} . \tag{3.5}$$

Equation (3.4) could be transformed into a quartic equation for X; however, the requirement about the determination of $R^{1/2}$ selects only two acceptable roots, which correspond to the two independent solutions of the homogeneous equation.

It is easy to check that, if $X_+(\eta)$ is a solution, $X_-(\eta)\equiv X_+^*(\eta^*)$ is also a solution corresponding to the correct determination of $R^{1/2}$. [Here we are analytically extending $X_\pm(\eta)$ into the complex η plane.] Furthermore, an analysis of (3.4) in a neighborhood of $\eta=0$ shows that, if $\nu(1-\alpha)<\frac{1}{2}$, these two solutions are indeed distinct. In the same way, $X(\eta)\equiv -X_+(-\eta)$ is also a solution which must therefore be identified with either X_+ or X_-. The possibility $X_\pm(\eta)=-X_\pm(-\eta)$, which implies $X_\pm(0)=0$, is ruled out by an explicit check of (3.4) at $\eta=0$, and hence we conclude that

$$X_-(\eta)=X_+^*(\eta^*)=-X_+(-\eta) . \tag{3.6}$$

By using the last result we can see that both the solutions behave in a physically unacceptable way as $|\eta|\to\infty$ along the real η axis. Indeed, if we write $X=\eta+\psi(1+\eta^2)^{1/2}$ in (3.4), we can check that, in the limit $|\eta|\to\infty$, ψ is a constant determined by

$$\frac{1}{[1+(\eta/|\eta|)\psi](1-\psi^2)^{1/2}}=\nu(1-\alpha) ,$$
$$\mathrm{Re}(1-\psi^2)^{1/2}>0 . \tag{3.7}$$

But $S_{0\pm}'(\eta)\sim\frac{1}{2}\eta(1+\psi_\pm)$ as $\eta\to+\infty$, and $\psi_+=\psi_-^*$; hence the condition $\mathrm{Re}(1-\psi^2)^{1/2}>0$ implies

$$\mathrm{Re}\frac{1}{1+\psi_\pm}=\frac{1}{1+|\psi|^2}\mathrm{Re}(1+\psi_\mp)>0 .$$

Therefore, $H_\pm(\eta)\equiv\exp[(1/\epsilon)\int_0^\eta S_\pm'(\tilde\eta)d\tilde\eta]$ diverges exponentially as $\eta\to+\infty$. By using (3.6) we see that the same is true if $\eta\to-\infty$; we can also check that the presence of $S_{1\pm}$ does not change this situation, for the dominant contribution as $|\eta|\to\infty$ is given by $S_{0\pm}$.

Now that we have two independent solutions of the homogeneous equation we can obtain the particular solution of (2.6) which is regular at $\eta\to-\infty$. The result is

$$h_P(\eta)=i\int_{-\infty}^{\eta}d\tilde\eta\,\phi(\tilde\eta)F(\tilde\eta)\left[\frac{H_+(\tilde\eta)}{H_+(\eta)}-\frac{H_-(\tilde\eta)}{H_-(\eta)}\right] , \tag{3.8}$$

where $\phi(\eta)=\frac{3}{2}\epsilon(\eta_0^{(1)})^2+o(\epsilon)$ and

$$F(\eta)=\frac{i}{\epsilon}\frac{X_+X_-}{X_--X_+}=\frac{i}{\epsilon}\eta_0^{(1)}(\eta)\frac{S_{0+}'S_{0-}'}{S_{0+}'-S_{0-}'} . \tag{3.9}$$

The solvability condition is readily obtained by imposing that $h_P(\eta)$ be well behaved at $\eta\to+\infty$:

$$I(\nu,\epsilon,\alpha)\equiv\int_{-\infty}^{+\infty}d\eta\frac{\phi(\eta)F(\eta)}{H_+(\eta)}$$
$$=\int_{-\infty}^{+\infty}d\eta\frac{\phi(\eta)F(\eta)}{H_-(\eta)}=0 . \tag{3.10}$$

The first equality and the fact that $I(\nu,\epsilon,\alpha)$ is real follow from the symmetry properties of $F(\eta)$, $S_{0\pm}$, and $S_{1\pm}$ which are consequences of (3.1)–(3.6) and the definition $S(\eta)=\int_0^\eta S'(\tilde\eta)d\tilde\eta$:

$$\begin{aligned} &F(\eta)=F(-\eta) ,\\ &F^*(\eta)=F(\eta^*) ,\\ &S_{0,1\pm}(\eta)=S_{0,1\mp}(-\eta) ,\\ &S_{0,1\pm}^*(\eta)=S_{0,1\pm}(-\eta^*) . \end{aligned} \tag{3.11}$$

The analogous properties of ϕ are $\phi(\eta)=\phi(-\eta)$ and $\phi^*(\eta)=\phi(\eta^*)$; the possible interpretations of $I(\nu,\epsilon,\alpha)$ are discussed in Ref. 1.

