Coherent states for the damped harmonic oscillator

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Using the Caldirola-Kanai Hamiltonian for the damped harmonic oscillator, exact coherent states are constructed. These new coherent states satisfy the properties which coherent states should generally have.

Since the coherent states for the harmonic oscillator were first constructed by Schrödinger,[1] they have been widely used to describe many fields of physics.[2–5] Recently Nieto and Simmons have constructed coherent states for particles in general potentials[6] and have applied their formalism to confining one-dimensional systems,[7] such as the harmonic oscillator with centripetal barrier and the symmetric Pöschl-Teller potential, and also to nonconfining one-dimensional systems[8] with the symmetric Rosen-Morse potential and the Morse potential. For time-dependent systems Lewis and Risenfeld[9] have investigated the harmonic oscillator with time-dependent frequency $\omega(t)$. Hartley and Ray[10] have obtained exact coherent states for this time-dependent harmonic oscillator on the basis of Lewis and Risenfeld theory. Hartley-Ray results satisfy most, but not all, of the properties of the coherent states. In the case of a quantum-mechanical model of a damped forced harmonic oscillator, Dodonov and Man'ko[11] have introduced the Caldirola-Kanai Hamiltonian[12] with an external force term and constructed integrals of motion of this Hamiltonian, eigenstates, and coherent states. The main flaw of the Dodonov-Man'ko result is its uncertainty relation $\Delta p \Delta x \geq e^{-\gamma t}\hbar/2$, in which the uncertainty vanishes as $t \rightarrow \infty$. This contradiction is critically reviewed by Greenberger[13] and Cervero and Villarroel.[14] Greenberger introduced the variable mass $m = m_0 e^{\gamma t}$, and removed the violation of uncertainty.

In this paper we construct exact coherent states for the damped harmonic oscillator described by the Caldirola-Kanai Hamiltonian

$$\mathcal{H} = e^{-\gamma t}\frac{p^2}{2m} + e^{\gamma t}\tfrac{1}{2}m\omega_0^2 x^2 . \tag{1}$$

We first define a creation operator $a^\dagger$ and an annihilation operator a, and using these operators we will derive the representation of coherent states and investigate whether our coherent states satisfy the following properties: (i) They are eigenstates of the annihilation operator, (ii) they are created from the vacuum or the ground states by a unitary operator, (iii) they represent the minimum uncertainty states, and (iv) they are not orthogonal but complete and normalized.

In the preceding paper[15] (hereafter referred to as paper I) we have developed the quantum theory of the damped driven harmonic oscillator with the Caldirola-Kanai Hamiltonian with an external driving force $f(t)$ by the path-integral method. In paper I, setting $f(t)=0$, the Hamiltonian is reduced to Eq. (1) and all other results become those corresponding to the Hamiltonian [Eq. (1)] the Lagrangian, mechanical energy, and propagator are given by

$$\mathcal{L} = e^{\gamma t}(\tfrac{1}{2}m\dot{x}^2 - \tfrac{1}{2}m\omega_0^2 x^2) , \tag{2}$$

$$E = e^{-2\gamma t}\frac{p^2}{2m} + \tfrac{1}{2}m\omega_0^2 x^2 , \tag{3}$$

$$K(x,t;x_0,0) = \left[\frac{m\omega e^{(\gamma/2)t}}{2\pi i\hbar\sin(\omega t)}\right]^{1/2} \exp\left[\frac{im}{4\hbar}\left[\gamma(x_0^2 - e^{\gamma t}x^2) + \frac{2\omega}{\sin(\omega t)}[(x^2 e^{\gamma t} + x_0^2)\cos(\omega t) - 2e^{(\gamma/2)t}xx_0]\right]\right] , \tag{4}$$

with $\omega = (\omega_0^2 - \gamma^2/4)^{1/2}$. Here, the energy expression in Eq. (3) is not equal to the Hamiltonian itself. With the help of Eq. (4) and the wave function of the simple harmonic oscillator we obtain the wave function of the damped harmonic oscillator,

$$\psi_n(x,t) = \frac{N}{(2^n n!)^{1/2}} H_n(Dx) \exp\left[-i(n+\tfrac{1}{2})\cot^{-1}\left[\frac{\gamma}{2\omega} + \cot(\omega t)\right] - Ax^2\right] , \tag{5}$$

where

$$N=\left[\frac{m\omega}{\pi\hbar}\right]^{1/4}\frac{e^{-1/4\gamma t}}{\zeta(t)\sin^{1/2}(\omega t)}\ ,$$

$$\zeta^2(t)=\frac{\gamma^2}{4\omega^2}+\frac{\gamma}{\omega}\cot(\omega t)+\csc^2(\omega t)\ ,$$

$$A(t)=\frac{m\omega}{2\hbar}e^{\gamma t}\left[\frac{1}{\zeta(t)^2\sin^2(\omega t)}+i\left[\frac{\gamma}{2\omega}-\cot(\omega t)+\frac{\frac{\gamma}{2\omega}+\cot(\omega t)}{\zeta(t)^2\sin^2(\omega t)}\right]\right]\ ,$$

$$D(t)=\left[\frac{m\omega}{\hbar}\right]^{1/2}\frac{e^{1/2\gamma t}}{\zeta(t)\sin(\omega t)}\ . \tag{6}$$

The quantum-mechanical expectation values of mechanical energy E [Eq. (3)] take the form

$$\langle E\rangle_{mn}=-\frac{\hbar^2}{2m}e^{-2\gamma t}\left\langle\frac{\partial^2}{\partial x^2}\right\rangle_{mn}+\tfrac{1}{2}m\omega_0^2\langle x^2\rangle_{mn}\ . \tag{7}$$

The evaluation of Eq. (7) gives the nonzero matrix elements which occur only in the principal diagonal and the two second off diagonal,

$$\langle E\rangle_{n+2,n}=[(n+2)(n+1)]^{1/2}\Theta(t)\ , \tag{8}$$

$$\langle E\rangle_{nn}=\tfrac{1}{2}(n+\tfrac{1}{2})\hbar\omega e^{-\gamma t}\left[\frac{\omega_0^2}{\omega^2}\zeta(t)^2\sin^2(\omega t)+\frac{1}{\zeta(t)^2\sin^2(\omega t)}\right]\ , \tag{9}$$

where

$$\theta(t)=\tfrac{1}{4}\hbar\omega e^{-\gamma t}\exp\left[\left[2i\cot^{-1}\left[\frac{\gamma}{2\omega}+\cot(\omega t)\right]\right]\right]\left\{\zeta(t)^2\sin^2(\omega t)-\frac{1}{\zeta(t)^2\sin^2(\omega t)}+\frac{1}{\zeta(t)^2\sin^2(\omega t)}\left[\left[\frac{\gamma}{2\omega}-\cot(\omega t)\right]\zeta(t)^2\sin^2(\omega t)+\frac{\gamma}{2\omega}+\cot(\omega t)\right]^2-2i\left[\left[\frac{\gamma}{2\omega}-\cot(\omega t)\right]\zeta(t)^2\sin^2(\omega t)+\frac{\gamma}{2\omega}+\cot(\omega t)\right]\right\}\ . \tag{10}$$

Taking the complex conjugate and changing n into $(n-2)$ in Eq. (8) we can easily obtain the energy expectation value in the $(n-2,n)$ state.

In a similar way to that used to obtain Eq. (7) we can obtain the uncertainty relations in the various states,

$$[(\Delta p)(\Delta x)]_{n+2,n}=\frac{\hbar}{2}[(n+2)(n+1)]^{1/2}\beta(t)\ , \tag{11}$$

$$[(\Delta p)(\Delta x)]_{n+1,n}=\frac{\hbar}{2}(n+1)\beta(t)\ , \tag{12}$$

$$[\Delta p\,\Delta x]_{nn}=(n+\tfrac{1}{2})\hbar\beta(t)\ , \tag{13}$$

and

$$\beta(t)=\left[1+\left\{\left[\frac{1}{8}\left[\frac{\gamma}{\omega}\right]^3+\left[\frac{\gamma}{\omega}\right]\right]\sin^2(\omega t)+\frac{1}{8}\left[\frac{\gamma}{\omega}\right]^2\sin(2\omega t)\right\}^2\right]^{1/2}\ . \tag{14}$$

Changing n into $(n-1)$ and $(n-2)$, respectively, in Eqs. (12) and (11) we can obtain the uncertainty relations in the $(n-1,n)$ and $(n-2,n)$ states.