IV. SADDLE-POINT EVALUATION OF I

Because we are interested in the limit $\epsilon\to0$, we can evaluate

$$I(\nu,\epsilon,\alpha)=\int_{-\infty}^{+\infty}d\eta\,\phi(\eta)F(\eta)\times\exp\left[-S_{1+}(\eta)-\frac{1}{\epsilon}S_{0+}(\eta)\right]$$

by using a saddle-point approximation. Therefore, we need to find the stationary points of $S_{0+}(\eta)$ in the complex η plane, i.e., the points where $S'_{0+}(\eta)$ vanishes.

If we look back at (3.4) we immediately see that $X=0$ implies $\nu f(\eta)=\frac{1}{2}$. As one could have guessed by the analysis of the BLM, the saddle points are precisely the points η_s in the complex plane where $\eta_0^{(1)}(\eta)$ diverges. Note that we only need the behavior of the relevant functions in a neighborhood of these points.

It is worth noticing that $S'_0=0$ does not necessarily imply $X=0$ if $\eta_0^{(1)}(\eta)\to\infty$ in the limit $\eta\to\eta_s$. As we shall see explicitly by expanding X in a neighborhood of $\eta=\eta_s$, we are simply defining $S'_{0+}(\eta)$ to be the solution of (3.4) such that $\lim_{\eta\to\eta_s}\eta_0^{(1)}(\eta)S'_{0+}(\eta)=0$ for a particular η_s. Once this η_s has been chosen, then $\lim_{\eta\to\eta_s}\eta_0^{(1)}(\eta)S'_{0-}(\eta)\neq 0$ and the behavior of S_{0+} at the other saddle points of interest will be determined by (3.6).

At this point it should be clear that the mechanism for selecting the value ν^* when $\alpha>0$ is the same one as in the $\epsilon\to 0$ limit of the BLM.[8] Indeed, the functions which determine the position of the saddle points are the same in the two cases.

It is easy to obtain the leading behavior of S_{0+} in a neighborhood of η_s; if $z=\eta-\eta_s$, we get

$$X_+(z)=-\frac{\nu}{\eta_s}\left.\frac{df}{d\eta}\right|_{\eta_s} z+o(z)\ , \tag{4.1}$$

$$S_{0+}(\eta)=\int_0^{\eta_s} S'_{0+}(\eta)d\eta+\frac{\nu^2}{3\eta_s}\left[\left.\frac{df}{d\eta}\right|_{\eta_s}\right]^2 z^3+o(z^3)\ . \tag{4.2}$$

Notice that we find only one solution of (3.4) corresponding to $\eta=\eta_s$ and $X=0$ because $X_-(\eta_s)\neq 0$; this last result can be used to compute

$$\phi(\eta)F(\eta)=-i\frac{3}{8}\left[\nu\eta_s\left.\frac{df}{d\eta}\right|_{\eta_s} z\right]^{-1}+O(z^0)\ . \tag{4.3}$$

Finally, by considering (3.2) and (3.5), we obtain a similar contribution from $\exp[-S_1(\eta)]$. Indeed, $S'_{1+}\sim 1/z$ and hence, in a neighborhood of η_s,

$$\exp[-S_1(\eta)]=\frac{C(\nu,\alpha)}{z}+O(z^0)\ , \tag{4.4}$$

where C is a constant whose exact dependence on ν and α is not important for our purposes.

The final result for $I(\nu,\epsilon,\alpha)$ in the limit $\epsilon\to 0$ is

$$I(\nu,\epsilon,\alpha)=-i\sum_{\text{relevant }\eta_s}\exp\left[-\frac{1}{\epsilon}S_{0+}(\eta_s,\nu,\alpha)\right]\times\int_{\Gamma_{\eta_s}} dz\frac{\rho(\eta_s)}{z^2}\exp\left[-\frac{1}{\epsilon}\gamma z^3\right]\ , \tag{4.5}$$

where

$$\rho=\frac{3}{8}\left[\nu\eta_s\left.\frac{df}{d\eta}\right|_{\eta_s}\right]^{-1}C(\nu,\alpha)\ ,$$

$$\gamma(\eta_s,\nu,\alpha)=\frac{\nu^2}{3\eta_s}\left[\left.\frac{df}{d\eta}\right|_{\eta_s}\right]^2 .$$

Γ_{η_s} is the path of steepest descent across η_s, running from $\eta=-\infty$ to $\eta=+\infty$. The reason for the sum will be clear in a second after discussing the position of the saddle points in the $\alpha=0$ and $\alpha>0$ cases.