Before we construct the annihilation operator a and creation operator $a^\dagger$, we give the properties of the coherent states. The states can be defined by the eigenstates of the nonhermitian operator a,

$$a\,|\alpha\rangle=\alpha\,|\alpha\rangle\ . \tag{15}$$

Using the completeness relation for the number representations, we can expand $|\alpha\rangle$ as

$$|\alpha\rangle=e^{-(1/2)|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle=e^{-(1/2)|\alpha|^2}e^{\alpha a^\dagger}|0\rangle\ , \tag{16}$$

where $|0\rangle$ is the vacuum or ground state and is independent of n. The calculation of $\langle\beta|\alpha\rangle$ in Eq. (16) gives

$$\langle\beta|\alpha\rangle=e^{-1/2(|\alpha|^2+|\beta|^2)+\alpha\beta^*}\ . \tag{17}$$

Since Eq. (17) has nonzero values for $\alpha\neq\beta$, the states are not orthogonal, but when $|\alpha-\beta|^2\to\infty$ the states become orthogonal.

The eigenvalues α of coherent states are complex numbers $u+iv$, and thus the completeness relation of coherent states is written as

$$\int|\alpha\rangle\langle\alpha|\frac{d^2\alpha}{\pi}=1\ , \tag{18}$$

where 1 is the identity operator and $d^2\alpha$ is given by $d(\mathrm{Re}u)d(\mathrm{Im}v)$.

To define $a^\dagger$ and a for the damped harmonic oscillator we make use of Eq. (5) for $\langle x\rangle_{mn}$ and $\langle p\rangle_{mn}$,

$$\begin{aligned}\langle x\rangle_{mn}&=\int_{-\infty}^{\infty}\psi_m^*(x)x\psi_n(x)dx\\&=\tfrac{1}{2}(n+1)^{1/2}(\mathrm{Re}A)^{-1/2}\exp\left[i\cot^{-1}\left[\frac{\gamma}{2\omega}+\cot(\omega t)\right]\right]\delta_{m,n+1}+\tfrac{1}{2}n^{1/2}(\mathrm{Re}A)^{-1/2}\\&\quad\times\exp\left[-i\cot^{-1}\left[\frac{\gamma}{2\omega}+\cot(\omega t)\right]\right]\delta_{m,n-1}\\&=(n+\tfrac{1}{2})^{1/2}\mu(t)\delta_{m,n+1}+n^{1/2}\mu(t)^*\delta_{m,n-1}\ ,\end{aligned}\tag{19}$$

$$\begin{aligned}\langle p\rangle_{mn}&=\int_{-\infty}^{\infty}\psi_m^*(x)\frac{\hbar}{i}\frac{\partial}{\partial x}\psi_n(x)dx\\&=i\hbar\sqrt{2}(n+1)^{1/2}\frac{A}{D}\exp\left[i\cot^{-1}\left[\frac{\gamma}{2\omega}+\cot(\omega t)\right]\right]\delta_{m,n+1}\\&\quad+i\hbar\sqrt{2}n^{1/2}\left[\frac{A}{D}-D\right]\exp\left[-i\cot^{-1}\left[\frac{\gamma}{2\omega}+\cot(\omega t)\right]\right]\delta_{m,n-1}\\&=(n+\tfrac{1}{2})^{1/2}\eta(t)\delta_{m,n+1}+n^{1/2}\eta^*(t)\delta_{m,n-1}\ ,\end{aligned}\tag{20}$$

where

$$\mu(t)=\tfrac{1}{2}(\mathrm{Re}A)^{-1/2}\exp\left[i\cot^{-1}\left[\frac{\gamma}{2\omega}+\cot(\omega t)\right]\right]\ ,\tag{21}$$

$$\eta(t)=\sqrt{2}i\hbar\frac{A}{D}\exp\left[i\cot^{-1}\left[\frac{\gamma}{2\omega}+\cot(\omega t)\right]\right]\ ,\tag{22}$$

and we have the relation

$$\eta\mu^*-\eta^*\mu=2i\ \mathrm{Im}\left[\tfrac{1}{2}(Re A)^{+1/2}\sqrt{2}i\hbar\frac{A}{D}\right]=i\hbar\ .\tag{23}$$

Therefore, we define annihilation operator a and creation operator $a^\dagger$ for the damped harmonic oscillator as

$$a=\frac{1}{i\hbar}(\eta x-\mu p)\ ,\tag{24}$$

$$a^\dagger=\frac{1}{i\hbar}(\mu^* p-\eta^* x)\ ,\tag{25}$$

where the expressions of x and p by a and $a^\dagger$ are

$$x=\mu^* a+\mu a^\dagger\ ,\tag{26}$$

$$p=\eta^* a+\eta a^\dagger\ .\tag{27}$$

Since η is not equal to μ in Eqs. (21) and (22), we can easily confirm that a and $a^\dagger$ are not Hermitian operators, but the following relations are preserved:

$$[x,p]=i\hbar\ ,\tag{28}$$

$$[a,a^\dagger]=1\ .\tag{29}$$

Now we evaluate the transformation function $\langle x\mid\alpha\rangle$ from coherent states to the coordinate representation $|x\rangle$. From Eqs. (15) and (24) we have

$$\left[\eta x'-\mu\frac{\hbar}{i}\frac{\partial}{\partial x'}\right]\langle x'\mid\alpha\rangle=i\hbar\alpha\langle x'\mid\alpha\rangle\ .\tag{30}$$

For convenience we change the variable x' into x and solve this differential equation to obtain

$$\langle x\mid\alpha\rangle=N\exp\left[\frac{1}{\mu}\alpha x-(2i\hbar\mu)^{-1}\eta x^2\right]\ ,\tag{31}$$

where N is the integral constant. Taking N to satisfy Eq. (18), we find the eigenvectors of operator a given in the coordinate representation $|x\rangle$,

$$\langle x\mid\alpha\rangle=(2\pi\mu\mu^*)^{-1/4}\exp\left[-\frac{1}{2i\hbar}\frac{\eta}{\mu}x^2+\frac{\alpha}{\mu}x-\frac{1}{2}|\alpha|^2-\frac{1}{2}\frac{\mu^*}{\mu}\alpha^2\right]\ .\tag{32}$$

Next we show that a coherent state represents a minimum uncertainty state. With the help of the relations between a, $a^\dagger$, x, and p we evaluate the expectation values of x, p, x^2, and p^2 in state $|\alpha\rangle$ as follows:

$$\begin{aligned}\langle x\rangle&=\langle\alpha\mid\mu^* a+\mu a^\dagger\mid\alpha\rangle=\mu^*\alpha+\mu\alpha^*\ ,\\\langle p\rangle&=\langle\alpha\mid\eta^* a+\eta a^\dagger\mid\alpha\rangle=\eta^*\alpha+\eta\alpha^*\ ,\\\langle x^2\rangle&=\mu^{*2}\alpha^2+\mu\mu^*(1+2\alpha\alpha^*)+\mu^2\alpha^{*2}\ ,\\\langle p^2\rangle&=\eta^{*2}\alpha^2+\eta\eta^*(1+2\alpha\alpha^*)+\eta^2\alpha^{*2}\ .\end{aligned}\tag{33}$$