Notice the similarity with the BLM in the limit $\epsilon\to 0$ where the term $\gamma_s(\eta_s,\nu,\alpha)z^3$ is exactly the same—except for a numerical factor—as the one obtained here.[8] We are now in the position to discuss the two cases, $\alpha=0$ and $\alpha>0$, in more detail.

In the case of zero anisotropy the saddle points are determined by the equation $1=2\nu f(\eta,\alpha=0)=2\nu(1+\eta^2)^{-3/2}$; this gives us, if $\nu<\frac{1}{2}$, two purely imaginary, complex-conjugate stationary points. If we take the path of steepest descent to lie in the upper half plane, the relevant η_s is $\eta_s=i[1-(2\nu)^{3/2}]^{1/2}$ so that we get

$$I(\epsilon,\nu,\alpha=0)=g(\nu)\epsilon^{-1/3}\exp\left[-\frac{1}{\epsilon}S_{0+}(\eta_s,\nu,\alpha=0)\right]. \tag{4.6}$$

It is easy to show, because $iS'_{0+}(\eta)$ is real on the imaginary axis and its only possible zeros are at $\pm\eta_s$, that $S_{0+}(\eta_s)>0$. The explicit form of $g(\nu)$ is not particularly important except it shows that $g(\nu)\neq 0$ if $0<\nu<\frac{1}{2}$; as a consequence, $I(\epsilon,\nu,\alpha=0)\neq 0$ so that the continuous family of modified-Ivantsov solutions is completely destroyed by the presence of the singular perturbation parameter ϵ.

Before going to the $\alpha>0$ case, it is worth mentioning how the result derived here can be reconciled with the apparently different result obtained in Ref. 1. The point is that η_s moves towards i as $\nu\to 0$, and it can be checked that $X(\eta)$ has a branch point at $\eta=i$. In a neighborhood of this point

$$S_0(\eta)-S_0(i)\propto\frac{1}{\sqrt{\nu}}(\eta-i)^{7/4}\quad\text{as }\nu\to 0$$

and therefore, in this limit, the integral (3.9) is actually dominated by the behavior of S_{0+} close to $\eta=i$; by taking this into account the results of Ref. 1 can be recovered.

If $\alpha>0$ the analysis of Ref. 8 can be applied. The saddle points are determined by the equation

$$\frac{\lambda^3}{2\nu}=1-\alpha+\frac{8\alpha(\lambda^2-1)}{\lambda^4},\quad \mathrm{Re}\lambda>0 \tag{4.7}$$

where $\lambda=(1+\eta^2)^{1/2}$. The condition on $\mathrm{Re}\lambda$ comes from our being interested in the solutions of (3.7) on the Riemann sheet defined by $(1+\eta^2)^{1/2}>0$ for $\mathrm{Re}\eta$. If $\nu(1-\alpha)<\frac{1}{2}$ and $0<\alpha<\frac{3}{19}$, the only real solutions occur when $0<\lambda<1$ and hence, as promised, both (2.6) and (3.8) are well defined in the whole range $-\infty<\eta<+\infty$.

Let us consider stationary points in the upper half-plane only; we see that, as ν decreases from $[2(1-\alpha)]^{-1}$ toward zero, two distinct saddle points, originally on the imaginary axis, merge together and acquire a real part when $\nu<\nu_m(\alpha)\neq 0$. (ν_m will be computed shortly.)

We can therefore distinguish two cases.

(a) $\nu_m(\alpha)<\nu<[2(1-\alpha)]^{-1}$. Here we have two purely imaginary saddle points η_1 and η_2 with $-i\eta_1<-i\eta_2<1$. Again $S_{0+}(\eta_1)$ and $S_{0+}(\eta_2)$ are real and, by an argument similar to the one used to show that $S_{0+}(\eta_s,\alpha=0)>0$, $S_{0+}(\eta_1)\neq S_{0+}(\eta_2)$. Because $\eta_2\to\eta_s(\alpha=0)$ in the limit $\alpha\to 0$, by continuity in α we conclude that the only relevant point in the $\epsilon\to 0$ limit is η_2. The situation is analogous to the $\alpha=0$ case and $I(\nu,\epsilon,\alpha)\neq 0$.

(b) $0<\nu<\nu_m(\alpha)$. In this case the two distinct points η_1 and η_2 satisfy the relation $\eta_1=-\eta_2^*$ and

$$S_{0+}(\eta_1)=S_{0+}(-\eta_2^*)=S_{0+}^*(\eta_2)\ . \tag{4.8}$$

Hence the contribution from both the saddle points must be taken into account. If η_1 and η_2 are separated enough we can perform separated steepest-descent integrations at each point; the result will then be qualitatively very different from the case (a).

Because of (3.10)–(3.12) we see that Γ_{η_1} and Γ_{η_2} are obtained one from the other by simple translation. Furthermore,

$$\begin{aligned} I(\nu,\epsilon,\alpha)&=-i\sum_{i=1,2}\exp\left[-\frac{1}{\epsilon}S_{0+}(\eta_s,\nu,\alpha)\right]\int_{\Gamma_{\eta_i}}dz\frac{\rho(\eta_i)}{z^2}\exp\left[-\frac{1}{\epsilon}\gamma(\eta_i)z^3\right]\\ &=-i\exp\left[-\frac{1}{\epsilon}S_{0+}(\eta_1,\nu,\alpha)\right]\int_{\Gamma_{\eta_1}}dz\frac{\rho(\eta_1)}{z^2}\exp\left[-\frac{1}{\epsilon}\gamma(\eta_i)z^3\right]+\text{c.c.}\\ &=-2\epsilon^{-1/3}r(\nu,\alpha)\exp\left[-\frac{1}{\epsilon}a(\nu,\alpha)\right]\cos\left[\frac{1}{\epsilon}b(\nu,\alpha)+\vartheta(\nu,\alpha)\right]\ , \end{aligned} \tag{4.9}$$

where we have written $S_{0+}=a-ib$ and

$$i\int_{\Gamma_{\eta_1}}dz\frac{\rho(\eta_1)}{z^2}\exp[-\gamma(\eta_1,\nu,\alpha)z^3]=r(\nu,\alpha)\exp[i\vartheta(\nu,\alpha)]\ .$$

Since $b(\nu,\alpha)\neq 0$, we see explicitly that, in the limit $\epsilon\to 0$, $I(\nu,\epsilon,\alpha)$ is a rapidly oscillating function of ν and therefore the solvability condition can indeed be satisfied. The naturally selected value of ν should then be given by that $\nu=\nu^*(\epsilon,\alpha)$ at which the first oscillation occurs. (Smaller values are likely to correspond to unstable solutions.[11])

The algebraic analysis is the same as in the BLM and we merely need to report the result; $\nu_m(\alpha)$ is obtained by solving (4.7) in the case when the two saddle points merge. The value of λ at this point is

$$\lambda_m^2(\alpha)=\frac{20\alpha}{3(1-\alpha)}\left[-1+\left[1+\frac{21(1-\alpha)}{50\alpha}\right]^{1/2}\right]\ ,$$

so that

$$\nu_m(\alpha)=\frac{3\lambda_m^7}{32\alpha(2-\lambda_m^2)}\ .$$

We can then write $\nu^*(\alpha,\epsilon)=\nu_m(\alpha)-\delta\nu$, and find that $\delta\nu$ is determined by the relation

$$\left[-\frac{d^2f}{d\lambda^2}\bigg|_{\lambda=\lambda_m}\right]^{-1/2}\frac{\lambda_m(\delta\nu)^{5/2}}{(1-\lambda_m^2)[\nu_m(\alpha)]^3}\approx\epsilon\ . \tag{4.10}$$

In the limit $\alpha\ll 1$ we have finally

$$\nu^*(\alpha,\epsilon)\simeq\frac{7}{8}\left[\frac{56\alpha}{3}\right]^{3/4}\left[1-\beta_0\left[\frac{\epsilon^2}{\alpha}\right]^{1/5}\right]\ , \tag{4.11}$$

where β_0 is a constant of order unity.

Let us summarize the results obtained in this paper. We have considered the two-dimensional symmetric model of solidification in the limit of large undercooling. One of the motivations for this analysis is that although, at present, the limit $\Delta\to 1$ does not seem to be within reach of experiments (not to mention the problems in obtaining effectively two-dimensional dendrites), nevertheless it represents a physically meaningful limit in which to check the validity of the solvability mechanism. Furthermore, in this way we can compare the results obtained from this fully nonlocal and, in certain cases, fairly realistic model of solidification, with those derived in the context of models of the boundary-layer type[12] which, being completely local, are much more tractable both analytically and numerically. The analysis presented shows that the agreement with the original version of the BLM (Ref. 4) is indeed very good.

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