From Eq. (33) we have

$$(\Delta x)^2=\langle x^2\rangle-\langle x^2\rangle=\mu\mu^*\ ,\tag{34}$$

$$(\Delta p)^2=\langle p^2\rangle-\langle p^2\rangle=\eta\eta^*\ ,\tag{35}$$

and thus the uncertainty relation becomes

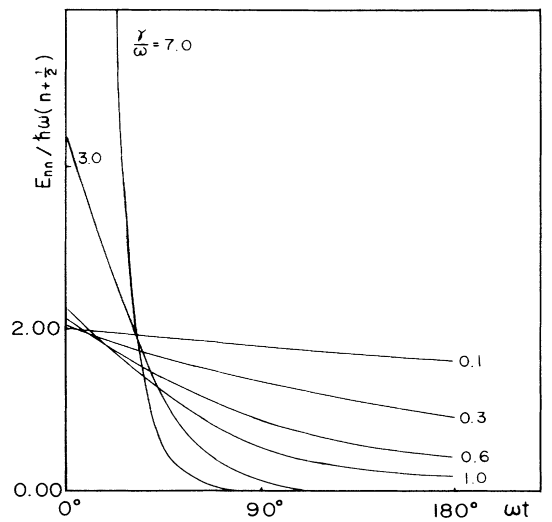

FIG. 1. Energy expectation value for (n,n) state as a function of ωt at the various value of γ/ω. As γ/ω tends to zero, the energy approaches the constant values.

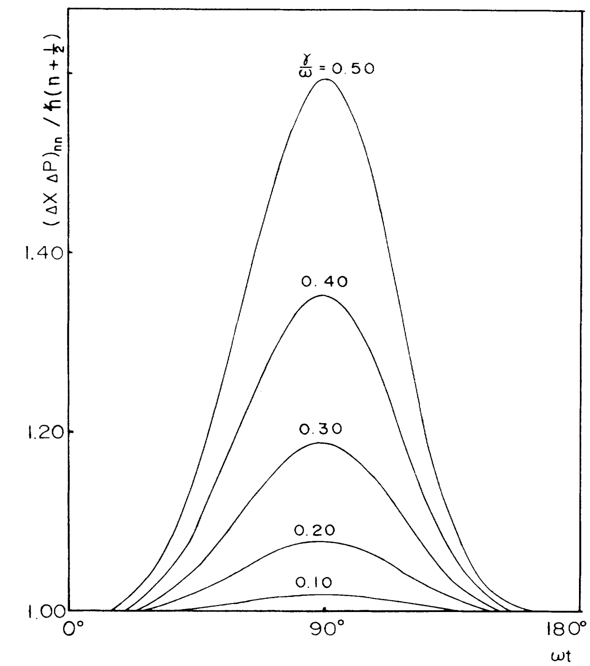

FIG. 2. Uncertainty relation for the (n,n) state vs ωt at various values of γ/ω.

$$(\Delta x)(\Delta p)=\{\,|\eta|^2|\mu|^2\}^{1/2}=\frac{\hbar}{2}\beta(t)\ . \tag{36}$$

Equation (36) is the minimum uncertainty corresponding to Eq. (13) in (0,0) state.

All of the formulas we have derived are reduced to those of the simple harmonic oscillator when $\gamma=0$. The propagator [Eq. (4)] has a very similar form to those of Cheng[16] and others,[17] but the wave function [Eq. (5)] is of new form.

We should note that the same classical equation of motion can be obtained from many different actions and thus one may have many different propagators corresponding to the actions. Therefore it is very important to get the correct propagator. The mechanical energy [Eq. (3)] is not identical to the Hamiltonian operator [Eq. (1)]. Hence, we assume that this Hamiltonian represents the quantum-mechanical dissipative system.

Figures 1 and 2 illustrate the decay of the energy expectation value and the uncertainty relation as a function of γ/ω in the (n,n) state. Although we have shown only the principal diagonal element, i.e., $\langle E\rangle_{nn}$ of the energy expectation values, there are four off diagonals adjacent to the principal diagonal, which are involved in the exponential decaying term $e^{-\gamma t}$. $\langle E\rangle_{nn}$ approaches the constant value as $\gamma/\omega\rightarrow 0$. The uncertainty for the (n,n) state with period π [Eq. (13)] is reduced to that of the harmonic oscillator of 0° and 180°.

From all of the above we conclude that the coherent states for the damped harmonic oscillator with the Caldirola-Kanai Hamiltonian we have constructed satisfy the properties of coherent states (i)–(iv).

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